FROM DISCRETE VELOCITY BOLTZMANN EQUATIONS TO GAS DYNAMICS BEFORE SHOCKS

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Abstract. This article is devoted to the proof of the hydrodynamic limit for a discrete velocity Boltzmann equation before appearance of shocks in the limit system.

1. Introduction

We consider the system of discrete velocity Boltzmann equations

$$\partial_t f_i + v_i \partial_x f_i = \frac{1}{\varepsilon} Q_i(f,f), \qquad \text{for } i = 1, \dots, N, \tag{1.1}$$

where

$$Q_i(f,f) = \sum_{jkl} S_{ijkl}(f_k f_l - f_i f_j), \tag{1.2}$$

and $N \geq 3$. Such systems have been extensively studied in the literature (see *e.g.* Cabannes, Gatignol and Luo [7] or Platkwoski and Illner [17] and references therein) because they offer a simplification and approximation of the Boltzmann equation that shares remarkable similarities to the latter model. Nevertheless, these systems are quite simpler than the Boltzmann equation and for instance their existence theory is relatively well understood in both the cases of one dimension [1, 3, 12] as well as in several space dimensions [4, 11]. Discrete velocity models present certain pathologies in several space dimensions and we will refrain from working with them here.

The parameter ε is called mean free path or Knudsen number and. under certain conditions on the interaction coefficients S_{ijkl} that will be precised later, the system formally converges as $\varepsilon \to 0$ to equations,

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0, \\ \partial_t(\rho u) + \partial_x(\rho E) &= 0, \\ \partial_t(\rho E) + \partial_x(\rho J(u,E)) &= 0, \end{aligned} \tag{1.3}$$

The limiting procedure has been justified for the case of the Broadwell model by Calfisch and Papanicolaou [8] and the related problem of the asymptotic in time convergence of (1.1) to global Maxwellians is established in works of Beale [1] and Kawashima [14].

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The objective of this article is to establish the hydrodynamic limit from discrete Boltzmann equations (1.1) to the gas dynamics system in the form (1.3) in the regime where the solutions of (1.3) remain smooth. We will develop and estimate a relative entropy identity following ideas in Berthelin-Vasseur [2] and Tzavaras [19]. These articles concern kinetic or relaxation limits for BGK-type of collision operators. The ingredients, required in order to account for Boltzmann collison operators, are an estimation of the entropy dissipation and certain structural properties that pertain to system (1.3). The relative entropy method was developed in the context of uniqueness and stability for hyperbolic conservation laws by Dafermos [9] and DiPerna [10], and the context of hydrodynamics for stochastic particle systems by Yau [20] and Olla-Varadhan-Yau [16]. In addition to the aforementioned references the reader is referred to [5, 6, 13, 15, 18] for the application of relative entropy in a variety of contexts.

We begin in section 2 with a description of the model, an outline of the formalism of its hydrodynamic limit and the statement of the main result Theorem 2.3. In section 3 we develop links between the kinetic and the macroscopic entropies and prove certain structural properties of the limit system, the entropy consistency property and hyperbolicity. Section 4 contains the key estimation of the entropy dissipation (Proposition 4.1), and section 5 contains the derivation of the relative entropy identity and the conclusion of the proof of Theorem 2.3.

2. Description of the model and statement of results

The interaction coefficients S_{ijkl} entering the definition of the collision operator (1.2) are assumed to satisfy the properties of symmetry and microreversibility,

$$S_{ijkl} = S_{jikl}, \qquad S_{ijkl} = S_{ijlk}, \tag{2.4}$$

$$S_{ijkl} = S_{klij}, \tag{2.5}$$

and to describe the probability of the elastic collision $(i,j) \to (k,l)$ conserving the microscopic mass, momentum and energy

$$v_k + v_l = v_i + v_j, \qquad v_k^2 + v_l^2 = v_i^2 + v_j^2 \quad \text{if } S_{ijkl} \neq 0. \tag{2.6}$$

For any $f \in \mathbb{R}^N$, we have from (2.6):

$$\sum_i Q_i = 0, \qquad \sum_i v_i Q_i = 0, \qquad \sum_i v_i^2 Q_i = 0\,, \tag{2.7}$$

which entail conservation laws for the total mass, momentum and energy. Consider the collision matrix $B \in \{-1,0,1\}^{N^2 \times N}$ given by:

$$\begin{aligned} &B_{ij,i} = B_{ij,j} = -B_{ij,k} = -B_{ij,l} = 1 \qquad \text{if} \quad S_{ijkl} \neq 0, \\ &B_{ij,k} = 0 \qquad \text{everywhere else.} \end{aligned}$$

Note that (2.7) implies that $(1,\cdots,1)$, $(v_1,\cdots,v_N)$ and $(v_1^2,\cdots,v_N^2)$ are in the kernel of B. We pose the additional hypothesis:

(H) $\quad N(B) = \operatorname{span}\big\{(1,\cdots,1),(v_1,\cdots,v_N),(v_1^2,\cdots,v_N^2)\big\}$ and $\dim N(B) = 3\,.$

This implies that the only conserved quantities are precisely the mass, momentum and energy and that the model does not have any extraneous conservation laws.

Finally, we define the entropy of f via the usual relation

$$\mathcal{H}(f) = \sum_{i=1}^{N} f_i \ln f_i$$

For any C^1 function $F : \mathcal{D} \to \mathbb{R}$ defined on a convex set $\mathcal{D} \subset \mathbb{R}^k$, we define the associated relative function:

$$F(U_1|U_2) = F(U_1) - F(U_2) - F'(U_2)(U_1 - U_2).$$

We also set

$$s(y) = y \ln y\,,$$

and use in the sequel the following notations:

$$\begin{aligned}
&a * b = a_1b_1 + a_2b_2 + a_3b_3 \qquad a, b \in \mathbb{R}^3,\\
&f \cdot g = \sum_{i=1}^{N} f_i g_i \qquad f, g \in \mathbb{R}^N,\\
&|f| = \sum_{i=1}^{N} |f_i| \qquad f \in \mathbb{R}^N,\\
&Pf = \sum_{i=1}^{N} (1, v_i, v_i^2) f_i \qquad f \in \mathbb{R}^N,\\
&|Pf| = \sum_{\beta=0}^{2} \sum_{i=1}^{N} \left| v_i^\beta f_i \right| \qquad f \in \mathbb{R}^N.
\end{aligned}$$

Our first lemma concerns tha structure of Maxwellians associated to discrete velocity Boltzmann equations:

Lemma 2.1. *A vector $(M_1, \cdots, M_N) \in (\mathbb{R}^+)^N$ verifies $Q(M, M) = 0$ if and only if there exists $a, b, c \in \mathbb{R}$ such that*

$$M_i = e^{a+bv_i+cv_i^2} \qquad \text{for any } 1 \le i \le N.$$

Setting $\psi(b, c) = \sum_{i=1}^{N} e^{bv_i+cv_i^2}$, we express the Maxwellians in the form

$$M_i = \rho \frac{e^{bv_i+cv_i^2}}{\sum_{i=1}^{N} e^{bv_i+cv_i^2}}$$

and note the relations

$$\rho = \sum_{i=1}^{N} M_i = e^a \psi(b, c), \quad \sum_{i=1}^{N} v_i M_i = \rho \frac{\partial_b \psi}{\psi}(b, c), \quad \sum_{i=1}^{N} v_i^2 M_i = \rho \frac{\partial_c \psi}{\psi}(b, c). \tag{2.8}$$

For a given Maxwellian M, we define

$$(\rho, \rho u, \rho E) = PM,$$

that is to say

$$\rho = \sum_{i=1}^{N} M_i = \sum_{i=1}^{N} e^{a+bv_i+cv_i^2} = e^a \psi(b,c),$$

$$\rho u = \sum_{i=1}^{N} v_i M_i = \sum_{i=1}^{N} v_i e^{a+bv_i+cv_i^2} = \rho \frac{\partial_b \psi}{\psi}(b,c),$$

$$\rho E = \sum_{i=1}^{N} v_i^2 M_i = \sum_{i=1}^{N} v_i^2 e^{a+bv_i+cv_i^2} = \rho \frac{\partial_c \psi}{\psi}(b,c).$$

We notice that

$$u = \partial_b(\ln \psi), \qquad E = \partial_c(\ln \psi),$$

and then we denote $\mathcal{U}$ the set of admissible value of (u, E), that is:

$$\mathcal{U} = \{(\partial_b(\ln \psi), \partial_c(\ln \psi)) \mid b, c \in \mathbb{R}\}.$$

The hydrodynamic limit system can be written formally as

$$\left\{ \begin{array}{l} \partial_t \sum_i M_i + \partial_x \sum_i v_i M_i = 0, \\ \partial_t \sum_i v_i M_i + \partial_x \sum_i v_i^2 M_i = 0, \\ \partial_t \sum_i v_i^2 M_i + \partial_x \sum_i v_i^3 M_i = 0, \end{array} \right. \tag{2.9}$$

which leads to

$$\left\{ \begin{array}{l} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho E) = 0, \\ \partial_t(\rho E) + \partial_x(\rho J) = 0, \end{array} \right. \tag{2.10}$$

where we set

$$J(u,E) = \frac{\partial_{bc}\psi}{\psi}. \tag{2.11}$$

The flux $J = J(u, E)$ is well defined thanks to the following lemma.

Lemma 2.2. *The function $\ln \psi$ is smooth and strictly convex and so the map: T : $(b, c) \to (u, E)$ defined by*

$$T(b,c) = \nabla_{(b,c)} \ln \psi(b,c)$$

is a C^1 diffeomorphism from $\mathbb{R}^2$ to $\mathcal{U}$.

We introduce also the entropy of the system:

$$\eta(\rho, \rho u, \rho E) = \mathcal{H}(M) = \sum_{i=1}^{N} M_i \ln M_i.$$

Conversely, for any $U = (\rho, \rho u, \rho E)$ with $\rho > 0$ and $(u, E) \in \mathcal{U}$, we define

$$M(U) = (M_i(U))_{i=1,\cdots,N} = (e^{a+bv_i+cv_i^2})_{i=1,\cdots,N}$$

with ρ, u, E and a, b, c related as in Lemma 2.1.

The article is devoted to the proof of the following theorem:

Theorem 2.3. *Let (ρ_0, u_0, E_0), be a Lipshitzian function on $\mathbb{R}$ with values in $\mathbb{R}^+ \times \mathcal{U}$ such that $U_0 = (\rho_0, \rho_0 u_0, \rho_0 E_0)$ and $\eta(\rho_0, \rho_0 u_0, \rho_0 E_0)$ lie altogether in $L^1(\mathbb{R})$ and $\partial_x U_0 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then, there exists a maximal time T^* such that the solution $(\rho, \rho u, \rho E)$ to the limit system (2.10) with initial values (ρ_0, u_0, E_0) stays Lipshitzian on $[0, T^*) \times \mathbb{R}$. Denote $\overline{M}$ the Maxwellian associated to $(\rho, \rho u, \rho E)$. Consider $f_\varepsilon^0 \in (L^1(\mathbb{R}))^N$ such that each component is nonnegative and verifying $\mathcal{H}(f_\varepsilon^0)$ bounded in $L^1(\mathbb{R})$. We denote f_ε the solution of (1.1) with initial value f_ε^0. If f_ε^0 converges strongly to $\overline{M}^0$, Maxwellian associated to (ρ_0, u_0, E_0) in the sense that*

$$\int_{\mathbb{R}} \mathcal{H}(f_\varepsilon^0 | \overline{M}^0)(x)\, dx \underset{\varepsilon \to 0}{\longrightarrow} 0,$$

then f_ε converges strongly to $\overline{M}$ in the sense that for any $T < T^*$:

$$\sup_{0 \le t \le T} \int_{\mathbb{R}} \mathcal{H}(f_\varepsilon | \overline{M})(t, x)\, dx \underset{\varepsilon \to 0}{\longrightarrow} 0,$$

where

$$\mathcal{H}(f|g) = \sum_i f_i \ln(f_i/g_i) - (f_i - g_i) \ge 0.$$

The proof is based on the results of Tzavaras [19] and Berthelin-Vasseur[2] on the relative entropy method, and on an estimation of the entropy-dissipation developed in section 4.

3. Preliminaries

In this section, we gather certain structural properties of the model (1.1) and its hydrodynamic limit. Especially, we introduce a link between kinetic relative entropies and macroscopic ones, we show that the system is hyperbolic, entropy consistent and obtain properties on the domain $\mathcal{U}$.

First, we prove Lemma 2.1 and Lemma 2.2.

Proof of Lemma 2.1. If $Q(M, M) = 0$ then in particular

$$D[M] = \sum_{i=1}^N \ln(M_i) Q_i(M) = \frac{1}{4} \sum_{ijkl} S_{ijkl} [\ln(M_k M_l) - \ln(M_i M_j)](M_k M_l - M_i M_j) = 0.$$

But each terms of the last sum is nonnegative so, for any i, j, k, l such that $S_{ijkl} \neq 0$, we have $M_k M_l = M_i M_j$ which means:

$$\ln M_k + \ln M_l = \ln M_i + \ln M_j.$$

This implies that $\ln M = (\ln M_i)_{i=1,\cdots,N}$ lies in $N(B)$. Hypothesis (H) implies that there exists (a, b, c) such that

$$\ln M_i = a + b v_i + c v_i^2 \qquad \text{for any } 1 \le i \le N.$$

Conversely, note that if $\ln M$ is given by such a formula, then $M_k M_l = M_i M_j$ for any $ijkl$ verifying $S_{ijkl} \neq 0$ and $Q(M, M) = 0$ as well.

To show the second part of the statement. Note that from the definition of ψ we have $e^a = \dfrac{\rho}{\psi}$, that is to say $e^{-a} = \dfrac{\psi}{\rho}$. Noting that

$$\partial_b \psi = \sum_{i=1}^N v_i e^{bv_i + cv_i^2} = e^{-a} \sum_{i=1}^N v_i M_i,$$
$$\partial_c \psi = \sum_{i=1}^N v_i^2 e^{bv_i + cv_i^2} = e^{-a} \sum_{i=1}^N v_i^2 M_i,$$

gives the result. □

We list the useful formulas

$$\begin{aligned} \sum_{i=1}^N e^{bv_i + cv_i^2} &= \psi(b,c), \\ \sum_{i=1}^N v_i e^{bv_i + cv_i^2} &= \psi(b,c)\, u, \\ \sum_{i=1}^N v_i^2 e^{bv_i + cv_i^2} &= \psi(b,c)\, E. \end{aligned} \tag{3.12}$$

Proof of Lemma 2.2. The matrix of the second derivatives of $\ln \psi$ is:

$$\frac{1}{\psi^2} \begin{pmatrix} \psi \partial_{bb}\psi - (\partial_b \psi)^2 & \psi \partial_{bc}\psi - \partial_b \psi \partial_c \psi \\ \psi \partial_{bc}\psi - \partial_b \psi \partial_c \psi & \psi \partial_{cc}\psi - (\partial_c \psi)^2 \end{pmatrix}$$

which can be rewritten

$$\frac{1}{\psi} \begin{pmatrix} \sum_{i=1}^N (v_i - u)^2 e^{bv_i + cv_i^2} & \sum_{i=1}^N (v_i - u)(v_i^2 - E) e^{bv_i + cv_i^2} \\ \sum_{i=1}^N (v_i - u)(v_i^2 - E) e^{bv_i + cv_i^2} & \sum_{i=1}^N (v_i^2 - E)^2 e^{bv_i + cv_i^2} \end{pmatrix}.$$

Indeed, we have

$$\begin{aligned} \psi \partial_{bb}\psi - (\partial_b \psi)^2 \quad &= \quad \psi \sum_{i=1}^N v_i^2 e^{bv_i + cv_i^2} - \psi u \sum_{i=1}^N v_i e^{bv_i + cv_i^2} \\ &\overset{(3.12)}{=} \quad \psi \sum_{i=1}^N (v_i - u)^2 e^{bv_i + cv_i^2}, \end{aligned}$$

$$\begin{aligned} \psi \partial_{bc}\psi - (\partial_b \psi)(\partial_c \psi) \quad &= \quad \psi \sum_{i=1}^N v_i^3 e^{bv_i + cv_i^2} - \sum_{i=1}^N v_i e^{bv_i + cv_i^2} \sum_{i=1}^N v_i^2 e^{bv_i + cv_i^2} \\ &\overset{(3.12)}{=} \quad \psi \sum_{i=1}^N v_i^3 e^{bv_i + cv_i^2} - \psi^2 u E \\ &\overset{(3.12)}{=} \quad \psi \sum_{i=1}^N (v_i - u)(v_i^2 - E) e^{bv_i + cv_i^2}, \end{aligned}$$

and similarly for the last entry of the matrix.

The trace of this matrix is positive. Its determinant is also positive as can be seen by applying the Cauchy-Schwarz inequality,

$$\Big[\sum_{i=1}^{N}(v_i-u)(v_i^2-E)e^{bv_i+cv_i^2}\Big]^2 \le \Big(\sum_{i=1}^{N}(v_i-u)^2e^{bv_i+cv_i^2}\Big)\Big(\sum_{i=1}^{N}(v_i^2-E)^2e^{bv_i+cv_i^2}\Big)$$

If the determinant is equal to 0 then equality holds in the Cauchy-Schwarz inequality which, in turn, implies that the vectors $(1,...,1)$, $(v_1,...,v_N)$ and $(v_1^2,...v_N^2)$ are linearly dependent. The latter is ruled out by hypothesis (H), and thus the matrix of the second derivatives of $\ln\psi$ is strictly positive, and $\ln\psi$ is strictly convex. The mapping T is a $C^{1,1}$ diffeomorphism from $\mathbb{R}^2$ to $\mathcal{U}$. □

We prove now the following lemma related to relative quantities.

Lemma 3.1. *Let $F:\mathcal{D}\to\mathbb{R}$ be a C^2 function on a convex set $\mathcal{D}\subset\mathbb{R}^k$. The function F is convex on $\mathcal{D}$ if and only if the associated relative function is nonnegative on $\mathcal{D}\times\mathcal{D}$.*

Proof. For any $U_1,U_2\in\mathcal{V}$ we have:

$$F(U_1|U_2)=\int_0^1\int_0^1 F''(U_1+st(U_2-U_1)):[(U_1-U_2)\otimes(U_1-U_2)]t\,ds\,dt.$$

Hence, if F is convex then F'' is positive and so $F(U_1|U_2)$ is nonnegative. Conversely, for $|U_1-U_2|$ small, we have:

$$F(U_1|U_2)=F''(U_2):[(U_1-U_2)\otimes(U_1-U_2)]+o(|U_1-U_2|^2).$$

Then, if $F(\cdot|\cdot)$ is nonnegative everywhere, then $F''(U_2)$ is a nonnegative matrix for any U_2 and F is convex. □

In particular, $s'(y)=1+\ln y$ leads to the usual relation:

$$s(y|z)=y\ln\frac{y}{z}-(y-z)\ge 0,$$

since s is convex.

Let us show now the following lemma which gives the link between the relative entropy at the kinetic level and at the macroscopic level.

Lemma 3.2. *For any $U=(\rho,\rho u,\rho E)$ with $\rho>0$ and $(u,E)\in\mathcal{U}$, we set*

$$M(U)=(M_i(U))_{i=1,\cdots,N}=(e^{a+bv_i+cv_i^2})_{i=1,\cdots,N}$$

with ρ,u,E and a,b,c related as in Lemma 2.1. We then have

$$i)\quad PM(U)=U,$$

$$ii)\quad \eta(U)=\mathcal{H}(M(U))=\inf_{Pf=U}\mathcal{H}(f),$$

and

$$iii)\quad \frac{\partial\eta}{\partial U}(U)*w=\frac{\partial\mathcal{H}}{\partial f}(M(U))\cdot f,$$

for any $w\in\mathbb{R}^3$ and any $f\in\mathbb{R}^N$ such that $w=Pf$.
Especially

$$iv)\quad \eta(U|\overline{U})=\mathcal{H}(M(U)|M(\overline{U})),$$

and

$$v)\quad \eta(U|\overline{U}) \le \mathcal{H}(f|M(\overline{U})),$$

for any $Pf = U$.

Proof. $i)$ We have

$$PM(U) = \sum_{i=1}^{N}(1, v_i, v_i^2)M_i(U) = \sum_{i=1}^{N}(1, v_i, v_i^2)e^{a+bv_i+cv_i^2} = (\rho, \rho u, \rho E) = U.$$

$ii)$ By definition, we have

$$\eta(U) = \mathcal{H}(M(U)) = \sum_{i=1}^{N} s(M_i(U)).$$

For any f such that $Pf = U$, we have

$$\begin{aligned} 0 \le \mathcal{H}(f|M(U)) &= \mathcal{H}(f) - \mathcal{H}(M(U)) - \partial_f\mathcal{H}(M(U))\cdot(f - M(U)) \\ &= \mathcal{H}(f) - \mathcal{H}(M(U)) - \sum_{i=1}^{N}(1 + \ln M_i(U))(f_i - M_i(U)), \end{aligned}$$

with

$$\sum_{i=1}^{N}(1 + \ln M_i(U))(f_i - M_i(U)) = (1 + a, b, c) * P(f - M(U)) = 0$$

since $P(f - M(U)) = U - PM(U) = 0$. Hence:

$$\mathcal{H}(f) \ge \mathcal{H}(M(U)) \quad \text{for any } Pf = U,$$

which gives the result.

$iii)$ By differentiation of $U = PM(U)$ with respect to U, we get, with P linear,

$$Id = P\frac{\partial M}{\partial U}.$$

We denote

$$e_\beta = P\frac{\partial M}{\partial U_\beta}.$$

Let $f \in \mathbb{R}^N$ and $w \in \mathbb{R}^3$ such that $w = Pf$. Decomposing w on the basis (e_β) we have:

$$Pf = w = \sum_{\beta=1}^{3} w_\beta e_\beta = \sum_{\beta=1}^{3} w_\beta P\frac{\partial M}{\partial U_\beta},$$

and so

$$P\left(f - \sum_{\beta=1}^{3} w_\beta\frac{\partial M}{\partial U_\beta}\right) = 0.$$

This gives the existence of g such that:

$$\begin{aligned} f &= \sum_{\beta=1}^{3} w_\beta\frac{\partial M}{\partial U_\beta} + g \\ Pg &= 0. \end{aligned}$$

But
$$\frac{\partial \eta}{\partial U_\beta} = \frac{\partial \mathcal{H}}{\partial f}(M(U)) \cdot \frac{\partial M}{\partial U_\beta},$$
Hence:
$$\begin{aligned}\frac{\partial \eta}{\partial U} * w = \sum_{\beta=1}^{3} \frac{\partial \eta}{\partial U_\beta} w_\beta &= \sum_{\beta=1}^{3} w_\beta \frac{\partial \mathcal{H}}{\partial f}(M(U)) \cdot \frac{\partial M}{\partial U_\beta} \\ &= \frac{\partial \mathcal{H}}{\partial f}(M(U)) \cdot (f-g).\end{aligned}$$
We conclude with the argument that
$$\frac{\partial \mathcal{H}}{\partial f}(M(U)) \perp N(P).$$
This comes from the fact that
$$\eta(U) = \min_{Pf=U} \mathcal{H}(f) = \mathcal{H}(M(U))$$
(see [19, Proposition 2.1]).

iv) We have
$$\begin{aligned}\eta(U|\overline{U}) &= \eta(U) - \eta(\overline{U}) - \partial_U \eta(\overline{U}) * (U - \overline{U}) \\ &= \mathcal{H}(M(U)) - \mathcal{H}(M(\overline{U})) - \partial_f \mathcal{H}(M(\overline{U})) \cdot (M(U) - M(\overline{U})) \\ &= \mathcal{H}(M(U)|M(\overline{U}))\end{aligned}$$
using *iii*) with $w = U - \overline{U}$ and $f = M(U) - M(\overline{U})$.

v) For f such that $U = Pf$, we have
$$\begin{aligned}\eta(U|\overline{U}) &= \mathcal{H}(M(U)) - \mathcal{H}(M(\overline{U})) - \partial_f \mathcal{H}(M(\overline{U})) \cdot (M(U) - M(\overline{U})) \\ &\le \mathcal{H}(f) - \mathcal{H}(M(\overline{U})) - \partial_f \mathcal{H}(M(\overline{U})) \cdot (M(U) - M(\overline{U})) \\ &\le \mathcal{H}(f|M(\overline{U})) - \partial_f H(M(\overline{U})) \cdot (M(U) - f).\end{aligned}$$
Now
$$\partial_f H(M(\overline{U})) \cdot g = \sum_{i=1}^{N} (1 + \ln M_i(U)) g_i = \sum_{i=1}^{N} (1 + a + bv_i + cv_i^2) g_i = 0$$
whenever $Pg = 0$. Since $P(M(U) - f) = 0$, we conclude. □

We can now show the main proposition of this section.

Proposition 3.3. *The system (1.3) is hyperbolic, admissible (in the sense of Berthelin-Vasseur* [2]*), i.e. there exists* $C > 0$ *such that*
$$|A(U|\overline{U})| \le C\eta(U|\overline{U}) \qquad \text{for any } \rho > 0, (u, E) \in \mathcal{U},$$
η *is a convex entropy and*
$$\eta(U|\overline{U}) = \mathcal{H}(M(U)|M(\overline{U})) = s(\rho|\overline{\rho}) + \rho \ln \psi((\overline{b}, \overline{c})|(b, c))$$
for any $U, \overline{U}$ *with* ρ, u, E *and* a, b, c *related as in Lemma 2.1. Finally, there exists a constant* $C > 0$ *such that*
$$|u| + |E| \le C \qquad \text{for any } (u, E) \in \mathcal{U}.$$

Proof. Let us first check that η is an entropy of the limit system with entropy flux $\sum_{i=1}^N v_i M_i(U)\ln M_i(U)$. Indeed

$$\begin{aligned}
&\partial_t \sum_{i=1}^N M_i \ln M_i + \partial_x \sum_{i=1}^N v_i M_i \ln M_i \\
= &\sum_{i=1}^N (1+\ln M_i)(\partial_t M_i + v_i \partial_x M_i) \\
= &\sum_{i=1}^N (1+a+bv_i+cv_i^2)(\partial_t M_i + v_i \partial_x M_i) \\
= &(1+a)\left(\partial_t \sum_{i=1}^N M_i + \partial_x \sum_{i=1}^N v_i M_i\right) + b\left(\partial_t \sum_{i=1}^N v_i M_i + \partial_x \sum_{i=1}^N v_i^2 M_i\right) \\
&+c\left(\partial_t \sum_{i=1}^N v_i^2 M_i + \partial_x \sum_{i=1}^N v_i^3 M_i\right) \\
= &\ 0.
\end{aligned}$$

Let us now calculate $\mathcal{H}(M|\overline{M})$ for two Maxwellians $M, \overline{M}$. We set

$$\begin{aligned}
\ln M_i &= a + bv_i + cv_i^2 \\
\ln \overline{M}_i &= \overline{a} + \overline{b}v_i + \overline{c}v_i^2.
\end{aligned}$$

Then,

$$\begin{aligned}
\mathcal{H}(M|\overline{M}) &= \mathcal{H}(M) - \mathcal{H}(\overline{M}) - \partial_f \mathcal{H}(\overline{M})\cdot(M-\overline{M}) \\
&= \sum_{i=1}^N M_i \ln M_i - \sum_{i=1}^N \overline{M_i}\ln \overline{M}_i - \sum_{i=1}^N (1+\ln \overline{M}_i)\cdot(M_i - \overline{M}_i) \\
&= \sum_{i=1}^N M_i(\ln M_i - \ln \overline{M}_i) - \sum_{i=1}^N (M_i - \overline{M}_i) \\
&= (a-\overline{a})\sum_{i=1}^N M_i + (b-\overline{b})\sum_{i=1}^N v_i M_i + (c-\overline{c})\sum_{i=1}^N v_i^2 M_i - \sum_{i=1}^N (M_i - \overline{M}_i) \\
&= (a-\overline{a})\rho + (b-\overline{b})\rho u + (c-\overline{c})\rho E - (\rho - \overline{\rho}) \\
&= \rho(\ln(\rho/\psi) - \ln(\overline{\rho}/\overline{\psi})) + (b-\overline{b})\rho u + (c-\overline{c})\rho E - (\rho-\overline{\rho}) \\
&= s(\rho|\overline{\rho}) + \rho\left[\ln\overline{\psi} - \ln\psi - \partial_b(\ln\psi)(b-\overline{b}) - \partial_c(\ln\psi)(c-\overline{c})\right] \\
&= s(\rho|\overline{\rho}) + \rho(\ln\psi)((\overline{b},\overline{c})|(b,c)).
\end{aligned}$$

The function s and $(-\ln\psi)$ are convex, and thus, thanks to Lemma 3.1,

$$\mathcal{H}(M|\overline{M}) \geq 0 \qquad \text{for any } M, \overline{M}.$$

Lemma 3.2 gives that the relative entropy of η is nonnegative, and thanks to Lemma 3.1 again, we conclude that η is convex. Hence, the limit system is hyperbolic.

Note that for any $(u,E)\in\mathcal{U}$, since

$$u=\frac{\sum_{i=1}^N v_i M_i}{\sum_{i=1}^N M_i},\qquad E=\frac{\sum_{i=1}^N v_i^2 M_i}{\sum_{i=1}^N M_i},$$

we have

$$|u|\le\sup_{i=1,\cdots,N}|v_i|,\qquad |E|\le\sup_{i=1,\cdots,N}|v_i^2|.$$

Hence $\mathcal{U}$ is bounded in $\mathbb{R}^2$. Let us write the limit system as

$$\partial_t U+\partial_x A(U)=0,$$

where

$$A(\rho,\rho u,\rho E)=(\rho u,\rho E,\rho J(u,E)).$$

First note that the two first component of A are linear in U, so the associated relative quantity are 0. For the third one we calculate:

$$A_3(U|\overline{U})=\rho J((u,E)|(\overline{u},\overline{E})).$$

Thanks to the Taylor expansion, since $J\in C^2$ and $\mathcal{U}$ is bounded, there exists a constant $C>0$ such that for any $(u,E)\in\mathcal{U}$, we have

$$J((u,E)|(\overline{u},\overline{E}))\le C(|u-\overline{u}|^2+|E-\overline{E}|^2).$$

We also have

$$\eta(U|\overline{U})\ge\rho(\ln\psi)((\overline{b},\overline{c})|(b,c))\ge c\rho(|u-\overline{u}|^2+|E-\overline{E}|^2),$$

with $c>0$ thanks to the strict convexity of $\ln\psi$ and the boundedness of $\mathcal{U}$. Hence

$$|A(U|\overline{U})|\le\frac{C}{c}\eta(U|\overline{U})\qquad\text{for any }\rho>0,(u,E)\in\mathcal{U},$$

which means that the system is admissible. □

4. Estimation of the dissipation

This section is dedicated to the estimation of the dissipation

$$D(f)=\frac{1}{4}\sum_{ijkl}S_{ijkl}\ln\left(\frac{f_kf_l}{f_if_j}\right)(f_kf_l-f_if_j)\ge 0 \tag{4.13}$$

via the proposition:

Proposition 4.1. *There exists a constant C such that for any $f\in\mathbb{R}^N$ we have*

$$\sum_{i=1}^N|f_i-M_i|\le C\sqrt{D(f)}, \tag{4.14}$$

where $M=M(Pf)$ is the associated Maxwellian.

We first prove three lemmas.

Lemma 4.2. *Let $0 < \alpha < \beta$. For any $f \in \mathbb{R}^N$, we set $\rho = \sum_{i=1}^N f_i$, and $M = M(Pf)$. There exists $C_{\alpha\beta}$ such that for any $f \in \mathbb{R}^N$, if $0 < \alpha\rho \le f_i \le \beta\rho$ for any i, then*

$$D(f) \ge C_{\alpha\beta} \sum_{i=1}^N |f_i - M_i|^2. \tag{4.15}$$

Proof. Since $D(f/\rho) = D(f)/\rho^2$ and $\sum_{i=1}^N \left| \frac{f_i}{\rho} - M_i(P\left(\frac{f}{\rho}\right)) \right|^2 = \sum_{i=1}^N \left| \frac{f_i}{\rho} - \frac{M_i(Pf)}{\rho} \right|^2 = \frac{1}{\rho^2} \sum_{i=1}^N |f_i - M_i|^2$, we can assume that $\rho = 1$.
From $\left|\ln \frac{A}{B}\right| \le \max\left(\frac{1}{A}, \frac{1}{B}\right) |A - B|$ with $A = f_i f_j$ and $B = f_k f_l$, we get $D(f) \ge \overline{D}(f)$ with

$$\overline{D}(f) = \frac{\alpha^2}{4} \sum_{ijkl} S_{ijkl} \left(\ln f_k + \ln f_l - \ln f_i - \ln f_j\right)^2.$$

Since the kernel of $\overline{D}(f)$ is $V = \text{vect}\left((1, \cdots, 1), (v_1, \cdots, v_N), (v_1^2, \cdots, v_N^2)\right)$ from property (H), denoting by $\mathbb{P}$ the linear projection from $\mathbb{R}^N$ onto V, there exists C such that

$$\overline{D}(f) \ge C \sum_i |\ln f_i - \mathbb{P}(\ln f_i)|^2.$$

Now, since $\exp(\mathbb{P}(\ln f)) = M(\exp(\mathbb{P}(\ln f)))$, we have

$$\begin{aligned} f - M(f) &= \exp(\ln f) - \exp(\mathbb{P}(\ln f)) + M(\exp(\mathbb{P}(\ln f))) - M(\exp(\ln f)) \\ &= (\text{Id} - M) \circ \exp(\ln f) - (\text{Id} - M) \circ \exp(\mathbb{P} \ln f). \end{aligned}$$

Using that exp is lipschitz on every $]-\infty, R]$ and that $\mathbb{P} \ln f$ do not goes to $-\infty$, there exists $K_{\alpha\beta} > 0$ such that $(\text{Id} - M) \circ \exp$ is lipschitz on $\ln[\alpha, \beta]$ and on $\ln(\mathbb{P}[\alpha, \beta])$. Thus

$$|f_i - M_i(f)| \le K_{\alpha\beta} |\ln f_i - \mathbb{P} \ln f_i|$$

and therefore

$$\overline{D}(f) \ge \frac{C}{K_{\alpha\beta}^2} \sum_i |f_i - M_i|^2. \ \square$$

Lemma 4.3. *There exists γ_1, C_1 such that for any $f \in \mathbb{R}^N$, setting $\rho = \sum_{i=1}^N f_i$, if there exists i_0 such that $f_{i_0} \le \gamma_1 \rho$, then*

$$D(f) \ge C_1 \rho^2. \tag{4.16}$$

Proof. Since $D(f/\rho) = D(f)/\rho^2$, we may assume with no loss of generality that $\rho = 1$.
The proof proceeds by contradiction. Let us assume that for any γ, C, there exists f and i_0 such that $f_{i_0} \le \gamma\rho$ and $D(f) \le C$. From Proposition 3.3, $\mathcal{U}$ is bounded, that is to say $(u = \sum_{i=1}^N v_i f_i, E = \sum_{i=1}^N |v_i|^2 f_i)$ is bounded. Thus there exists γ such that $0 < \gamma < M_i$ for any i.
With this γ, for any $n \in \mathbb{N}^*$, taking $C = 1/n$, there exists f^n and $i_0(n)$ such that

$f^n_{i_0(n)} \leq \gamma$ and $D(f^n) \leq 1/n$. Since $i_0(n)$ takes finitely many values, we can extract a subsequence such that $i_0(n)$ remains constant. For this index i_0, we have for a subsequence $f^n_{i_0} \to f_{i_0} \in [0, \gamma]$, and extracting successively further subsequences, $f^n_j \to f_j \in [0, 1]$ for all other j. Now $D(f^n) \to 0$ gives $D(f) = 0$, and Lemma 2.1 implies that $f = M$ and then $\gamma < M_{i_0} = f_{i_0}$ which is a contradiction. □

By similar arguments, we also prove that

Lemma 4.4. *There exists γ_2, C_2 such that for any $f \in \mathbb{R}^N$, setting $\rho = \sum_{i=1}^{N} f_i$, if there exists i_0 such that $f_{i_0} \geq \gamma_2 \rho$, then*

$$D(f) \geq C_2 \rho^2. \tag{4.17}$$

Based on these three properties, we can now show the Proposition 4.1.

Proof of Proposition 4.1. Let $\varepsilon > 0$ and set $I = \sum_{i=1}^N |f_i - M_i|$. If $\rho < \varepsilon$ then $I \leq 2\varepsilon$.

For $\rho \geq \varepsilon$, we select γ_1, γ_2 as in Lemmas 4.3 and 4.4 and distinguish three possibilities: either (i) $\gamma_1 \rho < f_i < \gamma_2 \rho$ for all indices i, or (ii) there exists i_0 so that $f_{i_0} > \gamma_2 \rho$, or finally (iii) there is i_0 such that $f_{i_0} < \gamma_1 \rho$. In each case I is estimated as follows:

$$\begin{aligned}
\sum_{i=1}^{N} |f_i - M_i| &\leq \sum_{i=1}^{N} |f_i - M_i| \mathbb{1}_{\rho \leq \varepsilon} + \sum_{i=1}^{N} |f_i - M_i| \mathbb{1}_{\rho \geq \varepsilon} \\
&\leq 2\varepsilon + \sum_{i=1}^{N} |f_i - M_i| \mathbb{1}_{\exists i_0; f_{i_0} \leq \gamma_1 \rho} \mathbb{1}_{\rho \geq \varepsilon} \\
&\quad + \sum_{i=1}^{N} |f_i - M_i| \mathbb{1}_{\exists i_0; f_{i_0} \geq \gamma_2 \rho} \mathbb{1}_{\rho \geq \varepsilon} \\
&\quad + \sum_{i=1}^{N} |f_i - M_i| \mathbb{1}_{\forall i; \gamma_1 \rho \leq f_i \leq \gamma_2 \rho} \mathbb{1}_{\rho \geq \varepsilon} \\
&\leq 2\varepsilon + 2\rho \mathbb{1}_{\exists i_0; f_{i_0} \leq \gamma_1 \rho} + 2\rho \mathbb{1}_{\exists i_0; f_{i_0} \geq \gamma_2 \rho} \\
&\quad + \sqrt{\sum_i |f_i - M_i|^2 \mathbb{1}_{\forall i; \gamma_1 \rho \leq f_i \leq \gamma_2 \rho} \mathbb{1}_{\rho \geq \varepsilon}} \sqrt{N} \\
&\leq 2\varepsilon + 2\sqrt{\frac{D(f)}{C_1}} + 2\sqrt{\frac{D(f)}{C_2}} + \sqrt{\frac{N D(f)}{C_{\gamma_1 \gamma_2}}}
\end{aligned}$$

Finally, we take $\varepsilon \to 0$. □

5. Hydrodynamic limit

In this section, we prove Theorem 2.3. We denote by f_ε the solution of (1.1), by $U_\varepsilon = (\rho_\varepsilon, \rho_\varepsilon u_\varepsilon, \rho_\varepsilon E_\varepsilon) = P f_\varepsilon = \sum_{i=1}^{N} (1, v_i, v_i^2)(f_\varepsilon)_i$, by $M_\varepsilon = M(U_\varepsilon)$, by $\overline{U}$ the smooth solution to the limit system and by $\overline{M} = \overline{M}(\overline{U})$ the associated Maxwellian.

Multiplying (1.1) by $\ln(f_\varepsilon)_i$ and summing in i gives

$$\partial_t \sum_{i=1}^N (f_\varepsilon)_i \ln(f_\varepsilon)_i + \partial_x \sum_{i=1}^N v_i (f_\varepsilon)_i \ln(f_\varepsilon)_i + \frac{D(f_\varepsilon)}{\varepsilon} = 0. \tag{5.18}$$

Thanks to Proposition 3.3, we have:

$$\partial_t \sum_{i=1}^N \overline{M}_i \ln \overline{M}_i + \partial_x \sum_{i=1}^N v_i \overline{M}_i \ln \overline{M}_i = 0. \tag{5.19}$$

We can now study the evolution of the relative entropy between f_ε and $\overline{M}$:

$$\begin{aligned}
& \partial_t \mathcal{H}(f_\varepsilon|\overline{M}) + \partial_x \sum_{i=1}^N v_i s((f_\varepsilon)_i|\overline{M}_i) \\
= \; & \partial_t \sum_{i=1}^N s((f_\varepsilon)_i|\overline{M}_i) + \partial_x \sum_{i=1}^N v_i s((f_\varepsilon)_i|\overline{M}_i) \\
= \; & \partial_t \sum_{i=1}^N (f_\varepsilon)_i \ln(f_\varepsilon)_i + \partial_x \sum_{i=1}^N v_i (f_\varepsilon)_i \ln(f_\varepsilon)_i \\
& -\partial_t \sum_{i=1}^N \overline{M}_i \ln \overline{M}_i - \partial_x \sum_{i=1}^N v_i \overline{M}_i \ln \overline{M}_i \\
& -\partial_t \sum_{i=1}^N (1 + \ln \overline{M}_i)((f_\varepsilon)_i - \overline{M}_i) - \partial_x \sum_{i=1}^N v_i (1 + \ln \overline{M}_i)((f_\varepsilon)_i - \overline{M}_i).
\end{aligned}$$

Since

$$\sum_{i=1}^N (1 + \ln \overline{M}_i)((f_\varepsilon)_i - \overline{M}_i) = \partial_f \mathcal{H}(\overline{M}) \cdot (f_\varepsilon - \overline{M}),$$

and using the notation $V : \mathbb{R}^N \to \mathbb{R}^N$ defined by:

$$V f_i = v_i f_i \qquad 1 \le i \le N,$$

we also have

$$\sum_{i=1}^N v_i (1 + \ln \overline{M}_i)((f_\varepsilon)_i - \overline{M}_i) = \partial_f \mathcal{H}(\overline{M}) \cdot (V f_\varepsilon - V\overline{M}).$$

Combining this with (5.18) and (5.19) , we get

$$\begin{aligned}
& \partial_t \mathcal{H}(f_\varepsilon|\overline{M}) + \partial_x \sum_{i=1}^N v_i s((f_\varepsilon)_i|\overline{M}_i) + \frac{D(f_\varepsilon)}{\varepsilon} \\
& \qquad = -\partial_t \Big(\partial_f \mathcal{H}(\overline{M}) \cdot (f_\varepsilon - \overline{M}) \Big) - \partial_x \Big(\partial_f \mathcal{H}(\overline{M}) \cdot (V f_\varepsilon - V\overline{M}) \Big).
\end{aligned}$$

Using Lemma 3.2, we get

$$\begin{aligned}(5.20)\qquad &\partial_t\mathcal{H}(f_\varepsilon|\overline{M})+\partial_x\sum_{i=1}^N v_i s((f_\varepsilon)_i|\overline{M}_i)+\frac{D(f_\varepsilon)}{\varepsilon}\\ &=-\partial_t\Big(\partial_U\eta(\overline{U})*P(f_\varepsilon-\overline{M})\Big)-\partial_x\Big(\partial_U\eta(\overline{U})*P(Vf_\varepsilon-V\overline{M})\Big)\\ &=-\partial_t(\partial_U\eta(\overline{U}))*P(f_\varepsilon-\overline{M})-\partial_U\eta(\overline{U})*\partial_t(P(f_\varepsilon-\overline{M}))\\ &\quad-\partial_x(\partial_U\eta(\overline{U}))*P(Vf_\varepsilon-V\overline{M})-\partial_U\eta(\overline{U})*\partial_x(P(Vf_\varepsilon-V\overline{M})).\end{aligned}$$

For $k=0,1,2$, multiplying (1.1) by v_i^k, summing over i and using (2.7), we have

$$\partial_t\sum_{i=1}^N v_i^k(f_\varepsilon)_i+\partial_x\sum_{i=1}^N v_i^{k+1}(f_\varepsilon)_i=0,$$

that is to say

$$\partial_t Pf_\varepsilon+\partial_x P(Vf_\varepsilon)=0. \tag{5.21}$$

Furthermore,

$$\partial_t\sum_{i=1}^N v_i^k\overline{M}_i+\partial_x\sum_{i=1}^N v_i^{k+1}\overline{M}_i=0,$$

that is to say

$$\partial_t P(\overline{M})+\partial_x P(V\overline{M})=0. \tag{5.22}$$

It gives

$$\begin{aligned}&\partial_t\mathcal{H}(f_\varepsilon|\overline{M})+\partial_x\sum_{i=1}^N v_i s((f_\varepsilon)_i|\overline{M}_i)+\frac{D(f_\varepsilon)}{\varepsilon}\\ &\quad=-\partial^2_{UU}\eta(\overline{U})\partial_t(\overline{U})*P(f_\varepsilon-\overline{M})-\partial^2_{UU}\eta(\overline{U})\partial_x(\overline{U})*P(Vf_\varepsilon-V\overline{M})\\ &\quad=\partial^2_{UU}\eta(\overline{U})A'(\overline{U})\partial_x(\overline{U})*P(f_\varepsilon-\overline{M})-\partial^2_{UU}\eta(\overline{U})\partial_x(\overline{U})*P(Vf_\varepsilon-V\overline{M})\\ &\quad=\partial^2_{UU}\eta(\overline{U})\partial_x(\overline{U})*\Big(A'(\overline{U})(U_\varepsilon-\overline{U})-P(Vf_\varepsilon-V\overline{M})\Big)\\ &\quad=\partial^2_{UU}\eta(\overline{U})\partial_x(\overline{U})*\Big(A'(\overline{U})(U_\varepsilon-\overline{U})-P(Vf_\varepsilon-VM_\varepsilon)-P(VM_\varepsilon-V\overline{M})\Big),\end{aligned}$$

where we used the fact that, since $\eta(U)$ is an entropy for (2.10), the flux $A(U)$ satisfies $(\partial_{uu}\eta)A'=(A')^T\partial_{uu}\eta$. Now

$$P(VM_\varepsilon-V\overline{M})=\sum_{i=1}^N(1,v_i,v_i^2)v_i((M_\varepsilon)_i-\overline{M}_i)=A(U_\varepsilon)-A(\overline{M}),$$

therefore

$$\begin{aligned}(5.23)\qquad &\partial_t\mathcal{H}(f_\varepsilon|\overline{M})+\partial_x\sum_{i=1}^N v_i s((f_\varepsilon)_i|\overline{M}_i)+\frac{D(f_\varepsilon)}{\varepsilon}\\ &\qquad=-\partial^2_{UU}\eta(\overline{U})\partial_x(\overline{U})*\Big(A(U_\varepsilon|\overline{U})+P(Vf_\varepsilon-VM_\varepsilon)\Big).\end{aligned}$$

We exploit this evolution equation in order to get the bound. First we want to bound $D(f_\varepsilon)$ with respect to ε. Integrating (5.20) with respect to (t,x) gives

$$\int_{\mathbb{R}} \mathcal{H}(f_\varepsilon|\overline{M})(t,x)\,dx - \int_{\mathbb{R}} \mathcal{H}(f_\varepsilon^0|\overline{M}^0)(x)\,dx + \int_0^t \int_{\mathbb{R}} \frac{D(f_\varepsilon)}{\varepsilon}\,dx\,ds$$
$$= -\int_{\mathbb{R}} \partial_U \eta(\overline{U}) * P(f_\varepsilon - \overline{M})\,dx + \int_{\mathbb{R}} \partial_U \eta(\overline{U}) * P(f_\varepsilon^0 - \overline{M^0})\,dx.$$

For every $T < T^*$, there exists C_T such that $|\partial_U \eta(\overline{U})|(t,x) \leq C_T$ for any $x \in \mathbb{R}$, $0 \leq t \leq T$. Thus we have

$$\int_0^t \int_{\mathbb{R}} \frac{D(f_\varepsilon)}{\varepsilon}\,dx\,ds \leq \int_{\mathbb{R}} \mathcal{H}(f_\varepsilon^0|\overline{M}^0)(x)\,dx + C_T \int_{\mathbb{R}} |P(f_\varepsilon - \overline{M})| + |P(f_\varepsilon^0 - \overline{M^0})|\,dx.$$

Integrating (5.21) and (5.22) with respect to (t,x) gives in particular

$$\int_{\mathbb{R}} |f_\varepsilon(t,x)|\,dx = \int_{\mathbb{R}} f_\varepsilon(t,x)\,dx = \int_{\mathbb{R}} f_\varepsilon^0(x)\,dx,$$

and

$$\int_{\mathbb{R}} |\overline{M}(t,x)|\,dx = \int_{\mathbb{R}} \overline{M}(t,x)\,dx = \int_{\mathbb{R}} \overline{M}^0(x)\,dx.$$

Thus

$$\int_{\mathbb{R}} |P(f_\varepsilon - \overline{M})|\,dx.$$
$$\leq (1 + \sup_{i=1,\cdots,N} |v_i| + \sup_{i=1,\cdots,N} |v_i^2|) \left(\int_{\mathbb{R}} f_\varepsilon^0(x)\,dx + \int_{\mathbb{R}} \overline{M}^0(x)\,dx \right),$$

and

$$\int_0^t \int_{\mathbb{R}} D(f_\varepsilon)\,dx\,ds \leq C_T^0 \varepsilon, \qquad \text{for } 0 \leq t \leq T, \tag{5.24}$$

with

$$C_T^0 = \sup_\varepsilon \Big(\int_{\mathbb{R}} \mathcal{H}(f_\varepsilon^0|\overline{M}^0)(x)\,dx \Big)$$
$$+ 4C_T \max(1, \sup_{i=1,\cdots,N} |v_i^2|) \sup_\varepsilon \Big(\int_{\mathbb{R}} f_\varepsilon^0(x)\,dx + \int_{\mathbb{R}} \overline{M}^0(x)\,dx \Big).$$

We turn now to the estimation of $\mathcal{H}(f_\varepsilon|\overline{M})$ with respect to ε. For every $T < T^*$, there exists $\tilde{C}_T$ such that

$$|\partial_{UU}^2 \eta(\overline{U})|(t,x) \leq \tilde{C}_T, \qquad |\partial_x \overline{U}|(t,x) \leq \tilde{C}_T,$$

for any $x \in \mathbb{R}$, $0 \leq t \leq T$ and

$$\int_0^T \int_{\mathbb{R}} |\partial_x \overline{U}|^2(s,x)\,dx\,ds \leq \tilde{C}_T. \tag{5.25}$$

Then integrating (5.23) with respect to (t,x) gives

$$\int_{\mathbb{R}} \mathcal{H}(f_\varepsilon|\overline{M})(t,x)\,dx - \int_{\mathbb{R}} \mathcal{H}(f_\varepsilon^0|\overline{M}^0)(x)\,dx + \int_0^t \int_{\mathbb{R}} \frac{D(f_\varepsilon)}{\varepsilon}\,dx\,ds$$
$$= -\int_0^t \int_{\mathbb{R}} \partial_{UU}^2 \eta(\overline{U}) \partial_x(\overline{U}) * \Big(A(U_\varepsilon|\overline{U}) + P(V f_\varepsilon - V M_\varepsilon) \Big)\,dx\,ds.$$

Thanks to Proposition 3.3 and Lemma 3.2, we get

$$|A(U_\varepsilon|\overline{U})| \le C_1\eta(U_\varepsilon|\overline{U}) \le C_1\mathcal{H}(f_\varepsilon|\overline{M}).$$

Thanks to Proposition 4.1, we get

$$|PV(f_\varepsilon - M_\varepsilon)|^2 \le C_2\left(\sum_{i=1}^N |(f_\varepsilon)_i - (M_\varepsilon)_i|\right)^2 = C_2|f_\varepsilon - M_\varepsilon|^2 \le C_3D(f_\varepsilon).$$

Thus, it gives

$$\begin{aligned}
&\int_{\mathbb{R}} \mathcal{H}(f_\varepsilon|\overline{M})(t,x)\,dx \\
&\le \int_{\mathbb{R}} \mathcal{H}(f_\varepsilon^0|\overline{M}^0)(x)\,dx + (\tilde{C}_T)^2C_1\int_0^t\int_{\mathbb{R}} \mathcal{H}(f_\varepsilon|\overline{M})(s,x)\,dx\,ds \\
&+\left(\int_0^t\int_{\mathbb{R}} |\partial^2_{UU}\eta(\overline{U})\partial_x(\overline{U})|^2\,dx\,ds\right)^{1/2}\left(\int_0^t\int_{\mathbb{R}} |PV(f_\varepsilon - M_\varepsilon)|^2\,dx\,ds\right)^{1/2} \\
&\le \int_{\mathbb{R}} \mathcal{H}(f_\varepsilon^0|\overline{M}^0)(x)\,dx + \tilde{C}_T^2C_1\int_0^t\int_{\mathbb{R}} \mathcal{H}(f_\varepsilon|\overline{M})(s,x)\,dx\,ds \\
&+\tilde{C}_T^{3/2}C_3^{1/2}\left(\int_0^t\int_{\mathbb{R}} D(f_\varepsilon)(s,x)\,dx\,ds\right)^{1/2} \\
&\le \int_{\mathbb{R}} \mathcal{H}(f_\varepsilon^0|\overline{M}^0)(x)\,dx + \tilde{C}_T^2C_1\int_0^t\int_{\mathbb{R}} \mathcal{H}(f_\varepsilon|\overline{M})(s,x)\,dx\,ds + \tilde{C}_T^{3/2}C_3^{1/2}\sqrt{C_T^0\varepsilon}
\end{aligned}$$

using (5.24). Setting $w_\varepsilon(t) = \int_{\mathbb{R}} \mathcal{H}(f_\varepsilon|\overline{M})(t,x)\,dx$, it writes

$$w_\varepsilon(t) \le w_\varepsilon(0) + C_4\int_0^t w_\varepsilon(s)\,ds + C_5\sqrt{\varepsilon}.$$

Using Gronwall's lemma, we get

$$\sup_{0\le t\le T}\int_{\mathbb{R}} \mathcal{H}(f_\varepsilon|\overline{M})(t,x)\,dx \le \left(\int_{\mathbb{R}} \mathcal{H}(f_\varepsilon^0|\overline{M}^0)\,dx + C_5\sqrt{\varepsilon}\right)e^{C_4T}. \quad \square$$

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