Program of the Cours Poupaud

Four lectures on "ODEs with rough coefficients and hyperbolic systems of conservation laws"

One lecture on "Rigidity estimates for almost umbilical surfaces"

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Wednesday Afternoon
Lecture 1. Three questions of Bressan on ODEs and hyperbolic systems of conservation laws

Consider the following hyperbolic system of conservation laws

$$\begin{cases} \partial_t u^i + \sum_{\alpha=1}^{m} \partial_{x_\alpha}(g^\alpha(|u|)u^i) \;=\; 0 \\ \\ u^i(0,\cdot) \;=\; \overline{u}^i(\cdot) \end{cases} \qquad (1)$$

where $u = (u^1, \ldots, u^k) : \mathbf{R}^+_t \times \mathbf{R}^m_x \to \mathbf{R}^k$ is the unknown vector map, $\overline{u} = (\overline{u}^1, \ldots, \overline{u}^k)$ the initial data, and $g^\alpha : \mathbf{R} \to \mathbf{R}$ are given (sufficiently smooth) scalar functions. Besides arising in some physical models, (1) (called Keyfitz and Kranzer system) is an example of multidimensional system of conservation laws which has many features, but which still (generically) develops singularities in finite time

Motivated by the study of system (1) and, more in general, of hyperbolic systems of conservation laws in more than one space dimension, Alberto Bressan raised three interesting questions on ODEs with rough coefficients.

In this lecture we will survey on recent results on global well–posedness and ill–posedness of the Cauchy problem (1) and on related progress about Bressan's questions.

Thursday Morning

Lecture 2. The Ambrosio Renormalization Theorem and the Keyfitz and Kranzer system

In a recent groundbreaking paper, Ambrosio succeeded in extending the DiPerna–Lions theory of transport equations with Sobolev coefficients to BV coefficients whose divergence is in L^1. The core of his result is a new Renormalization Theorem.

In this lecture we will state this theorem and give some ideas about its proof. Then we will apply it to show that, for a suitable notion of solution, the Keyfitz and Kranzer system enjoys global well–posedness in the space $X = \{u \in L^\infty(\mathbf{R}^m, \mathbf{R}^k) : |u| \in BV_{loc}\}$ (a result joint with Luigi Ambrosio and François Bouchut).

Then we will address the question whether it is possible to find well–posedness for the Keyfitz and Kranzer system in more "classical" function spaces. In particular:

- We will show examples in which (1) is ill–posed in the L^∞ setting (a result of Alberto Bressan);
- We will explain why in 3 space dimensions (1) is well–posed in BV only when g is a constant, i.e. when the system reduces to k decoupled transport equations with constant coefficients (a result of mine).

THURSDAY AFTERNOON

Lecture 3. Rigidity estimates for almost umbilical surfaces

Consider the following classical rigidity theorem:

Theorem 1. *Let $\Sigma \subset \mathbf{R}^3$ be a smooth compact connected surface of $\mathbf{R}^3$ without boundary. If its principal curvatures coincide at every point, then Σ is a round sphere.*

Denote by A the second fundamental of Σ (considered as a $(1,1)$ tensor) and by $\mathring{A}$ the tensor $A - \frac{\operatorname{tr} A}{2}\mathrm{Id}$. Then Theorem 1 can be restated as: If $\mathring{A} = 0$, then Σ is a round sphere, and therefore $A = \lambda \mathrm{Id}$ for some constant λ. Motivated by problems arising in General Relativity and Geometric Flows, Huisken asked whether inequality (2) below holds. In this lecture we will present the following result, contained in two joint works with Stefan Müller:

Theorem 2. *There exists a universal constant C such that*

$$\min_{\lambda} \|A - \lambda \mathrm{Id}\|_{L^2(\Sigma)} \;\leq\; C\|\mathring{A}\|_{L^2(\Sigma)} \tag{2}$$

for every smooth surface $\Sigma \subset \mathbf{R}^3$ connected, compact, and without boundary. If, in addition, $\|\mathring{A}\|^2_{L^2(\Sigma)} \leq 4\pi$ and $\mathrm{ar}(\Sigma) = 4\pi$, then there exists a conformal parameterization $\psi : \mathbf{S}^2 \to \Sigma$ and a vector $c_\Sigma \in \mathbf{R}^3$ such that

$$\|\psi - (c_\Sigma + \mathrm{id})\|_{W^{2,2}(\mathbf{S}^2)} \;\leq\; C\|\mathring{A}\|_{L^2(\Sigma)}\,, \tag{3}$$

where $\mathrm{id} : \mathbf{S}^2 \to \mathbf{R}^3$ denotes the standard isometric embedding.

Moreover, if we denote by h the conformal factor of the metric on Σ (that is, $h > 0$ and $\psi_\# \sigma = h^2\sigma$, where σ is the standard metric on $\mathbf{S}^2$), then $\|h - 1\|_{C^0(\Sigma)} \leq C\|\mathring{A}\|_{L^2(\Sigma)}$.

Inequality (2) is scaling invariant and one can check immediately that, if we replace the exponent 2 with a different one, this invariance is lost. However 2 is sharp for a deeper reason: Indeed one can show that for every $p < 2$, the corresponding inequality is false even at a fixed scale. In the lecture we will also mention recent related results of Metzger on foliations by mean curvature spheres and their applications to General Relativity.

Friday Morning

Lecture 4. Bressan's Compactness Conjecture

Motivated by the study of the Keyfitz and Kranzer system, Bressan raised the following compactness conjecture:

Conjecture 3 (Bressan's compactness conjecture)**.** *Let* $b_n : \mathbf{R}_t \times \mathbf{R}^m_x \to \mathbf{R}^m$*,* $n \in \mathbb{N}$*, be smooth maps and denote by* Φ_n *the solutions of the ODEs:*

$$\begin{cases} \dfrac{d}{dt}\Phi_n(t,x) \;=\; b_n(t,\Phi_n(t,x)) \\[2ex] \Phi_n(0,x) \;=\; x\,. \end{cases} \tag{4}$$

Assume that $\|b_n\|_\infty + \|\nabla b_n\|_{L^1}$ *is uniformly bounded and that the fluxes* Φ_n *are nearly incompressible, i.e. that*

$$C^{-1} \;\le\; \det(\nabla_x \Phi_n(t,x)) \;\le\; C \qquad \textit{for some constant } C > 0. \tag{5}$$

Then the sequence $\{\Phi_n\}$ *is strongly precompact in* L^1_{loc}*.*

Though we have been able to approach the Keyfitz and Kranzer system bypassing it, Conjecture 3 remains an interesting and challenging problem in the theory of ODEs with rough coefficients.

In this lecture we will present a Renormalization Conjecture advanced in a joint work with Luigi Ambrosio and François Bouchut, which would yield a robust renormalization theory for transport equations with nearly incompressible BV coefficients (and in particular would imply Conjecture 3).

Then we will present positive answers to some cases of this Renormalization Conjecture (joint work with Luigi Ambrosio and Jan Malý). These resulys build on a work of Luigi Ambrosio, Gianluca Crippa and Stefania Maniglia on trace properties for solutions to transport equations.

Friday Afternoon

Lecture 5. A direct lagrangian approach to ODEs with Sobolev coefficients and Bressan's closure problem

Motivated by going beyond the Keyfitz and Kranzer system, in this lecture we will address two further questions of Bressan. The first one concerns the possibility of recovering the conclusions of the DiPerna–Lions theory for ODEs with Sobolev coefficients in a direct "lagrangian" way, showing estimates in appropriate function spaces. This is indeed possible for coefficients in $W^{1,p}$ with $p > 1$, as we show in a forthcoming joint work with Gianluca Crippa. We will discuss the basic idea behind these new lagrangian estimates for ODEs, inspired by a paper of Luigi Ambrosio, Miriam Lécumberry, and Stefania Maniglia on differentiability properties of the DiPerna–Lions flow.

The second question concerns the possibility of finding an appropriate function space, containing the space of BV functions, and where one can solve recursively transport equations. In another joint work with Gianluca Crippa we answer negatively to this question, building on an elegant example of DePauw.

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