Solutions of kinetic equations related to non-local conservation laws

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Abstract

Conservation laws are well known to be a crucial part of modeling. Considering such models with the inclusion of non-local flows is becoming increasingly important in many models. On the other hand, kinetic equations provide interesting theoretical results and numerical schemes for the usual conservation laws. Therefore, studying kinetic equations associated to conservation laws for non-local flows naturally arises and is very important. The aim of this paper is to propose kinetic models associated to conservation laws with a non-local flux in dimension d and to prove the existence of solutions for these kinetic equations. This is the very first result of this kind. In order for the paper to be as general as possible, we have highlighted the properties that a kinetic model must verify in order that the present study applies. Thus the result can be applied to various situations. We present two sets of properties on a kinetic model and two different techniques to obtain an existence result. Finally, we present two examples of kinetic model for which our results apply, one for each set of properties.

Key-words: Scalar conservation laws – non-local flux – kinetic equations

Mathematics Subject Classification: 35L65, 35A35, 35D30, 82C40

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1 Introduction

1.1 Context

Conservation laws are now well known to model a wide range of situations in physics, biology, economics, engineering, etc. Non-local fluxes have been introduced recently to model pedestrian or vehicle traffic [15], [16], [18], [12]. These fields of application are also emerging for biology with, for example, the article [1] and non-local models should appear in the coming years to model more phenomena. We also refer to [10] for more bibliography on this subjet. Indeed, it is the natural extension of the conservation laws to take into account phenomena that are not all necessarily local. On the other hand, it is known that kinetic models associated with conservation laws provide interesting theoretical results and numerical schemes for usual conservation laws. (see for example [22], [25], [2]). It is therefore natural to propose and study kinetic models for non-local conservation laws. This is the object of this paper: to propose kinetic models associated with conservation laws with a non-local flux and to prove the existence of solutions for these kinetic equations. This paper is to our knowledge the first result of this kind of problem, that is to say where the non-local term is considered in the kinetic framework. Note that this extension is not trivial because the usual kinetic equations only account for the local character of the variables of the unknowns of the conservation law. Also one of the ideas of this paper is to consider kinetic equations with some kind of second kinetic variable which will account for non-local quantities. We have highlighted the properties that the kinetic equation has to verify for the proposed method to apply, so that it can be used for various models. We present two types of properties on the kinetic model and two different techniques to obtain an existence result. Finally, we present two examples for which our results apply. This paper is the first study of this kind of kinetic model associated with non-local laws with also the idea of adding this additional variable.

1.2 Models

First of all, let us specify the kind of models that we will study both from the point of view of the law of conservation with non-local flux then from the point of view of the kinetic equation.

For the non-local scalar conservation law, we consider the following equation. A function $\rho : [0, +\infty[\times \mathbb{R}^d \to \mathbb{R}$ have to satisfy

$$\partial_t \rho + \mathrm{div}_x(F(\rho)G(\eta * \rho)) = 0, \tag{1.1}$$

where the product $F(\rho)G(\eta * \rho)$ have to be understood in the sense that $(F(\rho)G(\eta * \rho))_i = F_i(\rho(t,x))G_i((\eta * \rho)(t,x))$ for any $i = 1, \dots, d$, which means

$$\partial_t \rho(t,x) + \sum_{i=1}^{d} \partial_{x_i}(F_i(\rho(t,x))G_i((\eta * \rho)(t,x))) = 0, \quad (t,x) \in [0, +\infty[\times \mathbb{R}^d, \tag{1.2}$$

where $F, G \in C^1(\mathbb{R}, \mathbb{R}^d)$, $\eta \in C^1(\mathbb{R}^d, \mathbb{R}) \cap L^\infty(\mathbb{R}^d)$ and

$$(\eta * \rho)(t,x) = \int_{\mathbb{R}^d} \eta(x-y)\rho(t,y)\, dy. \tag{1.3}$$

This term is well defined if $y \mapsto \rho(t,y) \in L^1(\mathbb{R}^d)$ for a.e. t.

For the kinetic equation, we consider the following model. A function $f_\varepsilon : [0, +\infty[\times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ have to satisfy

$$\partial_t f_\varepsilon + \mathrm{div}_x(a(v,\xi) f_\varepsilon) = \frac{\mathcal{M}_{\rho_\varepsilon} - f_\varepsilon}{\varepsilon}, \tag{1.4}$$

which means

$$\partial_t f_\varepsilon(t,x,v,\xi) + \sum_{i=1}^{d} \partial_{x_i}(a_i(v,\xi) f_\varepsilon(t,x,v,\xi)) = \frac{\mathcal{M}_{\rho_\varepsilon}(t,x,v,\xi) - f_\varepsilon(t,x,v,\xi)}{\varepsilon}, \tag{1.5}$$

for $(t,x,v,\xi) \in [0, +\infty[\times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$, where $a : \mathbb{R}^2 \to \mathbb{R}^d$,

$$\rho_\varepsilon(t,x) = \iint_{\mathbb{R}^2} f_\varepsilon(t,x,\tilde{v},\tilde{\xi})\, d\tilde{\xi} d\tilde{v} \tag{1.6}$$

and the Maxwelian $\mathcal{M}_{\tilde{\rho}} : [0, +\infty[\times \mathbb{R}^{d+2} \to \mathbb{R}$, defined for any $\tilde{\rho} : [0, +\infty[\times \mathbb{R}^d \to \mathbb{R}$, will have to be related with the non-local scalar conservation and will be defined later. The variable v is the classical kinetic variable and the variable ξ is the new added variable that must account for the non-local quantities of the associated non-local conservation law. For some practical conservation laws with non-local flux exemples, see the references [15], [16], [18], [12] and [1]. For BGK model with classical flux, see [23], [22], [6], [20] and [3]. Notice that in the present work, we are not in the classical BGK model framework as we need to extend the models by allowing some kind of second kinetic velocity to account the non-local effect.

1.3 Main results and organization of the paper

The organization of the paper is as follows. In section 2, we list the properties that a kinetic model must satisfy in order to be applied to the present study. First, we need two properties (2.1)-(2.2) to ensure consistence between the kinetic equation and the non-local scalar conservation equation. Then, we expose the necessary properties to obtain the existence of solutions for the kinetic equation according to the method used. We will present two proof methods, each one requiring specific properties. For the first existence result, we need (2.3). For the second existence result, we need (2.4)-(2.10). This section ends with a formal proof that justifies the need for consistence properties. Note that in the formal limit section, we also consider in the present situation the formal limit as the kernel approaches the Dirac delta.

In section 3, we study the well-posedness of the kinetic equation with property (2.3). In this framework, we can use a fixed point and the proof is relatively usual. We get the following result.

Theorem 1.1 *Let $f^0 \in L^1(\mathbb{R}^{d+2})$. We consider a Maxwellian $\mathcal{M}$ satisfying (2.1) and (2.3). Then, for any $\varepsilon > 0$, there exists $f_\varepsilon \in L^\infty([0,T], L^1(\mathbb{R}^{d+2}))$ for any $T > 0$ solution of (1.4) with initial data f^0. Furthermore this solution $f_\varepsilon \in L^\infty([0,T], L^1(\mathbb{R}^{d+2}))$ is unique with the initial data.*

This case is the easiest among the two that we study but most of the models won't verify (2.3) thus it requires the study of the second model.

This is why in section 4, we study the existence of a solution for the kinetic equation with properties (2.4)-(2.10). We get the following result.

Theorem 1.2 *Let $F, G \in C^1(\mathbb{R}, \mathbb{R}^d)$, $\eta \in C^1(\mathbb{R}^d, \mathbb{R}) \cap L^\infty(\mathbb{R}^d)$. Let $f^0 \in L^1(\mathbb{R}^{d+2}) \cap L^2(\mathbb{R}^{d+2})$ such that $xf^0, \xi f^0, vf^0, a(v,\xi)f^0 \in L^1(\mathbb{R}^{d+2})$ and*

$$\int_{\mathbb{R}^d} \left(\iint\limits_{\mathbb{R}^2} f^0(x,v,\xi)\, dv d\xi \right)^2 dx < +\infty.$$

We consider a maxwellian $\mathcal{M}$ satisfying (2.1) and (2.4)-(2.10). Assume that there exists a constant $K > 0$ such that

$$|F_i(z)| \leq K(|z| + |z|^2) \quad \text{for any } z \in \mathbb{R} \text{ and any } i = 1, \ldots, d. \tag{1.7}$$

Then, for any $\varepsilon > 0$, there exists $f_\varepsilon \in L^\infty([0,T], L^1(\mathbb{R}^{d+2}))$ for any $T > 0$ solution of (1.4) with initial data f^0.

This proof needs Schauder's theorem and is much more complex and requires solving numerous technical difficulties.

Then, in section 5, we present a model which satisfies the properties for the first theorem and in section 6, a model which satisfies the ones for the second theorem.

2 General framework for the kinetic model

This section defines the general framework and the properties that a kinetic model must satisfy in order to be applied to the present study. Then, we present the formal limit of the kinetic model to check that the limit equation is at least formally the scalar non-local conservation equation.

2.1 Properties that a kinetic model has to satisfy to be applied to our study

The first properties that a kinetic model has to satisfy are consistency type ones. They will ensure that the formal limit is indeed the non-local equation.
Consistency properties:

We assume that, for any $\rho : [0,+\infty[\times\mathbb{R}^d \to \mathbb{R}$, there exists $\mathcal{M}_\rho : [0,+\infty[\times\mathbb{R}^{d+2} \to \mathbb{R}$ such that for a.e. (t,x),

$$\iint\limits_{\mathbb{R}^2} \mathcal{M}_\rho(t,x,v,\xi)\, d\xi dv = \rho(t,x) \tag{2.1}$$

and

$$\iint\limits_{\mathbb{R}^2} a(v,\xi)\mathcal{M}_\rho(t,x,v,\xi)\, d\xi dv = F(\rho(t,x))G((\eta*\rho)(t,x)). \tag{2.2}$$

These two previous properties ensure the consistency between the kinetic and the non-local scalar equation.

For the existence of a solution to the kinetic equation, according to the model, we can consider two different lists of properties.

First set of properties to get an existence result:

We assume that for any $\rho_1,\rho_2 : [0,+\infty[\times\mathbb{R}^d \to \mathbb{R}$ such that $x\mapsto\rho_1(t,x)$, $\rho_2(t,x)\in L^1(\mathbb{R}^d)$ a.e. t, we have for a.e. t,

$$\int_{\mathbb{R}^d}\iint\limits_{\mathbb{R}^2} \left|\mathcal{M}_{\rho_1}(t,x,v,\xi) - \mathcal{M}_{\rho_2}(t,x,v,\xi)\right|\, dvd\xi\, dx \le K\int_{\mathbb{R}^d} \left|\rho_1-\rho_2\right|(t,x)\, dx. \tag{2.3}$$

Second set of properties to get an existence result:

We assume that there exists constants $K_2, K_3, K_4, K_5 > 0$ and $p = 1$ or 2 such that, for any $\rho : [0,+\infty[\times\mathbb{R}^d \to \mathbb{R}$, we have for a.e. (t,x),

$$\iint\limits_{\mathbb{R}^2} |\mathcal{M}_\rho(t,x,v,\xi)|\, d\xi dv \le |\rho(t,x)|, \tag{2.4}$$

$$\iint\limits_{\mathbb{R}^2} |a_i(v,\xi)\mathcal{M}_\rho(t,x,v,\xi)|\, d\xi dv \le K_2|F_i(\rho(t,x))G_i((\eta*\rho)(t,x))|, \quad \text{for any } i=1,\dots,d, \tag{2.5}$$

$$\iint\limits_{\mathbb{R}^2} |v||\mathcal{M}_\rho(t,x,v,\xi)|\, d\xi dv \le K_4\,\rho^2(t,x), \tag{2.6}$$

$$\iint\limits_{\mathbb{R}^2} |\mathcal{M}_\rho(t,x,v,\xi)|^2\, d\xi dv \le K_5\,|\rho(t,x)|^p, \tag{2.7}$$

for any $\rho : [0, +\infty[\times\mathbb{R}^d \to \mathbb{R}$ such that $x \mapsto \rho(t, x) \in L^2(\mathbb{R}^d)$ a.e. t, we have for a.e. (t, x),

$$\iint_{\mathbb{R}^2} |\xi||\mathcal{M}_\rho(t, x, v, \xi)|\, d\xi dv \le K_3\, |\rho(t, x)| \left(1 + \int_{\mathbb{R}} |\rho(t, y)|^2\, dy\right), \tag{2.8}$$

and also

$$\text{if } \rho_n \to \rho \text{ a.e. } (t, x) \in]0, T[\times\mathbb{R}^d \text{ and } |\rho_n| \le |h| \in L^1(]0, T[\times\mathbb{R}^d), \\ \text{then } \mathcal{M}_{\rho_n} \to \mathcal{M}_\rho \text{ a.e. } (t, x, v, \xi) \in]0, T[\times\mathbb{R}^d \times \mathbb{R}^2 \tag{2.9}$$

and

$$\text{the term } a(v, \xi) \text{ allows to apply an averaging lemma.} \tag{2.10}$$

Remark 2.1 Let's explicit what we mean by averaging lemma. It is such a result that if

$$\partial_t g_n + \mathrm{div}_x(a(v, \xi) g_n) = h_n$$

with $(g_n)_n$ and $(h_n)_n$ bounded in $L^1(]0, T[\times\mathbb{R}^d\times\mathbb{R}^2)$ then ρ_n is compact in $L^1_{loc}(]0, T[\times\mathbb{R}^d)$ where

$$\rho_n(t, x) = \iint_{\mathbb{R}^2} \psi(\xi, v) g_n(t, x, v, \xi)\, d\xi dv$$

with $\psi \in L^\infty(]0, T[\times\mathbb{R}^d \times \mathbb{R}^2)$. The function $a(v, \xi)$ must satisfy a non degeneracy condition to pretend getting this kind of result. We will come back to this item later in the paper.

The first property allows us to apply contraction technics and the second list of properties to apply Schauder's result.

Remark 2.2 Notice that (2.1) and (2.4) imply

$$\iint_{\mathbb{R}^2} |\mathcal{M}_\rho(t, x, v, \xi)|\, d\xi dv = |\rho(t, x)|$$

and if $K_2 = 1$, then (2.2) and (2.5) imply

$$\iint_{\mathbb{R}^2} |a_i(v, \xi)\mathcal{M}_\rho(t, x, v, \xi)|\, d\xi dv = |F_i(\rho(t, x)) G_i((\eta * \rho)(t, x))|, \quad \textit{for any } i = 1, \ldots, d.$$

Remark 2.3 Notice also that if we have a bound like

$$\iint_{\mathbb{R}^2} |\xi||\mathcal{M}_\rho(t, x, v, \xi)|\, d\xi dv \le K_3\, |\rho(t, x)| \int_{\mathbb{R}} |\rho(t, y)|\, dy,$$

then it implies (2.8).

Remark 2.4 Any additional property expected on the solution, as for example the positivity of the solution, must be reflected on the properties. Thus for the important case where we want $\rho \geq 0$, we take the adapted properties:

$$\iint_{\mathbb{R}^2} \mathcal{M}_\rho(t,x,v,\xi)\, d\xi dv = \rho(t,x), \tag{2.11}$$

$$\iint_{\mathbb{R}^2} a(v,\xi)\mathcal{M}_\rho(t,x,v,\xi)\, d\xi dv = F(\rho(t,x))G((\eta * \rho)(t,x)) \tag{2.12}$$

and

$$\int_{\mathbb{R}^d} \iint_{\mathbb{R}^2} |\mathcal{M}_{\rho_1}(t,x,v,\xi) - \mathcal{M}_{\rho_2}(t,x,v,\xi)|\, dv d\xi\, dx \leq K \int_{\mathbb{R}^d} |\rho_1 - \rho_2|\,(t,x)\, dx \tag{2.13}$$

for any $\rho, \rho_1, \rho_2 \geq \mathbf{0}$ *and with* $x \mapsto \rho_1(t,x), \rho_2(t,x) \in L^1(\mathbf{R}^d)$ *a.e.* t *for the last property.*

2.2 Formal limit

Consistency properties, that is to say (2.1) and (2.2) are related to the consistency between kinetic and non-local equation by the following formal limit. Indeed we assume that the limit f of (f_ε) exists when $\varepsilon \to 0$. From

$$\mathcal{M}_{\rho_\varepsilon} - f_\varepsilon = \varepsilon \left(\partial_t f_\varepsilon + \mathrm{div}_x(a(v,\xi) f_\varepsilon)\right),$$

we formally have when $\varepsilon \to 0$

$$\mathcal{M}_\rho = f.$$

On the other hand, an integration with respect to (v,ξ) of (1.4) yields

$$\partial_t \iint_{\mathbf{R}^2} f_\varepsilon\, d\xi dv + \mathrm{div}_x \iint_{\mathbf{R}^2} a(v,\xi) f_\varepsilon\, d\xi dv = 0,$$

since

$$\iint_{\mathbf{R}^2} \mathcal{M}_{\rho_\varepsilon}(t,x,v,\xi)\, d\xi dv = \rho_\varepsilon(t,x) = \iint_{\mathbf{R}^2} f_\varepsilon(t,x,v,\xi)\, d\xi dv.$$

At the limit, we have

$$\partial_t \iint_{\mathbf{R}^2} f\, d\xi dv + \mathrm{div}_x \iint_{\mathbf{R}^2} a(v,\xi) f\, d\xi dv = 0,$$

and thus

$$\partial_t \iint_{\mathbf{R}^2} \mathcal{M}_\rho\, d\xi dv + \mathrm{div}_x \iint_{\mathbf{R}^2} a(v,\xi)\mathcal{M}_\rho\, d\xi dv = 0.$$

Now

$$\iint_{\mathbf{R}^2} \mathcal{M}_\rho(t,x,v,\xi)\, d\xi dv = \rho(t,x)$$

and

$$\iint_{\mathbf{R}^2} a(v,\xi)\mathcal{M}_\rho(t,x,v,\xi)\,d\xi dv = F(\rho)G(\eta*\rho).$$

Finally we get

$$\partial_t\rho + \mathrm{div}_x(F(\rho)G(\eta*\rho)) = 0.$$

The finding of this formal limit has two goals: first to present what would be the theoretical justification but also to point out that the kinetic solutions obtained can be taken, at least formally, as an approximation at order one in ε of the solutions of the scalar conservation law model. Notice that non-local scalar conservation laws can present dispersive effect for certain kernels as can be seen in the work [26], kernel different from those considered here however. Indeed having therefore an approximated equation which is more stable can be useful from a numerical point of view even if it is only formal in this first step. Going further and considering entropic solutions in one dimension is actually a challenging problem. In this spirit, let's refer the papers [9] and [10] which get one entropy in the case of some very special non-local conservation laws with a kernel which is a kind of approximation of unity in order to be close to the classical scalar case with local flux. Formally for our model in one dimension, if we consider a sequence of kernels $\eta_\mu \in L^\infty(\mathbf{R})$ and

$$(\eta_\mu*\rho)(t,x) = \int_{\mathbf{R}} \eta_\mu(x-y)\rho(t,y)\,dy \tag{2.14}$$

such that $\eta_\mu*\rho \to \rho$, we have at the limit

$$\partial_t\rho + \partial_x(F(\rho)G(\eta_\mu*\rho)) = 0$$

and then, for every non negative regular test function E,

$$E'(\rho)\partial_t\rho + E'(\rho)\partial_x(F(\rho)G(\rho)) = E'(\rho)\partial_x(F(\rho)(G(\rho)-G(\eta_\mu*\rho)))$$

that is to say

$$\partial_t E(\rho) + \partial_x\psi(\rho) = E'(\rho)\partial_x(F(\rho)(G(\rho)-G(\eta_\mu*\rho)))$$

where $\psi'(\rho) = E'(\rho)F'(\rho)G'(\rho)$. Let φ be a test function. Then

$$\begin{aligned} I &= \int_{\mathbf{R}} E'(\rho)\partial_x(F(\rho)(G(\rho)-G(\eta_\mu*\rho)))\varphi\,dx \\ &= \int_{\mathbf{R}} E'(\rho)F(\rho)\,\partial_x(G(\rho)-G(\eta_\mu*\rho))\varphi\,dx + \int_{\mathbf{R}} E'(\rho)F'(\rho)\partial_x\rho\,(G(\rho)-G(\eta_\mu*\rho))\varphi\,dx. \end{aligned}$$

In the case with $F(\rho)=\rho$ and for the entropy $E(\rho)=\rho^2$, we get

$$\begin{aligned} I &= \int_{\mathbf{R}} 2\rho^2\,\partial_x(G(\rho)-G(\eta_\mu*\rho))\varphi\,dx + \int_{\mathbf{R}} 2\rho\partial_x\rho\,(G(\rho)-G(\eta_\mu*\rho))\varphi\,dx \\ &= \int_{\mathbf{R}} 2\rho^2\,\partial_x(G(\rho)-G(\eta_\mu*\rho))\varphi\,dx + \int_{\mathbf{R}} \partial_x(\rho^2)\,(G(\rho)-G(\eta_\mu*\rho))\varphi\,dx \\ &= \int_{\mathbf{R}} \rho^2\,\partial_x(G(\rho)-G(\eta_\mu*\rho))\varphi\,dx + \int_{\mathbf{R}} \partial_x(\rho^2(G(\rho)-G(\eta_\mu*\rho)))\varphi\,dx \\ &= \int_{\mathbf{R}} \rho^2\,\partial_x(G(\rho)-G(\eta_\mu*\rho))\varphi\,dx - \int_{\mathbf{R}} \rho^2(G(\rho)-G(\eta_\mu*\rho))\partial_x\varphi\,dx. \end{aligned}$$

For a choice like $\eta(z) = \mathbb{1}_{z<0} e^z$ and $\eta_\mu(z) = \eta(z/\mu)/\mu$, we get

$$(\eta_\mu * \rho)(t,x) = \frac{1}{\mu} \int_x^{+\infty} e^{(x-y)/\mu} \rho(t,y)\, dy$$

and $\partial_x(\eta_\mu * \rho) = (\eta_\mu * \rho - \rho)/\mu$ which can be rephrased as

$$\rho = \eta_\mu * \rho - \mu \partial_x(\eta_\mu * \rho).$$

In this case we have

$$\begin{aligned}
& \int_{\mathbf{R}} \rho^2\, \partial_x(G(\rho) - G(\eta_\mu * \rho))\varphi\, dx \\
= & \int_{\mathbf{R}} \rho^2\, \partial_x(G(\rho) - G(\eta_\mu * \rho))\varphi\, dx \\
= & \int_{\mathbf{R}} \rho^2\, (G'(\rho)\partial_x \rho - G'(\eta_\mu * \rho)\partial_x(\eta_\mu * \rho))\varphi\, dx \\
= & \int_{\mathbf{R}} \rho^2\, G'(\rho)\partial_x \rho\, \varphi\, dx - \int_{\mathbf{R}} \rho(\eta_\mu * \rho - \mu\partial_x(\eta_\mu * \rho))\, G'(\eta_\mu * \rho)\partial_x(\eta_\mu * \rho)\varphi\, dx \\
= & \int_{\mathbf{R}} \rho^2\, G'(\rho)\partial_x \rho\, \varphi\, dx - \int_{\mathbf{R}} \rho\, \eta_\mu * \rho\, G'(\eta_\mu * \rho)\partial_x(\eta_\mu * \rho)\varphi\, dx \\
& + \int_{\mathbf{R}} \rho\mu\, G'(\eta_\mu * \rho)(\partial_x(\eta_\mu * \rho))^2 \varphi\, dx \\
= & \int_{\mathbf{R}} \rho^2\, G'(\rho)\partial_x \rho\, \varphi\, dx - \int_{\mathbf{R}} (\eta_\mu * \rho - \mu\partial_x(\eta_\mu * \rho))\, (\eta_\mu * \rho)\, G'(\eta_\mu * \rho)\partial_x(\eta_\mu * \rho)\varphi\, dx \\
& + \int_{\mathbf{R}} \rho\mu\, G'(\eta_\mu * \rho)(\partial_x(\eta_\mu * \rho))^2 \varphi\, dx \\
= & \int_{\mathbf{R}} \rho^2\, G'(\rho)\partial_x \rho\, \varphi\, dx - \int_{\mathbf{R}} (\eta_\mu * \rho)^2\, G'(\eta_\mu * \rho)\partial_x(\eta_\mu * \rho)\varphi\, dx \\
& + \int_{\mathbf{R}} \mu\, \eta_\mu * \rho\, G'(\eta_\mu * \rho)(\partial_x(\eta_\mu * \rho))^2 \varphi\, dx + \int_{\mathbf{R}} \rho\mu\, G'(\eta_\mu * \rho)(\partial_x(\eta_\mu * \rho))^2 \varphi\, dx.
\end{aligned}$$

If $\rho \geq 0$ and $G' \leq 0$, the last two terms are non positive and we get

$$\begin{aligned}
& \int_{\mathbf{R}} \rho^2\, \partial_x(G(\rho) - G(\eta_\mu * \rho))\varphi\, dx \\
\leq & \int_{\mathbf{R}} \rho^2\, G'(\rho)\partial_x \rho\, \varphi\, dx - \int_{\mathbf{R}} (\eta_\mu * \rho)^2\, G'(\eta_\mu * \rho)\partial_x(\eta_\mu * \rho)\varphi\, dx.
\end{aligned}$$

We set H such that $H'(z) = z^2 G'(z)$, then

$$\begin{aligned}
\int_{\mathbf{R}} \rho^2\, \partial_x(G(\rho) - G(\eta_\mu * \rho))\varphi\, dx & \leq \int_{\mathbf{R}} \partial_x H(\rho)\, \varphi\, dx - \int_{\mathbf{R}} \partial_x H(\eta_\mu * \rho)\, \varphi\, dx \\
& \leq -\int_{\mathbf{R}} H(\rho)\, \partial_x \varphi\, dx + \int_{\mathbf{R}} H(\eta_\mu * \rho)\, \partial_x \varphi\, dx \\
& \leq \int_{\mathbf{R}} (H(\eta_\mu * \rho) - H(\rho))\, \partial_x \varphi\, dx
\end{aligned}$$

and we get

$$
\begin{aligned}
I &= \int_{\mathbf{R}} E'(\rho)\partial_x(F(\rho)(G(\rho)-G(\eta_\mu*\rho)))\varphi\,dx \\
&\leq \int_{\mathbf{R}} (G(\eta_\mu*\rho)-G(\rho))\,\partial_x\varphi\,dx - \int_{\mathbf{R}} \rho^2(G(\rho)-G(\eta_\mu*\rho))\partial_x\varphi\,dx.
\end{aligned}
$$

Formally both terms of the right hand side goes to 0 when $\mu \to 0$ and we get

$$\int_{\mathbf{R}} E'(\rho)\partial_x(F(\rho)(G(\rho)-G(\eta_\mu*\rho)))\varphi\,dx \leq 0$$

for any non negative test function φ. That is to say $E'(\rho)\partial_x(F(\rho)(G(\rho)-G(\eta_\mu*\rho))) \leq 0$, thus

$$\partial_t E(\rho) + \partial_x \psi(\rho) \leq 0.$$

It should be noted that during the revision of the paper, two articles dealing with situations with some links to these problems have appeared. One dealing with the well-posedness of the nonlocal problem in an appropriate class of functions which allows to perform the formal computation and a compactness result to perform the limit when the kernel approches the Dirac delta: the case of the exponential kernel has been completed in [13]. And for more general kernels the convergence to the entropy solutions is obtained in [14]. It would be of course interesting to investigate if the approach of the present paper produces more general results compared to the mentioned works.

Now from the kinetic point of view with a fixed kernel η, several tracks can be followed. If we ask the maxwellian to satisfy an additional condition as

$$\iint_{\mathbf{R}^2} e(v,\xi)\mathcal{M}_\rho(t,x,v,\xi)\,d\xi dv = E(\rho),$$

then by an integration with respect to (v,ξ) of

$$\partial_t e(v,\xi)f_\varepsilon + \partial_x(e(v,\xi)a(v,\xi)f_\varepsilon) = e(v,\xi)\frac{\mathcal{M}_{\rho_\varepsilon}-f_\varepsilon}{\varepsilon},$$

we get

$$\partial_t \iint_{\mathbf{R}^2} e(v,\xi)f_\varepsilon\,d\xi dv + \partial_x \iint_{\mathbf{R}^2} e(v,\xi)a(v,\xi)f_\varepsilon\,d\xi dv = \iint_{\mathbf{R}^2} e(v,\xi)\frac{\mathcal{M}_{\rho_\varepsilon}-f_\varepsilon}{\varepsilon}.\,d\xi dv. \quad (2.15)$$

Then a condition like

$$\iint_{\mathbf{R}^2} e(v,\xi)a(v,\xi)\mathcal{M}_\rho(t,x,v,\xi)\,d\xi dv = \mathcal{F}(\rho(t,x))\mathcal{G}((\eta*\rho)(t,x))$$

and a non positive sign on the right hand side of (2.15) gives, formally when $\varepsilon \to 0$, since $\mathcal{M}_\rho = f$, the entropy inequality

$$\partial_t E(\rho) + \mathrm{div}_x(\mathcal{F}(\rho)\mathcal{G}(\eta*\rho)) \leq 0.$$

This is a general framework which would be the next step of the study in a forthcoming work. We will have to specify which entropy we want to get for which conservation law. In the present paper we focus on the existence of solutions in the most general case so last, later, we'll be able to consider more specific equation and entropy.

3 Well-posedness of the kinetic equation with the first set of properties, namely (2.3)

We consider the case where we have a kinetic model satisfying the first of properties and then prove Theorem 1.1. We also need the first property of consistency. Then, we assume that (2.1) and (2.3) are satisfied and we prove that it allows to get existence and unicity of a solution to the kinetic equation.

Proof of Theorem 1.1. Equation (1.4) is equivalent to the following integral representation

$$f_\varepsilon(t,x,v,\xi) = e^{-t/\varepsilon} f_\varepsilon(0, x-a(v,\xi)t, v,\xi) + \frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon} \mathcal{M}_{\rho_\varepsilon}(s, x-a(v,\xi)(t-s), v, \xi)\, ds$$

with

$$\rho_\varepsilon(t,x) = \iint_{\mathbf{R}^2} f_\varepsilon(t,y,\tilde{v},\tilde{\xi})\, d\tilde{\xi} d\tilde{v}.$$

Let $\varepsilon > 0$ and $T > 0$. Denote by Φ the application from $L^\infty([0,T], L^1(\mathbf{R}^{d+2}))$ to $L^\infty([0,T], L^1(\mathbf{R}^{d+2}))$ which at f associates

$$\Phi(f)(t,x,v,\xi) = e^{-t/\varepsilon} f^0(x-a(v,\xi)t, v,\xi) + \frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon} \mathcal{M}_{\rho}(s, x-a(v,\xi)(t-s), v, \xi)\, ds,$$

where

$$\rho(t,x) = \iint_{\mathbf{R}^2} f(t,y,\tilde{v},\tilde{\xi})\, d\tilde{\xi} d\tilde{v}.$$

For $f_1, f_2 \in L^\infty([0,T], L^1(\mathbf{R}^{d+2}))$, we note

$$\rho_1(t,x) = \iint_{\mathbf{R}^2} f_1(t,y,v,\xi)\, d\xi dv \quad \text{and} \quad \rho_2(t,x) = \iint_{\mathbf{R}^2} f_2(t,y,v,\xi)\, d\xi dv.$$

We have

$$\begin{aligned}
&\iiint_{\mathbf{R}^{d+2}} |\Phi(f_1)(t,x,v,\xi) - \Phi(f_2)(t,x,v,\xi)|\, dx d\xi dv \\
\leq\ & \frac{1}{\varepsilon} \iiint_{\mathbf{R}^{d+2}} \int_0^t e^{(s-t)/\varepsilon} |\mathcal{M}_{\rho_1} - \mathcal{M}_{\rho_2}|(s, x-a(v,\xi)(t-s), v, \xi)\, ds\, dx d\xi dv \\
\leq\ & \frac{1}{\varepsilon}\int_0^t \iint_{\mathbf{R}^2} e^{(s-t)/\varepsilon} \left(\int_{\mathbf{R}^d} |\mathcal{M}_{\rho_1}(s,x,v,\xi) - \mathcal{M}_{\rho_1}(s,x,v,\xi)|\, dx \right) d\xi dv ds \\
\leq\ & \frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon} \int_{\mathbf{R}^d} \left(\iint_{\mathbf{R}^2} |\mathcal{M}_{\rho_1}(s,x,v,\xi) - \mathcal{M}_{\rho_1}(s,x,v,\xi)|\, d\xi dv \right) dx ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}\int_{\mathbf{R}^d} K|\rho_1-\rho_2|(s,x)\,dxds \\
&\leq K\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}\iiint_{\mathbf{R}^{d+2}} |f_1(s,x,v,\xi)-f_2(s,x,v,\xi)|\;dxd\xi dvds \\
&\leq K\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}\,ds\sup_{s\in[0,t]}\iiint_{\mathbf{R}^{d+2}} |f_1(s,x,v,\xi)-f_2(s,x,v,\xi)|\;dxd\xi dv \\
&\leq K\left(1-e^{-t/\varepsilon}\right)\sup_{s\in[0,t]}\iiint_{\mathbf{R}^{d+2}} |f_1(s,x,v,\xi)-f_2(s,x,v,\xi)|\;dxd\xi dv.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\sup_{t\in[0,T]}\iiint_{\mathbf{R}^{d+2}} |\Phi(f_1)(t,x,v,\xi)-\Phi(f_2)(t,x,v,\xi)|\;dxd\xi dv \\
&\leq K\left(1-e^{-T/\varepsilon}\right)\sup_{t\in[0,T]}\iiint_{\mathbf{R}^{d+2}} |f_1(t,x,v,\xi)-f_2(t,x,v,\xi)|\;dxd\xi dv.
\end{aligned}
$$

Taking

$$T_\varepsilon = -\varepsilon\ln\left(\frac{2K-1}{2K}\right) > 0,$$

we have

$$K\left(1-e^{-T_\varepsilon/\varepsilon}\right) = \frac{1}{2}$$

and Φ is a contraction on $L^\infty([0,T_\varepsilon],L^1(\mathbf{R}^{d+2}))$. Then we get the existence and uniqueness of a solution in $L^\infty([0,T_\varepsilon],L^1(\mathbf{R}^{d+2}))$ to (1.4) with initial data $f^0\geq 0$. Since the time T_ε does not depend on f^0, we can restart from the obtained solution at value T_ε and get a solution on $[T_\varepsilon,2T_\varepsilon]$ and so on. Finally we get existence and uniqueness of a solution in X on any $[0,T]$ with $T>0$. □

We also have a variant for the important case where $\rho\geq 0$.

Proposition 3.1 *Let $f^0\in L^1(\mathbf{R}^{d+2})$ such that $f^0\geq 0$. We consider a Maxwellian $\mathcal{M}$ satisfying (2.11) and (2.13). Then, for any $\varepsilon>0$, there exists $f_\varepsilon\in L^\infty([0,T],L^1(\mathbf{R}^{d+2}))$ for any $T>0$ solution of (1.4) with initial data f^0 and such that $f_\varepsilon\geq 0$.*

Proof. We adapt the previous proof by considering the space X of functions f in $L^\infty([0,T_\varepsilon],L^1(\mathbf{R}^{d+2}))$ such that $f\geq 0$. For $f\in X$, we have $\Phi(f)\in X$ since then $\rho\geq 0$ and $\mathcal{M}_\rho\geq 0$. □

4 Existence of a solution for the kinetic equation with the second set of properties, namely (2.4)-(2.10)

We consider the case where we have a kinetic model satisfying the second of properties and then prove Theorem 1.2. We also need the first property of consistency. Then, we assume that (2.1) and (2.4)-(2.10) are satisfied and we prove that it allows to get existence of a solution to the kinetic equation.

Proof of Theorem 1.2. Let $\varepsilon > 0$ and $T > 0$. Denote by Φ the application from $L^\infty([0,T], L^1(\mathbf{R}^{d+2}))$ to $L^\infty([0,T], L^1(\mathbf{R}^{d+2}))$ which at f associates

$$\Phi(f)(t,x,v,\xi) = e^{-t/\varepsilon} f^0(x - a(v,\xi)t, v, \xi) + \frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}\mathcal{M}_\rho(s, x - a(v,\xi)(t-s), v, \xi)\, ds,$$

where

$$\rho(t,x) = \iint_{\mathbf{R}^2} f(t,y,\tilde{v},\tilde{\xi})\, d\tilde{\xi} d\tilde{v}.$$

Since $f^0 \in L^1(\mathbf{R}^{d+2}) \cap L^2(\mathbf{R}^{d+2})$ is such that $xf^0, \xi f^0, vf^0, a(v,\xi)f^0 \in L^1(\mathbf{R}^{d+2})$ and

$$\int_{\mathbf{R}^d} \left(\iint_{\mathbf{R}^2} f^0(x,v,\xi)\, dv d\xi \right)^2 dx < +\infty,$$

then there exists constants $C_0^1, \ldots, C_0^6, C_0^a > 0$ such that

$$\iiint_{\mathbf{R}^{d+2}} |f^0(x,v,\xi)|\, dv d\xi dx = C_0^1 < +\infty, \tag{4.1}$$

$$\iiint_{\mathbf{R}^{d+2}} |x_i||f^0(x,v,\xi)|\, dv d\xi dx = C_0^2 < +\infty, \quad \text{for any } i = 1,\ldots,d, \tag{4.2}$$

$$\iiint_{\mathbf{R}^{d+2}} |\xi||f^0(x,v,\xi)|\, dv d\xi dx = C_0^3 < +\infty, \tag{4.3}$$

$$\iiint_{\mathbf{R}^{d+2}} |v||f^0(x,v,\xi)|\, dv d\xi dx = C_0^4 < +\infty, \tag{4.4}$$

$$\iiint_{\mathbf{R}^{d+2}} |f^0(x,v,\xi)|^2\, dv d\xi dx = C_0^5 < +\infty. \tag{4.5}$$

$$\int_{\mathbf{R}^d} \left(\iint_{\mathbf{R}^2} f^0(x,v,\xi)\, dv d\xi \right)^2 dx = C_0^6 < +\infty \tag{4.6}$$

and

$$\iiint_{\mathbf{R}^{d+2}} |a_i(v,\xi)||f^0(x,v,\xi)|\, dv d\xi dx = C_0^a < +\infty, \quad \text{for any } i = 1,\ldots,d. \tag{4.7}$$

We set

$$\mathcal{G} = 1 + \sup_{z \in \overline{B}(0, \|\eta\|_\infty C_0^1)} |G(z)| < +\infty \tag{4.8}$$

since G is continuous and $\overline{B}(0, \|\eta\|_\infty C_0^1)$ is compact. We also set

$$R_1 = C_0^1, \quad R_6 = \max\left(C_0^6, \frac{C_0^a}{K_2 K \mathcal{G}}\right), \tag{4.9}$$

$$R_2 = \max(C_0^2, \varepsilon K K_2 \mathcal{G}(R_1 + R_6)), \quad R_3 = \max(C_0^3, K_3 R_1(1 + R_6)) \tag{4.10}$$

and

$$R_4 = \max(C_0^4, K_4 R_6), \quad R_5 = \max(C_0^5, K_5 R_1, K_5 R_6). \tag{4.11}$$

We denote $R = (R_1, R_2, R_3, R_4, R_5, R_6)$ and C_R the set of all $f \in L^\infty([0,T], L^1(\mathbf{R}^{d+2}))$ such that for a.e. $t \in]0, T[$,

$$\iiint_{\mathbf{R}^{d+2}} |f(t,x,v,\xi)|\, dv d\xi dx \le R_1, \tag{4.12}$$

$$\iiint_{\mathbf{R}^{d+2}} |x_i||f(t,x,v,\xi)|\, dv d\xi dx \le R_2\left(1 + \frac{t}{\varepsilon}\right), \quad \text{for any } i = 1, \dots, d, \tag{4.13}$$

$$\iiint_{\mathbf{R}^{d+2}} |\xi||f(t,x,v,\xi)|\, dv d\xi dx \le R_3, \tag{4.14}$$

$$\iiint_{\mathbf{R}^{d+2}} |v||f(t,x,v,\xi)|\, dv d\xi dx \le R_4, \tag{4.15}$$

$$\iiint_{\mathbf{R}^{d+2}} |f(t,x,v,\xi)|^2\, dv d\xi dx \le R_5 \tag{4.16}$$

and

$$\int_{\mathbf{R}^d} \left(\iint_{\mathbf{R}^2} f(t,x,v,\xi)\, dv d\xi\right)^2 dx \le R_6. \tag{4.17}$$

We denote also $\tilde{C}_R$ the set of all $f \in C([0,T], L^1(\mathbf{R}^{d+2}))$ satisfying (4.12)-(4.17) with

$$\partial_t f + \operatorname{div}_x(a(v,\xi) f) + \frac{1}{\varepsilon} f \in \frac{C_R}{\varepsilon}. \tag{4.18}$$

The presentation of the proof is divided into seven parts.
Step 1. We prove that if $f \in C_R$, then $\mathcal{M}_\rho \in C_R$.
First, using (2.4), we have

$$\begin{aligned}
\iiint_{\mathbf{R}^{d+2}} |\mathcal{M}_\rho(t,x,v,\xi)|\, dv d\xi dx &\le \int_{\mathbf{R}^d} |\rho(t,x)| dx \\
&\le \int_{\mathbf{R}^d} \iint_{\mathbf{R}^2} |f(t,x,v,\xi)|\, d\xi dv dx \\
&\le R_1
\end{aligned}$$

and, using (2.1),

$$\int_{\mathbf{R}^d}\left(\iint\limits_{\mathbf{R}^2}\mathcal{M}_\rho(t,x,v,\xi)\,dvd\xi\right)^2 dx \;\leq\; \int_{\mathbf{R}^d}\rho(t,x)^2dx$$
$$\leq\; \int_{\mathbf{R}^d}\left(\iint\limits_{\mathbf{R}^2}|f(t,x,v,\xi)|\,d\xi dv\right)^2 dx$$
$$\leq\; R_6.$$

Now, using (2.7), we get

$$\iiint\limits_{\mathbf{R}^{d+2}}|\mathcal{M}_\rho(t,x,v,\xi)|^2\,dvd\xi dx \;\leq\; \int_{\mathbf{R}^d}K_5|\rho(t,x)|^p dx.$$

If $p=1$, it gives

$$\iiint\limits_{\mathbf{R}^{d+2}}|\mathcal{M}_\rho(t,x,v,\xi)|^2\,dvd\xi dx \;\leq\; \int_{\mathbf{R}^d}K_5\iint\limits_{\mathbf{R}^2}|f(t,x,v,\xi)|\,d\xi dvdx$$
$$\leq\; K_5R_1\leq R_5.$$

Otherwise $p=2$ and it gives

$$\iiint\limits_{\mathbf{R}^{d+2}}|\mathcal{M}_\rho(t,x,v,\xi)|^2\,dvd\xi dx \;\leq\; \int_{\mathbf{R}^d}K_5\left(\iint\limits_{\mathbf{R}^2}f(t,x,v,\xi)\,d\xi dv\right)^2 dx$$
$$\leq\; K_5R_6\leq R_5.$$

Now, using (2.4), we have, for any $i=1,\ldots,d$,

$$\iiint\limits_{\mathbf{R}^{d+2}}|x_i||\mathcal{M}_\rho(t,x,v,\xi)|\,dvd\xi dx \;\leq\; \int_{\mathbf{R}^d}|x_i||\rho(t,x)|dx$$
$$\leq\; \iiint\limits_{\mathbf{R}^{d+2}}|x_i||f(t,x,v,\xi)|\,dvd\xi dx$$
$$\leq\; R_2\left(1+\frac{t}{\varepsilon}\right).$$

Furthermore, using (2.8), we have

$$\iiint\limits_{\mathbf{R}^{d+2}}|\xi||\mathcal{M}_\rho(t,x,v,\xi)|\,dvd\xi dx$$
$$\leq\; \int_{\mathbf{R}^d}K_3|\rho(t,x)|\left(1+\int_{\mathbf{R}^d}|\rho(t,y)|^2dy\right)dx$$
$$\leq\; K_3\iiint\limits_{\mathbf{R}^{d+2}}|f(t,x,v,\xi)|\,d\xi dvdx\left(1+\int_{\mathbf{R}^d}\left(\iint\limits_{\mathbf{R}^2}|f(t,y,v,\xi)|\,d\xi dv\right)^2dy\right)$$
$$\leq\; K_3R_1(1+R_6)$$
$$\leq\; R_3.$$

Finally, using (2.6), we deduce

$$\begin{aligned}
\iiint\limits_{\mathbf{R}^{d+2}} |v||\mathcal{M}_\rho(t,x,v,\xi)|\,dvd\xi dx &\leq \int_{\mathbf{R}^d} K_4|\rho(t,x)|^2\,dx \\
&\leq K_4\int_{\mathbf{R}^d}\left(\iint\limits_{\mathbf{R}^2} f(t,x,v,\xi)\,d\xi dv\right)^2 dx \\
&\leq K_4R_6 \\
&\leq R_4.
\end{aligned}$$

Then we get that $\mathcal{M}_\rho \in C_R$.

Step 2. We prove that if $f \in C_R$, then $\Phi(f) \in C_R$.
First, we have

$$\begin{aligned}
&\iiint\limits_{\mathbf{R}^{d+2}} |\Phi(f)(t,x,v,\xi)|\,dvd\xi dx \\
\leq\;& e^{-t/\varepsilon}\iiint\limits_{\mathbf{R}^{d+2}} |f^0(x-a(v,\xi)t,v,\xi)|\,dvd\xi dx \\
&+\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}\iiint\limits_{\mathbf{R}^{d+2}} |\mathcal{M}_\rho(s,x-a(v,\xi)(t-s),v,\xi)|\,dvd\xi dx\,ds \\
\leq\;& e^{-t/\varepsilon}\iiint\limits_{\mathbf{R}^{d+2}} |f^0(x,v,\xi)|\,dvd\xi dx \\
&+\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}\iiint\limits_{\mathbf{R}^{d+2}} |\mathcal{M}_\rho(s,x,v,\xi)|\,dvd\xi dx\,ds \\
\leq\;& e^{-t/\varepsilon}R_1+\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}R_1\,ds \\
\leq\;& e^{-t/\varepsilon}R_1+R_1(1-e^{-t/\varepsilon}) = R_1.
\end{aligned}$$

Now notice that

$$\begin{aligned}
\Phi(f)(t,x,v,\xi) &= e^{-t/\varepsilon}f^0(x-a(v,\xi)t,v,\xi) \\
&\quad+(1-e^{-t/\varepsilon})\int_0^t \mathcal{M}_\rho(s,x-a(v,\xi)(t-s),v,\xi)\frac{e^{(s-t)/\varepsilon}\,ds}{\int_0^t e^{-\sigma/\varepsilon}d\sigma},
\end{aligned}$$

then for a convex function H, we have

$$\begin{aligned}
H(\Phi(f)(t,x,v,\xi)) &\leq e^{-t/\varepsilon}H(f^0(x-a(v,\xi)t,v,\xi)) \\
&\quad+(1-e^{-t/\varepsilon})H\left(\int_0^t \mathcal{M}_\rho(s,x-a(v,\xi)(t-s),v,\xi)\frac{e^{(s-t)/\varepsilon}\,ds}{\int_0^t e^{-\sigma/\varepsilon}d\sigma}\right)
\end{aligned}$$

and by Jensen's inequality, we get

$$\begin{aligned} H(\Phi(f)(t,x,v,\xi)) &\leq e^{-t/\varepsilon} H(f^0(x-a(v,\xi)t,v,\xi)) \\ &\quad +(1-e^{-t/\varepsilon})\int_0^t H(\mathcal{M}_\rho(s,x-a(v,\xi)(t-s),v,\xi))\frac{e^{(s-t)/\varepsilon}\,ds}{\int_0^t e^{-\sigma/\varepsilon}d\sigma} \\ &\leq e^{-t/\varepsilon} H(f^0(x-a(v,\xi)t,v,\xi)) \\ &\quad +\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon} H(\mathcal{M}_\rho(s,x-a(v,\xi)(t-s),v,\xi))\,ds. \end{aligned}$$

With $H(z)=z^2$, it gives

$$\begin{aligned} (\Phi(f)(t,x,v,\xi))^2 &\leq e^{-t/\varepsilon}(f^0(x-a(v,\xi)t,v,\xi))^2 \\ &\quad +\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}(\mathcal{M}_\rho(s,x-a(v,\xi)(t-s),v,\xi))^2\,ds \end{aligned}$$

and, using (2.7),

$$\begin{aligned} &\iiint\limits_{\mathbf{R}^{d+2}} \Phi(f)(t,x,v,\xi)^2\,dvd\xi dx \\ \leq\;& e^{-t/\varepsilon}\iiint\limits_{\mathbf{R}^{d+2}} |f^0(x-a(v,\xi)t,v,\xi)|^2\,dvd\xi dx \\ &+\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}\iiint\limits_{\mathbf{R}^{d+2}} |\mathcal{M}_\rho(s,x-a(v,\xi)(t-s),v,\xi)|^2\,dvd\xi dx\,ds \\ \leq\;& e^{-t/\varepsilon}\iiint\limits_{\mathbf{R}^{d+2}} |f^0(x,v,\xi)|^2\,dvd\xi dx \\ &+\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}\iiint\limits_{\mathbf{R}^{d+2}} |\mathcal{M}_\rho(s,x,v,\xi)|^2\,dvd\xi dx\,ds \\ \leq\;& e^{-t/\varepsilon}C_0^5+\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}R_5\,ds \\ \leq\;& e^{-t/\varepsilon}R_5+R_5(1-e^{-t/\varepsilon})=R_5. \end{aligned}$$

Furthermore

$$\begin{aligned} &\iint\limits_{\mathbf{R}^2}\Phi(f)(t,x,v,\xi)\,dvd\xi \\ =\;& e^{-t/\varepsilon}\iint\limits_{\mathbf{R}^2} f^0(x-a(v,\xi)t,v,\xi)\,dvd\xi \\ &+\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}\iint\limits_{\mathbf{R}^2}\mathcal{M}_\rho(s,x-a(v,\xi)(t-s),v,\xi)\,dvd\xi\,ds \end{aligned}$$

then, by convexity,

$$
\begin{aligned}
& \left(\iint_{\mathbf{R}^2} \Phi(f)(t,x,v,\xi)\, dv d\xi \right)^2 \\
\leq \;& e^{-t/\varepsilon} \left(\iint_{\mathbf{R}^2} f^0(x-a(v,\xi)t,v,\xi)\, dv d\xi \right)^2 \\
& + \frac{1}{\varepsilon} \int_0^t e^{(s-t)/\varepsilon} \left(\iint_{\mathbf{R}^2} \mathcal{M}_\rho(s,x-a(v,\xi)(t-s),v,\xi)\, dv d\xi \right)^2 \, ds.
\end{aligned}
$$

Thus

$$
\begin{aligned}
& \int_{\mathbf{R}^d} \left(\iint_{\mathbf{R}^2} \Phi(f)(t,x,v,\xi)\, dv d\xi \right)^2 \, dx \\
\leq \;& e^{-t/\varepsilon} \int_{\mathbf{R}^d} \left(\iint_{\mathbf{R}^2} f^0(x-a(v,\xi)t,v,\xi)\, dv d\xi \right)^2 \, dx \\
& + \frac{1}{\varepsilon} \int_0^t e^{(s-t)/\varepsilon} \int_{\mathbf{R}^d} \left(\iint_{\mathbf{R}^2} \mathcal{M}_\rho(s,x-a(v,\xi)(t-s),v,\xi)\, dv d\xi \right)^2 \, dx\, ds \\
\leq \;& e^{-t/\varepsilon} \int_{\mathbf{R}^d} \left(\iint_{\mathbf{R}^2} f^0(x,v,\xi)\, dv d\xi \right)^2 \, dx \\
& + \frac{1}{\varepsilon} \int_0^t e^{(s-t)/\varepsilon} \int_{\mathbf{R}^d} \left(\iint_{\mathbf{R}^2} \mathcal{M}_\rho(s,x,v,\xi)\, dv d\xi \right)^2 \, dx\, ds \\
\leq \;& e^{-t/\varepsilon} C_0^6 + \frac{1}{\varepsilon} \int_0^t e^{(s-t)/\varepsilon} R_6 \, ds \\
\leq \;& e^{-t/\varepsilon} R_6 + R_6(1-e^{-t/\varepsilon}) = R_6.
\end{aligned}
$$

Now, using (2.4) and (2.5), we have, for any $i = 1, \dots, d$,

$$
\begin{aligned}
& \iiint_{\mathbf{R}^{d+2}} |x_i||\Phi(f)(t,x,v,\xi)|\, dv d\xi dx \\
\leq \;& e^{-t/\varepsilon} \iiint_{\mathbf{R}^{d+2}} |x_i||f^0(x-a(v,\xi)t,v,\xi)|\, dv d\xi dx \\
& + \frac{1}{\varepsilon} \int_0^t e^{(s-t)/\varepsilon} \iiint_{\mathbf{R}^{d+2}} |x_i||\mathcal{M}_\rho(s,x-a(v,\xi)(t-s),v,\xi)|\, dv d\xi dx\, ds \\
\leq \;& e^{-t/\varepsilon} \iiint_{\mathbf{R}^{d+2}} |x_i + a_i(v,\xi)t||f^0(x,v,\xi)|\, dv d\xi dx
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon} \iiint\limits_{\mathbf{R}^{d+2}} |x_i + a_i(v,\xi)(t-s)||\mathcal{M}_\rho(s,x,v,\xi)|\, dvd\xi dx\, ds \\
\leq\ & e^{-t/\varepsilon}(C_0^2 + tC_0^a) \\
&+\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon} \int_{\mathbf{R}^d} \left(|x_i||\rho(s,x)| + (t-s)K_2|F_i(\rho(s,x))G_i((\eta*\rho)(s,x))|\right) dx\, ds \\
\leq\ & e^{-t/\varepsilon}(C_0^2 + tC_0^a) \\
&+\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon} \left(\iiint\limits_{\mathbf{R}^{d+2}} |x_i||f(s,x,v,\xi)|\, dvd\xi dx + (t-s)K_2 \int_{\mathbf{R}^d} |F_i(\rho(s,x))G_i((\eta*\rho)(s,x))| dx \right) ds \\
\leq\ & e^{-t/\varepsilon}(C_0^2 + tC_0^a) + \frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon} \left(R_2\left(1+\frac{s}{\varepsilon}\right) + (t-s)K_2 \int_{\mathbf{R}^d} |F_i(\rho(s,x))G_i((\eta*\rho)(s,x))| dx \right) ds.
\end{aligned}
$$

Since

$$
|(\eta*\rho)(s,x)| \leq \|\eta\|_\infty \int_{\mathbf{R}} |\rho(s,y)|\, dy \leq \|\eta\|_\infty R_1,
$$

we note that

$$
|G((\eta*\rho)(s,x))| \leq \mathcal{G},
$$

then, with relation (1.7),

$$
\begin{aligned}
&\iiint\limits_{\mathbf{R}^{d+2}} |x_i||\Phi(f)(t,x,v,\xi)|\, dvd\xi dx \\
\leq\ & e^{-t/\varepsilon}(C_0^2 + tC_0^a) + \frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon} \left(R_2\left(1+\frac{s}{\varepsilon}\right) + (t-s)K_2\mathcal{G} \int_{\mathbf{R}^d} K(|\rho(s,x)| + |\rho(s,x)|^2) dx \right) ds \\
\leq\ & e^{-t/\varepsilon}(C_0^2 + tC_0^a) + \frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon} \left(R_2\left(1+\frac{s}{\varepsilon}\right) + (t-s)K_2K(R_1+R_6)\mathcal{G} \right) ds.
\end{aligned}
$$

Now

$$
\frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}(\alpha + s\beta)\, ds = \alpha - \beta\varepsilon + t\beta + (\beta\varepsilon - \alpha)e^{-t/\varepsilon},
$$

then, with $\alpha = R_2 + tK_2K(R_1+R_6)\mathcal{G}$ and $\beta = \dfrac{R_2}{\varepsilon} - K_2K(R_1+R_6)\mathcal{G}$, we get

$$
\begin{aligned}
&\iiint\limits_{\mathbf{R}^{d+2}} |x||\Phi(f)(t,x,v,\xi)|\, dvd\xi dx \\
\leq\ & e^{-t/\varepsilon}(C_0^2 + tC_0^a) + R_2 + tK_2K(R_1+R_6)\mathcal{G} - R_2 + \varepsilon K_2K(R_1+R_6)\mathcal{G} + \frac{tR_2}{\varepsilon} - tK_2K(R_1+R_6)\mathcal{G} \\
&+(R_2 - K_2K(R_1+R_6)\mathcal{G}\varepsilon - R_2 - tK_2K(R_1+R_6)\mathcal{G})e^{-t/\varepsilon} \\
\leq\ & e^{-t/\varepsilon}C_0^2 + te^{-t/\varepsilon}(C_0^a - K_2K(R_1+R_6)\mathcal{G}) + \frac{tR_2}{\varepsilon} + \varepsilon K_2K(R_1+R_6)\mathcal{G}(1-e^{-t/\varepsilon}) \\
\leq\ & e^{-t/\varepsilon}R_2 + te^{-t/\varepsilon}(C_0^a - K_2K(R_1+R_6)\mathcal{G}) + \frac{tR_2}{\varepsilon} + R_2(1-e^{-t/\varepsilon})
\end{aligned}
$$

$$
\begin{aligned}
&\leq \quad R_2 + \frac{tR_2}{\varepsilon} + te^{-t/\varepsilon}(C_0^a - K_2K(R_1+R_6)\mathcal{G}) \\
&\leq \quad R_2\left(1+\frac{t}{\varepsilon}\right)
\end{aligned}
$$

since $C_0^a - K_2K(R_1+R_6)\mathcal{G} \leq 0$.

Furthermore we have

$$
\begin{aligned}
& \iiint_{\mathbf{R}^{d+2}} |\xi||\Phi(f)(t,x,v,\xi)|\,dvd\xi dx \\
\leq\ & e^{-t/\varepsilon} \iiint_{\mathbf{R}^{d+2}} |\xi||f^0(x-a(v,\xi)t,v,\xi)|\,dvd\xi dx \\
& + \frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon} \iiint_{\mathbf{R}^{d+2}} |\xi||\mathcal{M}_\rho(s,x-a(v,\xi)(t-s),v,\xi)|\,dvd\xi dx\,ds \\
\leq\ & e^{-t/\varepsilon} \iiint_{\mathbf{R}^{d+2}} |\xi||f^0(x,v,\xi)|\,dvd\xi dx \\
& + \frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon} \iiint_{\mathbf{R}^{d+2}} |\xi||\mathcal{M}_\rho(s,x,v,\xi)|\,dvd\xi dx\,ds \\
\leq\ & e^{-t/\varepsilon}C_0^3 + \frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}R_3\,ds \\
\leq\ & e^{-t/\varepsilon}R_3 + R_3(1-e^{-t/\varepsilon}) = R_3.
\end{aligned}
$$

Finally we have

$$
\begin{aligned}
& \iiint_{\mathbf{R}^{d+2}} |v||\Phi(f)(t,x,v,\xi)|\,dvd\xi dx \\
\leq\ & e^{-t/\varepsilon} \iiint_{\mathbf{R}^{d+2}} |v||f^0(x-a(v,\xi)t,v,\xi)|\,dvd\xi dx \\
& + \frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon} \iiint_{\mathbf{R}^{d+2}} |v||\mathcal{M}_\rho(s,x-a(v,\xi)(t-s),v,\xi)|\,dvd\xi dx\,ds \\
\leq\ & e^{-t/\varepsilon} \iiint_{\mathbf{R}^{d+2}} |v||f^0(x,v,\xi)|\,dvd\xi dx \\
& + \frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon} \iiint_{\mathbf{R}^{d+2}} |v||\mathcal{M}_\rho(s,x,v,\xi)|\,dvd\xi dx\,ds \\
\leq\ & e^{-t/\varepsilon}C_04 + \frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon}R_4\,ds \\
\leq\ & e^{-t/\varepsilon}R_4 + R_4(1-e^{-t/\varepsilon}) = R_4.
\end{aligned}
$$

Then we get that $\Phi(f) \in C_R$.

Step 3. We prove that if $f \in C_R$, then $\Phi(f) \in \tilde{C}_R$.
By step 1 and step 2 and since $\Phi(f)$ satisfies

$$\partial_t \Phi(f) + \mathrm{div}_x(a(v,\xi)\Phi(f)) + \frac{1}{\varepsilon}\Phi(f) = \frac{\mathcal{M}_\rho}{\varepsilon}, \tag{4.19}$$

we get (4.18) for $\Phi(f)$.

Step 4. We prove that Φ is continuous on C_R.
Let $g, g_n \in C_R$ such that $g_n \to g$ in $L^\infty([0,T], L^1(\mathbf{R}^{d+2}))$. Set

$$\rho_n(t,x) = \iint_{\mathbf{R}^2} g_n(t,x,v,\xi)\, d\xi dv \quad \text{ and } \quad \rho(t,x) = \iint_{\mathbf{R}^2} g(t,x,v,\xi)\, d\xi dv.$$

Since

$$\int_{]0,T[} \int_{\mathbf{R}^d} |\rho_n - \rho|\,(t,x)\, dx\, dt \leq \int_{]0,T[} \iiint_{\mathbf{R}^{d+2}} |g_n - g|(t,x,v,\xi)\, dx d\xi dv\, dt,$$

then $\rho_n \to \rho$ in $L^1(]0,T[\times\mathbf{R}^d)$ and there exists a subsequence $\rho_{\varphi(n)}$ and a function $h \in L^1(]0,T[\times\mathbf{R}^d)$ such that $\rho_{\varphi(n)} \to \rho$ and $|\rho_{\varphi(n)}| \leq |h|$ a.e. t,x. Thus $\mathcal{M}_{\rho_{\varphi(n)}} \to \mathcal{M}_\rho$ a.e. t,x,v,ξ by (2.9). Furthermore, the sequence $(\mathcal{M}_{\rho_{\varphi(n)}})_n$ is uniformly integrable thanks to (4.16) and tight thanks to (4.13)-(4.15). Then by Vitali's convergence theorem, we get $\mathcal{M}_{\rho_{\varphi(n)}} \to \mathcal{M}_\rho$ in $L^1(]0,T[\times\mathbf{R}^{d+2})$. Now

$$|\Phi(g_{\varphi(n)}) - \Phi(g)|(t,x,v,\xi) \leq \frac{1}{\varepsilon}\int_0^t e^{(s-t)/\varepsilon} \left|\mathcal{M}_{\rho_{\varphi(n)}} - \mathcal{M}_\rho\right|(s, x - a(v,\xi)(t-s), v, \xi)\, ds,$$

thus

$$\begin{aligned}
&\iiint_{\mathbf{R}^{d+2}} \left|\Phi(g_{\varphi(n)})(t,x,v,\xi) - \Phi(g)(t,x,v,\xi)\right| dx d\xi dv \\
\leq\ & \frac{1}{\varepsilon} \iint_{\mathbf{R}^{d+2}} \int_0^t e^{(s-t)/\varepsilon} |\mathcal{M}_{\rho_{\varphi(n)}} - \mathcal{M}_\rho|(s, x - a(v,\xi)(t-s), v, \xi)\, ds\, dx d\xi dv \\
\leq\ & \frac{1}{\varepsilon}\int_0^t \iint_{\mathbf{R}^2} e^{(s-t)/\varepsilon} \int_{\mathbf{R}^d} |\mathcal{M}_{\rho_{\varphi(n)}} - \mathcal{M}_\rho|(s,x,v,\xi)\, dx\, d\xi dv ds.
\end{aligned}$$

We obtain

$$\begin{aligned}
&\sup_{t\in[0,T]} \iiint_{\mathbf{R}^{d+2}} \left|\Phi(g_{\varphi(n)})(t,x,v,\xi) - \Phi(g)(t,x,v,\xi)\right| dx d\xi dv \\
\leq\ & \frac{1}{\varepsilon}\int_0^T \iiint_{\mathbf{R}^{d+2}} |\mathcal{M}_{\rho_{\varphi(n)}} - \mathcal{M}_\rho|(s,x,v,\xi)\, dx\, d\xi dv ds
\end{aligned}$$

and we get that $\Phi(g_{\varphi(n)}) \to \Phi(g)$ in $L^\infty([0,T], L^1(\mathbf{R}^{d+2}))$, or also in $C([0,T], L^1(\mathbf{R}^{d+2}))$. It is enough to get the continuity of Φ on C_R.

Step 5. We prove the following properties on the sets C_R and $\tilde{C}_R$: they are convex and not empty, the set C_R is compact for the weak topology of $L^1(]0,T[\times\mathbf{R}^{d+2})$ and the set $\tilde{C}_R$ is closed in $C([0,T], L^1(\mathbf{R}^{d+2}))$.

The sets C_R and $\tilde{C}_R$ are clearly convex. Since $f^0 \in C_R$, the set C_R is not empty. Since $f^0 \in C_R$, then $\Phi(f^0) \in \tilde{C}_R$ by step 2. Thus the set $\tilde{C}_R$ is not empty.

The uniform integrability comes from (4.16) and the tightness comes from (4.13)-(4.15), then the set C_R is relatively compact for the weak topology of $L^1(]0,T[\times\mathbf{R}^{d+2})$ by Dunford-Pettis' theorem.

Let us prove now that C_R is closed for the weak topology of $L^1(]0,T[\times\mathbf{R}^{d+2})$. Since C_R is convex, it is enough to prove that C_R is closed for the strong topology of $L^1(]0,T[\times\mathbf{R}^{d+2})$. Let $g_n \in C_R$ such that $g_n \to g$ in $L^1([0,T]\times\mathbf{R}^{d+2})$. After extraction of a subsequence, we have $g_{\varphi(n)} \to g$ a.e. (t,x,v,ξ) and $g_{\varphi(n)}(t,\cdot) \to g(t,\cdot)$ in $L^1(\mathbf{R}^{d+2})$ a.e. t. Since the sequence $(g_{\varphi(n)})_n$ satisfies (4.12)-(4.17) uniformly with respect to n, applying Fatou's lemma to each inequality, we get that $g \in C_R$.

We prove similarly that $\tilde{C}_R$ is closed in $C([0,T], L^1(\mathbf{R}^{d+2}))$.

Step 6. We prove that $\Phi(\tilde{C}_R)$ is relatively compact in $C([0,T], L^1(\mathbf{R}^{d+2}))$.

Let $f_n \in \Phi(\tilde{C}_R)$ define a sequence in $\Phi(\tilde{C}_R)$. Then there exists $g_n \in \tilde{C}_R$ such that $f_n = \Phi(g_n)$. Set

$$\rho_n(t,x) = \iint_{\mathbf{R}^2} g_n(t,x,v,\xi)\, d\xi dv.$$

Since $\tilde{C}_R \subset C_R$ and since C_R is compact for the weak topology of $L^1(]0,T[\times\mathbf{R}^{d+2})$, there exists a subsequence $g_{\varphi(n)}$ such that $g_{\varphi(n)} \rightharpoonup g$ in weak $L^1([0,T]\times\mathbf{R}^{d+2})$. Thus $\rho_{\varphi(n)} \rightharpoonup \rho$ in weak $L^1([0,T]\times\mathbf{R}^d)$ since the functions are in C_R where

$$\rho(t,x) = \iint_{\mathbf{R}^2} g(t,x,v,\xi)\, d\xi dv.$$

Since $g_{\varphi(n)} \in \tilde{C}_R$, then, by (4.18),

$$h_{\varphi(n)} = \varepsilon\partial_t g_{\varphi(n)} + \varepsilon\,\mathrm{div}_x(a(v,\xi)g_{\varphi(n)}) + g_{\varphi(n)} \in C_R.$$

By (2.10), we get that $\rho_{\varphi(n)}$ is compact in $L^1_{loc}(]0,T[\times\mathbf{R}^d)$, then for a subsequence $\rho_{\varphi\circ\psi(n)} \to \tilde{\rho}$ in $L^1(]0,T[\times K)$ for any compact K of $\mathbf{R}^d$. We deduce, since the functions are in C_R, that $\rho_{\varphi\circ\psi(n)} \to \rho$ in $L^1(]0,T[\times\mathbf{R}^d)$.

Finally we apply the same argument as in step 4 to get that for a subsequence $\Phi(g_{\varphi\circ\psi\circ\Gamma(n)}) \to \Phi(g)$ in $C([0,T], L^1(\mathbf{R}^{d+2}))$. This is how we finalise step 6.

Step 7. We conclude by applying Schauder's theorem in $C([0,T], L^1(\mathbf{R}^{d+2}))$ to $\Phi : \tilde{C}_R \to \tilde{C}_R$. There exists $f \in C([0,T], L^1(\mathbf{R}^{d+2}))$ such that $\Phi(f) = f$. This gives a solution in $[0,T]$ for any $T > 0$, and by extraction of a diagonal subsequence, we obtain a solution in $[0,+\infty[$. $\square$

Remark 4.1 Notice that (1.7) is satisfied for example if $F' \in L^\infty$ and $F(0) = 0$ since then, for any $i = 1, \ldots, d$,

$$|F_i(z)| = |F_i(z) - F_i(0)| \leq \|F_i'\|_\infty |z|.$$

But we can also consider more general cases.

5 A model satisfying the first set of properties

Let's explicit a model for which the properties (2.11)-(2.13) are satisfied and then for which Theorem 1.1, and ever better here the variant with Proposition 3.1, can be applied.

For the scalar non-local model, we assume that

$$F(0) = 0 \quad \text{and} \quad \eta, \frac{1}{\eta} \in L^\infty(\mathbf{R}^d, \mathbf{R}), \tag{5.1}$$

that is to say that there exists $\alpha, \beta > 0$ such that

$$\alpha \leq \eta(z) \leq \beta, \quad \text{for any } z \in \mathbf{R}^d. \tag{5.2}$$

Notice that the term $\eta * \rho$ is well defined as soon as $x \mapsto \rho(t, x) \in L^1(\mathbf{R}^d)$ for a.e. t.

For the kinetic model, we take, for $i = 1, \ldots, d$,

$$a_i(v, \xi) = b_i(v) c_i(\xi)$$

with

$$b_i(v) = F_i'(v), \quad c_i(\xi) = G_i(\xi) + \xi G_i'(\xi) \tag{5.3}$$

and

$$\mathcal{M}_\rho(t, x, v, \xi) = M_{\rho(t,x),(\eta*\rho)(t,x)}(v, \xi), \tag{5.4}$$

where

$$M_{\rho,q}(v, \xi) = M_1(v, \rho) M_2(\xi, q), \tag{5.5}$$

$$M_1(v, \rho) = \begin{cases} \text{sgn}(\rho) & \text{if } (\rho - v)v \geq 0, \\ 0 & \text{if } (\rho - v)v < 0, \end{cases} \tag{5.6}$$

$$M_2(\xi, q) = \begin{cases} \frac{\text{sgn}(q)}{q} & \text{if } (q - \xi)\xi > 0, \\ 0 & \text{if } (q - \xi)\xi \leq 0, \end{cases} \tag{5.7}$$

Remember that

$$\rho(t, x) = \iint_{\mathbf{R}^2} f(t, y, \tilde{v}, \tilde{\xi}) \, d\tilde{\xi} d\tilde{v}.$$

Notice that we write $\mathcal{M}_\rho(t, x, v, \xi)$ and not $\mathcal{M}_{\rho(t,x)}(v, \xi)$ because here the term $\mathcal{M}_\rho(t, x, v, \xi)$ depends on the function ρ for any value at (t, y) because of the term $\eta * \rho$. At the kinetic level, we also have a non-local taking into account of the values of ρ_ε. This choice of $\mathcal{M}_\rho$ is the most natural because for the classical part, that is to say $b(v)$, of the model we consider the classical physical BGK model $M_1(v, \rho)$ for scalar conservation law.

5.1 Preliminary properties

First, notice the following properties :

Proposition 5.1 *The functions M_1 and M_2 satisfy*

$$\int_{\mathbf{R}} M_1(v,\rho)\, dv = \rho,$$

$$\int_{\mathbf{R}} |M_1(v,\rho) - M_1(v,\tilde{\rho})|\; dv = |\rho - \tilde{\rho}|\,,$$

$$\int_{\mathbf{R}} C'(v) M_1(v,\rho)\, dv = C(\rho) - C(0), \quad \forall C \in C^1(\mathbf{R},\mathbf{R}),$$

$$\int_{\mathbf{R}} b_i(v) M_1(v,\rho)\, dv = F_i(\rho), \quad \textit{for any } i = 1,\dots,d,$$

$$\int_{\mathbf{R}} M_2(\xi,q)\, d\xi = \mathbb{1}_{q\neq 0},$$

$$\int_{\mathbf{R}} (C(\xi) + \xi C'(\xi)) M_2(\xi,q)\, d\xi = C(q)\mathbb{1}_{q\neq 0}, \quad \forall C \in C^1(\mathbf{R},\mathbf{R}),$$

$$\int_{\mathbf{R}} c_i(\xi) M_2(\xi,q)\, d\xi = G_i(q)\mathbb{1}_{q\neq 0}, \quad \textit{for any } i = 1,\dots,d.$$

Proof. The five first properties come from classical computations. The penultimate one comes from the following. For $q > 0$, we have

$$\begin{aligned}\int_{\mathbf{R}} (C(\xi) + \xi C'(\xi)) M_2(\xi,q)\, d\xi &= \int_0^q \frac{1}{q}(C(\xi) + \xi C'(\xi))\, d\xi = \frac{1}{q}\int_0^q (\xi C(\xi))'\, d\xi \\ &= \frac{1}{q}\left[\xi C(\xi)\right]_0^q = C(q)\end{aligned}$$

and for $q < 0$,

$$\int_{\mathbf{R}} (C(\xi) + \xi C'(\xi)) M_2(\xi,q)\, d\xi = \int_q^0 \frac{-1}{q}(C(\xi) + \xi C'(\xi))\, d\xi = C(q).$$

We deduce from this that

$$\int_{\mathbf{R}} c_i(\xi) M_2(\xi,q)\, d\xi = \int_{\mathbf{R}} (G_i(\xi) + \xi G_i'(\xi)) M_2(\xi,q)\, d\xi = G_i(q)\mathbb{1}_{q\neq 0}. \;\square$$

The most difficult property to deal with is :

Proposition 5.2 *The function M satisfies*

$$\iint_{\mathbf{R}^2} |M_{\rho,q}(v,\xi) - M_{\tilde{\rho},\tilde{q}}(v,\xi)|\; d\xi dv = |\rho - \tilde{\rho}| + 2\,\frac{\min(\rho,\tilde{\rho})}{\max(q,\tilde{q})}\,|q - \tilde{q}|$$

for any $\rho, \tilde{\rho} \geq 0$ and $q, \tilde{q} > 0$.

Proof. We have

$$\iint_{\mathbf{R}^2} |M_{\rho,q}(v,\xi) - M_{\tilde\rho,\tilde q}(v,\xi)|\, d\xi dv = \iint_{\mathbf{R}^2} |M_1(v,\rho)M_2(\xi,q) - M_1(v,\tilde\rho)M_2(\xi,\tilde q)|\, d\xi dv.$$

For $\tilde\rho > \rho > 0$ and $\tilde q > q > 0$, we get

$$\begin{aligned}
&\iint_{\mathbf{R}^2} |M_{\rho,q}(v,\xi) - M_{\tilde\rho,\tilde q}(v,\xi)|\, d\xi dv \\
=\ & \int_0^\rho \int_0^q |M_1(v,\rho)M_2(\xi,q) - M_1(v,\tilde\rho)M_2(\xi,\tilde q)|\, d\xi dv \\
&+ \int_0^\rho \int_q^{\tilde q} |M_1(v,\rho)M_2(\xi,q) - M_1(v,\tilde\rho)M_2(\xi,\tilde q)|\, d\xi dv \\
&+ \int_\rho^{\tilde\rho} \int_0^{\tilde q} |M_1(v,\rho)M_2(\xi,q) - M_1(v,\tilde\rho)M_2(\xi,\tilde q)|\, d\xi dv \\
=\ & \int_0^\rho \int_0^q \left|\frac{1}{q} - \frac{1}{\tilde q}\right| d\xi dv + \int_0^\rho \int_q^{\tilde q} \left|0 - \frac{1}{\tilde q}\right| d\xi dv + \int_\rho^{\tilde\rho} \int_0^{\tilde q} \left|0 - \frac{1}{\tilde q}\right| d\xi dv \\
=\ & 2\rho\frac{\tilde q - q}{\tilde q} + (\tilde\rho - \rho).
\end{aligned}$$

For $\tilde\rho > \rho > 0$ and $q > \tilde q > 0$, we get

$$\begin{aligned}
&\iint_{\mathbf{R}^2} |M_{\rho,q}(v,\xi) - M_{\tilde\rho,\tilde q}(v,\xi)|\, d\xi dv \\
=\ & \int_0^\rho \int_0^{\tilde q} |M_1(v,\rho)M_2(\xi,q) - M_1(v,\tilde\rho)M_2(\xi,\tilde q)|\, d\xi dv \\
&+ \int_0^\rho \int_{\tilde q}^{q} |M_1(v,\rho)M_2(\xi,q) - M_1(v,\tilde\rho)M_2(\xi,\tilde q)|\, d\xi dv \\
&+ \int_\rho^{\tilde\rho} \int_0^{\tilde q} |M_1(v,\rho)M_2(\xi,q) - M_1(v,\tilde\rho)M_2(\xi,\tilde q)|\, d\xi dv \\
&+ \int_\rho^{\tilde\rho} \int_{\tilde q}^{q} |M_1(v,\rho)M_2(\xi,q) - M_1(v,\tilde\rho)M_2(\xi,\tilde q)|\, d\xi dv \\
=\ & \int_0^\rho \int_0^{\tilde q} \left|\frac{1}{q} - \frac{1}{\tilde q}\right| d\xi dv + \int_0^\rho \int_q^{\tilde q} \left|\frac{1}{q} - 0\right| d\xi dv + \int_\rho^{\tilde\rho} \int_0^{\tilde q} \left|0 - \frac{1}{\tilde q}\right| d\xi dv + 0 \\
=\ & 2\rho\frac{q - \tilde q}{q} + (\tilde\rho - \rho).\ \square
\end{aligned}$$

5.2 First set of properties satisfied and existence result

We are now able to get the following result on this model.

Proposition 5.3 *Let $F, G \in C^1(\mathbf{R}, \mathbf{R}^d)$, $\eta \in C^1(\mathbf{R}^d, \mathbf{R})$ functions satisfying (5.1)-(5.2). Let $a(v,\xi) = b(v)c(\xi)$ be such that (5.3)-(5.7). Then the model satisfy (2.11)-(2.13).*

Proof. First we have

$$\iint_{\mathbf{R}^2} \mathcal{M}_\rho(t,x,v,\xi)\, d\xi dv = \int_{\mathbf{R}} M_1(v,\rho(t,x))\, dv \int_{\mathbf{R}} M_2(\xi,(\eta*\rho)(t,x))\, d\xi$$
$$= \rho(t,x)\mathbb{1}_{(\eta*\rho)(t,x)\neq 0} = \rho(t,x)$$

since $(\eta*\rho)(t,x)>0$ as soon as $\rho(t,x)>0$ (remember that $\eta>0$) and thus $\rho(t,x)=0$ a.e. if $(\eta*\rho)(t,x)=0$ a.e. Thus we get (2.11). Now we have, for any $i=1,\dots,d$,

$$\iint_{\mathbf{R}^2} b_i(v)c_i(\xi)\mathcal{M}_\rho(t,x,v,\xi)\, d\xi dv = \int_{\mathbf{R}} b_i(v)M_1(v,\rho_\varepsilon(t,x))\, dv \int_{\mathbf{R}} c_i(\xi)M_2(\xi,(\eta*\rho_\varepsilon)(t,x))\, d\xi$$
$$= F_i(\rho(t,x))G_i((\eta*\rho)(t,x))\mathbb{1}_{\eta*\rho(t,x)\neq 0}$$
$$= F_i(\rho(t,x))G_i((\eta*\rho)(t,x))$$

since $\eta*\rho(t,x)=0$ a.e. implies $\rho(t,x)=0$ a.e. and $F(0)=0$. Thus we get (2.12). Finally we have

$$\iint_{\mathbf{R}^2} |\mathcal{M}_{\rho_1}(t,x,v,\xi)-\mathcal{M}_{\rho_2}(t,x,v,\xi)|\, d\xi dv$$
$$= \iint_{\mathbf{R}^2} \left|M_{\rho_1(t,x),(\eta*\rho_1)(t,x)}(v,\xi)-M_{\rho_2(t,x),(\eta*\rho_2)(t,x)}(v,\xi)\right| d\xi dv$$
$$= |\rho_1-\rho_2|\,(t,x) + 2\left(\frac{\min(\rho_1,\rho_2)}{\max(\eta*\rho_1,\eta*\rho_2)}\right)(t,x)\,|\eta*\rho_1-\eta*\rho_2|(t,x)$$

from proposition 5.2. As a consequence,

$$\int_{\mathbf{R}^d}\iint_{\mathbf{R}^2} |\mathcal{M}_{\rho_1}(t,x,v,\xi)-\mathcal{M}_{\rho_2}(t,x,v,\xi)|\, d\xi dv\, dx$$
$$\leq \int_{\mathbf{R}^d} |\rho_1-\rho_2|\,(t,x)\,dx + 2\int_{\mathbf{R}^d}\left(\frac{\min(\rho_1,\rho_2)}{\max(\eta*\rho_1,\eta*\rho_2)}\right)(t,x)\,|\eta*\rho_1-\eta*\rho_2|(t,x)\,dx$$

Since

$$|(\eta*\rho_1-\eta*\rho_2)(t,x)| \leq \int_{\mathbf{R}^d}\eta(x-y)\,|(\rho_1-\rho_2)(t,y)|\,dy$$
$$\leq \|\eta\|_\infty\int_{\mathbf{R}^d}|(\rho_1-\rho_2)(t,y)|\,dy,$$

then we have

$$\int_{\mathbf{R}^d}\iint_{\mathbf{R}^2} |\mathcal{M}_{\rho_1}(t,x,v,\xi)-\mathcal{M}_{\rho_2}(t,x,v,\xi)|\, d\xi dv\, dx$$
$$\leq \int_{\mathbf{R}^d} |\rho_1-\rho_2|\,(t,x)\,dx + 2\|\eta\|_\infty\int_{\mathbf{R}^d}|(\rho_1-\rho_2)(t,y)|\,dy\int_{\mathbf{R}^d}\left(\frac{\min(\rho_1,\rho_2)}{\max(\eta*\rho_1,\eta*\rho_2)}\right)(t,x)\,dx$$

From

$$\frac{\min(\rho_1,\rho_2)}{\max(\eta*\rho_1,\eta*\rho_2)} = \frac{\rho_1+\rho_2-|\rho_1-\rho_2|}{\eta*\rho_1+\eta*\rho_2+|\eta*\rho_1-\eta*\rho_2|} \le \frac{\rho_1+\rho_2}{\eta*\rho_1+\eta*\rho_2}$$

and

$$\eta*\rho_1(t,x)+\eta*\rho_2(t,x) \ge \int_{\mathbf{R}^d} \eta(x-y)(\rho_1+\rho_2)(t,y)\,dy \ge \alpha \int_{\mathbf{R}^d} (\rho_1+\rho_2)(t,y)\,dy,$$

we get

$$\int_{\mathbf{R}^d} \frac{\min(\rho_1,\rho_2)}{\max(\eta*\rho_1,\eta*\rho_2)}\,dx \le \int_{\mathbf{R}^d} \frac{\rho_1+\rho_2}{\alpha\int_{\mathbf{R}^d}(\rho_1+\rho_2)(t,y)\,dy}\,dx = \frac{1}{\alpha}.$$

Therefore we obtain

$$\int_{\mathbf{R}^d} \iint_{\mathbf{R}^2} |\mathcal{M}_{\rho_1}(t,x,v,\xi)-\mathcal{M}_{\rho_2}(t,x,v,\xi)|\,d\xi dv\,dx \le \left(1+\frac{2\|\eta\|_\infty}{\alpha}\right)\int_{\mathbf{R}^d} |\rho_1-\rho_2|\,(t,x)\,dx$$

and (2.13). □

Finally, applying Proposition 3.1, which is a variant of Theorem 1.1, we obtain the following result.

Theorem 5.4 *Let $f^0 \in L^1(\mathbf{R}^{d+2})$ such that $f^0 \ge 0$. Let $F, G \in C^1(\mathbf{R},\mathbf{R}^d)$, $\eta \in C^1(\mathbf{R}^d,\mathbf{R})$ functions such that $F(0)=0$, $\eta, \frac{1}{\eta} \in L^\infty(\mathbf{R}^d,\mathbf{R})$. Let $a(v,\xi)=b(v)c(\xi)$ such that (5.3)-(5.7). Then, for any $\varepsilon>0$, there exists $f_\varepsilon \in L^\infty([0,T],L^1(\mathbf{R}^{d+2}))$ for any $T>0$ solution of (1.4) with initial data f^0 and such that $f_\varepsilon \ge 0$.*

6 A model satisfying the second set of properties

We present a model in one dimension, what is to say $d=1$, for which Theorem 1.2 can be applied.

We make the following assumptions on F, G and η:

$$F \in C^2(\mathbf{R},\mathbf{R}), F(0)=0 \text{ and } F, F' \text{ are strictly monotone functions,} \tag{6.1}$$

$$G \in C^1(\mathbf{R},\mathbf{R}), G, G' \text{ are strictly increasing functions, } G'>0, \tag{6.2}$$

and such that there exists $X_0<0$, $K_0>0$ and $\gamma>1$ for which

$$|G(x)| \le \frac{K_0}{|x|^\gamma} \quad \text{and} \quad |G'(x)| \le \frac{K_0}{|x|^{\gamma+1}} \quad \text{if } x \le X_0, \tag{6.3}$$

and

$$\eta \in C^1(\mathbf{R},\mathbf{R}) \cap L^\infty(\mathbf{R}) \cap L^2(\mathbf{R}). \tag{6.4}$$

The term $\eta*\rho$ is well defined as soon as $x \mapsto \rho(t,x) \in L^1(\mathbf{R})$ for a.e. t.

For the kinetic model, we consider

$$a(v,\xi) = b(v)c'(\xi) \tag{6.5}$$

where

$$b(v) = F'(v), \quad c(\xi) = 2\sum_{n=0}^{+\infty} G(\xi - 2n - 1) \tag{6.6}$$

and

$$\tilde{\mathcal{M}}_\rho(t,x,v,\xi) = \tilde{M}_{\rho(t,x),(\eta*\rho)(t,x)}(v,\xi), \tag{6.7}$$

where

$$\tilde{M}_{\rho,q}(v,\xi) = M_1(v,\rho)M_3(\xi,q), \tag{6.8}$$

$$M_1(v,\rho) = \begin{cases} \mathrm{sgn}(\rho) & \text{if } (\rho - v)v \geq 0, \\ 0 & \text{if } (\rho - v)v < 0, \end{cases} \tag{6.9}$$

$$M_3(\xi,q) = \frac{1}{2}\mathbb{1}_{|\xi - q|<1}(\xi). \tag{6.10}$$

Remember that

$$\rho(t,x) = \iint\limits_{\mathbf{R}^2} f(t,y,\tilde{v},\tilde{\xi})\, d\tilde{\xi} d\tilde{v}.$$

Remark 6.1 Notice that c is well defined thanks to assumption (6.3) because

$$|G(\xi - 2n - 1)| \leq \frac{K_0}{(2n+1-|\xi|)^\gamma} \qquad \textit{for } 2n-1 > |\xi| - X_0$$

and is C^1 on $\mathbf{R}$ since, for any set $]-\infty,\alpha]$ with $\alpha > 0$, we have, for any $n \geq n_0$ where $2n_0 + 1 > \alpha$,

$$|G'(\xi - 2n - 1)| \leq \frac{K_0}{(2n+1-\alpha)^{\gamma+1}} \qquad \textit{if } x \in]-\infty, \alpha].$$

Then

$$c'(\xi) = 2\sum_{n=0}^{+\infty} G'(\xi - 2n - 1) \qquad \textit{for any } x \in \mathbf{R},$$

and c, c' are strictly increasing functions and $c' > 0$.

Remark 6.2 We can also consider the case where G is a strictly decreasing function with assumptions on $+\infty$ this time.

We need to apply averaging lemma, thus we have to prove the following non degeneracy condition : for all $R > 0$, there is a constant $C = C(R)$ such that for $z \in \mathbf{R}$, $\tau \in \mathbf{R}$ with $\sigma^2 + \tau^2 = 1$, then

$$\mathrm{meas}\{(v,\xi) \in \mathbf{R}^2 \text{ s.t. } |v|,|\xi| \leq R \text{ and } |a(v,\xi)\sigma - \tau| \leq \varepsilon\} \leq C\varepsilon. \tag{6.11}$$

We refer to [17], [8], [19], [7], [24], [21] and references within for averaging lemmas.

6.1 Preliminary properties

Properties for M_1 are included in proposition 5.1. For M_3, we have the following result.

Proposition 6.1 *Let $F, G \in C^1(\mathbf{R},\mathbf{R})$ such that (6.3) is satisfied. Then we have, for any $q \in \mathbf{R}$,*

$$\int_{\mathbf{R}} M_3(\xi,q)\,d\xi = 1,$$

$$\int_{\mathbf{R}} C'(\xi)M_3(\xi,q)\,d\xi = \frac{1}{2}(C(q+1)-C(q-1)), \quad \forall C \in C^1(\mathbf{R},\mathbf{R})$$

and

$$\int_{\mathbf{R}} c'(\xi)M_3(\xi,q)\,d\xi = G(q).$$

Proof. For the first property, we write

$$\int_{\mathbf{R}} \frac{1}{2}\mathbb{1}_{|\xi-q|<1}(\xi)\,d\xi = \frac{1}{2}\int_{q-1}^{q+1} d\xi = \frac{2}{2} = 1.$$

The second equality comes from the following:

$$\int_{\mathbf{R}} C'(\xi)M_3(\xi,q)\,d\xi = \frac{1}{2}\int_{q-1}^{q+1} C'(\xi)\,d\xi = \frac{1}{2}(C(q+1)-C(q-1)).$$

Then we get the third one since

$$\begin{aligned} c(q+1)-c(q-1) &= 2\sum_{n=0}^{+\infty} G(q+1-2n-1) - 2\sum_{n=0}^{+\infty} G(q-1-2n-1) \\ &= 2G(q). \ \square \end{aligned}$$

Remark 6.3 Notice that we cannot apply contraction tools in this case since we have the following equalities. First

$$\iint_{\mathbf{R}^2} \left|\tilde{M}_{\rho,q}(v,\xi) - \tilde{M}_{\tilde{\rho},\tilde{q}}(v,\xi)\right| d\xi dv = \iint_{\mathbf{R}^2} |M_1(v,\rho)M_3(\xi,q) - M_1(v,\tilde{\rho})M_3(\xi,\tilde{q})|\, d\xi dv.$$

For $\tilde{\rho} > \rho > 0$ and $\tilde{q} > q > 0$, we get

$$\begin{aligned} &\iint_{\mathbf{R}^2} \left|\tilde{M}_{\rho,q}(v,\xi) - \tilde{M}_{\tilde{\rho},\tilde{q}}(v,\xi)\right| d\xi dv \\ &= \frac{1}{2}\int_0^{\rho}\int_{\mathbf{R}} |\mathbb{1}_{q-1<\xi<q+1}(\xi) - \mathbb{1}_{\tilde{q}-1<\xi<\tilde{q}+1}(\xi)|\, d\xi dv + \frac{1}{2}\int_{\rho}^{\tilde{\rho}}\int_{\mathbf{R}} \mathbb{1}_{\tilde{q}-1<\xi<\tilde{q}+1}(\xi)\,d\xi dv. \end{aligned}$$

If $q+2 \le \tilde{q}$, we have

$$\iint_{\mathbf{R}^2} \left|\tilde{M}_{\rho,q}(v,\xi) - \tilde{M}_{\tilde{\rho},\tilde{q}}(v,\xi)\right| d\xi dv = 2\rho + (\tilde{\rho}-\rho),$$

if $\tilde{q} < q+2$, *we have*

$$\iint\limits_{\mathbf{R}^2} \left|\tilde{M}_{\rho,q}(v,\xi) - \tilde{M}_{\tilde{\rho},\tilde{q}}(v,\xi)\right| \, d\xi dv = \rho(\tilde{q}-q) + (\tilde{\rho}-\rho).$$

Then, by studying similar cases, we get

$$\begin{aligned}\iint\limits_{\mathbf{R}^2} \left|\tilde{M}_{\rho,q}(v,\xi) - \tilde{M}_{\tilde{\rho},\tilde{q}}(v,\xi)\right| \, d\xi dv &= 2\min(\rho,\tilde{\rho})\mathbb{1}_{\min(q,\tilde{q})+2\leq \max(q,\tilde{q})} \\ &\quad + \min(\rho,\tilde{\rho})|\tilde{q}-q|\mathbb{1}_{0<\max(q,\tilde{q})<\min(q,\tilde{q})+2} + |\tilde{\rho}-\rho|.\end{aligned}$$

The term $2\min(\rho,\tilde{\rho})\mathbb{1}_{\min(q,\tilde{q})+2\leq \max(q,\tilde{q})}$ *does not allow a contraction study.*

6.2 Second set of properties satisfied and existence result

We are now able to get the following result on this model.

Proposition 6.2 *Let* $F, G, \eta : \mathbf{R} \to \mathbf{R}$, $a : \mathbf{R}^2 \to \mathbf{R}$ *satisfying (6.1)- (6.10). Then the model of this section satisfies (2.1)-(2.2) and (2.4)-(2.9) and also (6.11) and (2.10)*

Proof. First we have

$$\iint\limits_{\mathbf{R}^2} \tilde{\mathcal{M}}_\rho(t,x,v,\xi)\, d\xi dv = \int_{\mathbf{R}} M_1(v,\rho(t,x))\, dv \int_{\mathbf{R}} M_3(\xi,(\eta*\rho)(t,x))\, d\xi = \rho(t,x).$$

Thus we get (2.1). Now we have

$$\begin{aligned}\iint\limits_{\mathbf{R}^2} a(v,\xi)\tilde{\mathcal{M}}_\rho(t,x,v,\xi)\, d\xi dv &= \iint\limits_{\mathbf{R}^2} b(v)c'(\xi)\tilde{\mathcal{M}}_\rho(t,x,v,\xi)\, d\xi dv \\ &= \int_{\mathbf{R}} b(v)M_1(v,\rho(t,x))\, dv \int_{\mathbf{R}} c'(\xi)M_3(\xi,(\eta*\rho)(t,x))\, d\xi \\ &= F(\rho(t,))G((\eta*\rho)(t,x))\end{aligned}$$

and we obtain (2.2). Furthermore, we have

$$\begin{aligned}\iint\limits_{\mathbf{R}^2} |\tilde{\mathcal{M}}_\rho(t,x,v,\xi)|\, d\xi dv &= \int_{\mathbf{R}} |M_1(v,\rho(t,x))|\, dv \int_{\mathbf{R}} |M_3(\xi,(\eta*\rho)(t,x))|\, d\xi \\ &= |\rho(t,x)| \leq |\rho(t,x)|,\end{aligned}$$

that is to say (2.4) and

$$\begin{aligned}\iint\limits_{\mathbf{R}^2} |a(v,\xi)\tilde{\mathcal{M}}_\rho(t,x,v,\xi)|\, d\xi dv &= \iint\limits_{\mathbf{R}^2} |b(v)||c'(\xi)||M_1(v,\rho(t,x))||M_3(\xi,(\eta*\rho)(t,x))|\, d\xi dv \\ &= \int_{\mathbf{R}} |b(v)||M_1(v,\rho(t,x))|\, dv \int_{\mathbf{R}} |c'(\xi)||M_3(\xi,(\eta*\rho)(t,x))|\, d\xi \\ &= \int_{\mathbf{R}} |F'(v)||M_1(v,\rho(t,x))|\, dv \left|\int_{\mathbf{R}} c'(\xi)M_3(\xi,(\eta*\rho)(t,x))\, d\xi\right| \\ &\leq |F(\rho(t,x))G((\eta*\rho)(t,x))|\end{aligned}$$

thanks to the monotonicity properties of c and F and we have (2.5) with $K_2 = 1$. Now

$$\iint_{\mathbf{R}^2} |\xi||\tilde{\mathcal{M}}_\rho(t,x,v,\xi)|\, d\xi dv = \int_{\mathbf{R}} |M_1(v,\rho(t,x))|\, dv \int_{\mathbf{R}} |\xi| M_3(\xi,(\eta*\rho)(t,x))\, d\xi$$

and since

$$\int_{\mathbf{R}} |\xi| M_3(\xi,q)\, d\xi = \begin{cases} |q| & \text{if } q+1<0 \text{ or } q-1>0, \\ (q^2+1)/2 & \text{if } q-1 \le 0 \le q+1, \end{cases}$$

we get $\int_{\mathbf{R}} |\xi| M_3(\xi,q)\, d\xi \le q^2+1$ and

$$\begin{aligned}
\iint_{\mathbf{R}^2} |\xi||\tilde{\mathcal{M}}_\rho(t,x,v,\xi)|\, d\xi dv &\le |\rho(t,x)| \left((\eta*\rho)(t,x))^2+1\right) \\
&\le |\rho(t,x)| \left(1+\left(\int_{\mathbf{R}} \eta(x-y)\rho(t,y)\, dy\right)^2\right) \\
&\le |\rho(t,x)| \left(1+\int_{\mathbf{R}} \eta(x-y)^2\, dy \int_{\mathbf{R}} |\rho(t,y)|^2\, dy\right) \\
&\le |\rho(t,x)| \left(1+\int_{\mathbf{R}} \eta(y)^2\, dy \int_{\mathbf{R}} |\rho(t,y)|^2\, dy\right) \\
&\le \max\left(1,\int_{\mathbf{R}} \eta(y)^2\, dy\right) |\rho(t,x)| \left(1+\int_{\mathbf{R}} |\rho(t,y)|^2\, dy\right)
\end{aligned}$$

that is to say (2.8) with $K_3 = \max\left(1, \int_{\mathbf{R}} \eta(y)^2\, dy\right)$. For the following estimate, we have

$$\iint_{\mathbf{R}^2} |v||\tilde{\mathcal{M}}_\rho(t,x,v,\xi)|\, d\xi dv = \int_{\mathbf{R}} |v||M_1(v,\rho(t,x))|\, dv \int_{\mathbf{R}} M_3(\xi,(\eta*\rho)(t,x))\, d\xi$$

and

$$\int_{\mathbf{R}} |v||M_1(v,\rho(t,x))|\, dv = \begin{cases} \displaystyle\int_0^\rho v\, dv & \text{if } \rho>0, \\ \displaystyle\int_\rho^0 (-v)\, dv & \text{if } \rho \le 0, \end{cases}$$

thus $\int_{\mathbf{R}} |v||M_1(v,\rho(t,x))|\, dv = \rho(t,x)^2/2$ and

$$\iint_{\mathbf{R}^2} |v||\tilde{\mathcal{M}}_\rho(t,x,v,\xi)|\, d\xi dv = \frac{\rho(t,x)^2}{2}$$

that is (2.6) with $K_4 = 1/2$. After this, we write

$$\begin{aligned}
\iint_{\mathbf{R}^2} |\tilde{\mathcal{M}}_\rho(t,x,v,\xi)|^2\, d\xi dv &= \int_{\mathbf{R}} M_1(v,\rho(t,x))^2\, dv \int_{\mathbf{R}} M_3(\xi,(\eta*\rho)(t,x))^2\, d\xi \\
&= \int_{\mathbf{R}} |M_1(v,\rho(t,x))|\, dv \int_{\mathbf{R}} \frac{1}{2} M_3(\xi,(\eta*\rho)(t,x))\, d\xi \\
&= \frac{1}{2}|\rho(t,x)|
\end{aligned}$$

and we get (2.7) with $K_5 = 1/2$ and $p = 1$. Assuming now that we have functions satisfying $\rho_n \to \rho$ a.e. (t,x) and $|\rho_n| \le |h| \in L^1(\mathbf{R})$, then applying the theorem of dominated convergence, we get that

$$(\eta * \rho_n)(t,x) = \int_{\mathbf{R}} \eta(x-y)\rho_n(t,y)\,dy \to (\eta * \rho)(t,x) = \int_{\mathbf{R}} \eta(x-y)\rho(t,y)\,dy$$

since $\eta \in L^\infty$. Then we get (2.9).

Let $R > 0$. We set

$$K_R = \max\left(8 \sup_{z\in[-R,R]} |F'(z)| \sup_{z\in[-R,R]} |c'(z)|, \frac{4}{\sqrt{7}}\right) \frac{1}{R}.$$

Let $(\sigma,\tau) \in \mathbf{R}^2$ such that $\sigma^2 + \tau^2 = 1$. Let $\varepsilon \in]0, 1/2[$. We want to consider set where $|F'(v)c'(\xi)\sigma + \tau| < \varepsilon$. Notice that changing (σ,τ) by $(-\sigma,-\tau)$, we can assume that $\sigma \ge 0$. There exists $\theta \in]-\pi/2, \pi/2]$ such that $\sigma = \cos\theta$ and $\tau = \sin\theta$. Since c' is strictly increasing and strictly positive, we have

$$0 < c'(-R) < c'(\xi) < c'(R) \quad \text{for any } \xi \in [-R,R].$$

We consider $(v,\xi) \in \mathbf{R}^2$ such that $|v| \le R$ and $|\xi| \le R$ satisfying

$$\sin\theta - \varepsilon < F'(v)c'(\xi)\cos\theta < \sin\theta + \varepsilon.$$

If $\cos\theta = 0$, then the set of (v,ξ) satisfying $\pm 1 - \varepsilon < 0 < \pm 1 + \varepsilon$ is empty since $0 < \varepsilon < 1$. We consider now the case $\cos\theta > 0$. Then we have

$$\frac{\sin\theta - \varepsilon}{c'(\xi)\cos\theta} < F'(v) < \frac{\sin\theta + \varepsilon}{c'(\xi)\cos\theta}$$

and since F' is strictly monotone, we get

$$(F')^{-1}\left(\frac{\sin\theta - \varepsilon}{c'(\xi)\cos\theta}\right) < v < (F')^{-1}\left(\frac{\sin\theta + \varepsilon}{c'(\xi)\cos\theta}\right)$$

or

$$(F')^{-1}\left(\frac{\sin\theta + \varepsilon}{c'(\xi)\cos\theta}\right) < v < (F')^{-1}\left(\frac{\sin\theta - \varepsilon}{c'(\xi)\cos\theta}\right).$$

Consider for example the strictly increasing case. First case: if $0 < 1/\cos\theta \le K_R R$, then we get

$$\begin{aligned}
&\text{meas}\{(v,\xi) \in \mathbf{R}^2 \text{ s.t. } |v|,|\xi| \le R \text{ and } |a(v,\xi)\sigma - \tau| < \varepsilon\} \\
\le\ & \int_{c'(-R)}^{c'(R)} \int_{(F')^{-1}((\sin\theta-\varepsilon)/(c'(\xi)\cos\theta))}^{(F')^{-1}((\sin\theta+\varepsilon)/(c'(\xi)\cos\theta))} dv d\xi \\
\le\ & \int_{g(-R)}^{g(R)} \left((F')^{-1}\left(\frac{\sin\theta + \varepsilon}{c'(\xi)\cos\theta}\right) - (F')^{-1}\left(\frac{\sin\theta - \varepsilon}{c'(\xi)\cos\theta}\right)\right) d\xi \\
\le\ & \sup_{z\in I_R} |((F')^{-1})'(z)| \int_{g(-R)}^{g(R)} \left(\frac{\sin\theta + \varepsilon}{c'(\xi)\cos\theta} - \frac{\sin\theta - \varepsilon}{c'(\xi)\cos\theta}\right) d\xi
\end{aligned}$$

where $I_R = [(-1-\varepsilon)K_R R/c'(R), (1+\varepsilon)K_R R/c'(-R)]$. It leads to

$$\begin{aligned} & \text{meas}\{(v,\xi)\in \mathbf{R}^2 \text{ s.t. } |v|,|\xi|\le R \text{ and } |F'(v)c'(\xi)\sigma-\tau|<\varepsilon\} \\ \le\ & 2\varepsilon K_R R \sup_{z\in I_R} |((F')^{-1})'(z)| \int_{g(-R)}^{g(R)} \frac{1}{c'(\xi)}\, d\xi. \end{aligned}$$

Second case: if $1/\cos\theta > K_R R$, then we get

$$|F'(v)c'(\xi)\sigma| \le \sup_{z\in[-R,R]} |F'(z)| \sup_{z\in[-R,R]} |c'(z)| \frac{1}{K_R R} \le \frac{1}{8}$$

and

$$|\tau| = \sqrt{1-\cos^2\theta} > \sqrt{1-\frac{1}{K_R^2 R^2}} \le \frac{3}{4}.$$

Thus $|a(v,\xi)\sigma-\tau| > 3/4 - 1/8 > 1/2 > \varepsilon$ and

$$\text{meas}\{(v,\xi)\in \mathbf{R}^2 \text{ s.t. } |v|,|\xi|\le R \text{ and } |a(v,\xi)\sigma-\tau|<\varepsilon\} = 0.$$

Finally, we get

$$\sup_{\sigma^2+\tau^2=1} \text{meas}\{(v,\xi)\in \mathbf{R}^2 \text{ s.t. } |v|,|\xi|\le R \text{ and } |a(v,\xi)\sigma-\tau|<\varepsilon\} \le C_R\varepsilon$$

where

$$\begin{aligned} C_R &= 2K_R R \sup_{z\in I_R} |((F')^{-1})'(z)| \int_{g(-R)}^{g(R)} \frac{1}{c'(\xi)}\, d\xi \\ &= 2\max\left(8 \sup_{z\in[-R,R]} |F'(z)| \sup_{z\in[-R,R]} |c'(z)|, \frac{4}{\sqrt{7}}\right) \sup_{z\in I_R} |((F')^{-1})'(z)| \int_{g(-R)}^{g(R)} \frac{1}{c'(\xi)}\, d\xi. \end{aligned}$$

It gives (6.11) and we get (2.10). □

Then applying Theorem 1.2 we settle the following result.

Theorem 6.3 *Let $f^0 \in L^1(\mathbf{R}^3)\cap L^2(\mathbf{R}^3)$ such that $xf^0, \xi f^0, vf^0, a(v,\xi)f^0 \in L^1(\mathbf{R}^3)$ and*

$$\int_{\mathbf{R}} \left(\iint_{\mathbf{R}^2} f^0(x,v,\xi)\, dv d\xi \right)^2 dx < +\infty.$$

Consider $F, G, \eta : \mathbf{R}\to\mathbf{R}$ satisfying (6.1)-(6.4) and assume that $\eta\in L^\infty(\mathbf{R})$ and that there exists a constant $K>0$ such that

$$|F(z)| \le K(|z|+|z|^2) \quad \text{for any } z\in\mathbf{R}.$$

Let $a(v,\xi) = b(v)c'(\xi)$ such that (6.6)-(6.10). Then, for any $\varepsilon>0$, there exists $f_\varepsilon \in L^\infty([0,T], L^1(\mathbf{R}^3))$ for any $T>0$ solution of (1.4) with initial data f^0.

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