From Isometric Embeddings to Turbulence

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Programme for the Cours Poupaud
15-17 March 2010, Nice

Monday Morning
Lecture 1. The Nash-Kuiper Theorem

In 1954 J.Nash shocked the world of differential geometry with the following theorem [12]:

Theorem 1. *Given a closed Riemannian n-manifold M^n, any smooth, strictly short immersion (embedding) $M^n \hookrightarrow \mathbb{R}^{n+2}$ can be uniformly approximated by C^1 isometric immersions (embeddings).*

A year later N.Kuiper [10] modified the arguments of Nash to extend to the case $M^n \hookrightarrow \mathbb{R}^{n+1}$. Thus, in particular, it is possible to "crumple" the 2-sphere $S^2 \subset \mathbb{R}^3$ into an arbitrarily small volume **in a** C^1 **way**!. The reason this was (and still is) a shock, is that a classical theorem states that the **only** C^2 embedding of the sphere is the standard embedding, modulo rigid motion.

In the lecture I will give a proof of the Nash-Kuiper theorem and indicate how the rigidity for the sphere in C^2 arises.

Monday Afternoon

Lecture 2. The Baire Category Method

In some sense the Nash-Kuiper construction relies on a local perturbation technique. A simple model for this is the construction of maps $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ with

$$\nabla u^T \nabla u = I.$$

If u is C^1, it is not difficult to see that any solution has to be affine. On the other hand Lipschitz solutions can be very wild. Indeed, it turns out that a generic short map, i.e. where $\nabla u^T \nabla u \leq I$ a.e., is a nowhere C^1 almost-everywhere solution of the system above. Moreover, the argument is the same as for showing that a generic function $u : [0,1] \to [-1,1]$ is everywhere discontinuous with $u(x) \in \{-1, 1\}$ a.e.

In this lecture I will show the Baire category method, and show how it applies to partial differential inclusions of the type

$$\nabla u \in K \text{ a.e.} \tag{1}$$

This will involve conditions on the rank-one convex hull, which I will introduce. This lecture will mostly be based on Chapter 3 of [9].

Tuesday Afternoon

Lecture 3. Laminates

Somewhat surprisingly, the framework introduced so far for solving differential inclusions applies to certain problems concerning elliptic equations and systems. The first example of this, the construction of irregular critical points to quasiconvex functionals, was given by Müller and Šverák in [11]. In this lecture I will prove the following two results, following [17, 2]. In what follows $\Omega \subset \mathbb{R}^2$ is a bounded domain.

Theorem 2. *There exists a smooth convex function* $f : \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathbb{R}$ *and* $\gamma > 0$ *such that the functional*

$$\int_\Omega \gamma|\nabla u|^2 + f(\nabla u, \det \nabla u)\, dx$$

possesses critical points $u : \Omega \to \mathbb{R}^2$ *which are Lipschitz, but nowhere* C^1.

Functionals of the form above are called strongly polyconvex. In contrast, minimizers of such functionals are regular outside a closed set of Lebesgue-measure zero.

Theorem 3. *For any* $\Lambda > 1$ *and any* $\alpha < 1$ *there exists* $\sigma \in L^\infty(\Omega)$ *with* $\sigma(x) \in \{\Lambda, \Lambda^{-1}\}$ *and a solution* $u \in C_0^\alpha(\overline{\Omega}) \cap W^{1,1}(\Omega)$ *to the equation*

$$div\; \sigma \nabla u = 0$$

such that $\nabla u \in L^{\frac{2\Lambda}{\Lambda+1},\infty}$ *(the Marcinkiewicz space).*

The interest in this theorem is the sharpness of the exponent: it follows from the theory of quasiregular maps [1, 14] that if instead $\nabla u \in L^{\frac{2\Lambda}{\Lambda+1}}$, then automatically $u \in W^{1,2}$ and hence must be identically zero.

Both proofs involve the method of convex integration and the concept of laminates.

Wednesday Morning

Lecture 4. Wild solutions for the Euler equations

The framework of solving differential inclusions with the Baire category method has an obvious extension to more general systems of the form

$$\begin{aligned} \sum_{i=1}^{N} A_i \partial_i z &= 0 \\ z(y) &\in K \text{ a.e.} \end{aligned}$$

where $z : \Omega \subset \mathbb{R}^N \to \mathbb{R}^d$ (see [18]). In particular, the method applies to the incompressible Euler equations. In this way one can recover the following theorem, originally due to Scheffer [15]:

Theorem 4. *There exist (non-trivial) weak solutions of the Euler equations which are compactly supported in space* **and time***.*

As is well known, for classical solutions the energy $E(t) = \frac{1}{2} \int |v|^2 \, dx$ is constant in time. Although a solution such as in the theorem is clearly unphysical, there is a certain amount of physical relevance of weak solutions of the Euler equations which **dissipate** energy. In fact, motivated by the phenomenon of anomalous dissipation, Onsager [13] conjectured that there could be weak solutions in the space C^α with $\alpha < 1/3$ for which the energy is strictly decreasing on some time interval. In joint work with Camillo De Lellis in [6, 7] we gave a construction of such dissipating solutions in L^∞, using the Baire category scheme. In particular, we obtain non-uniqueness for the initial value problem for dissipative solutions, even for entropy solutions. This lecture will be devoted to explaining the details of this construction.

Wednesday Afternoon

Lecture 5. Isometric Immersions revisited: rigidity vs flexibility

Coming back to isometric immersions, one might ask where is the sharp borderline between C^1 flexibility and C^2 rigidity. This question is still unanswered, but there are some partial results. In particular, the Nash-Kuiper construction can be extended to $C^{1,\alpha}$ immersions for some α. This is a result which was first announced by Borisov in [3]. Subsequently a rather intransparent proof of the 2-dimensional case with analytic metric appeared in [4]. Motivated mainly by the obvious connection to Onsager's conjecture, with Sergio Conti and Camillo De Lellis [5] we looked at this problem and gave a cleaner proof of the general case. In particular, we obtain (for immersions of a "single chart"):

Theorem 5. *Let $\Omega \subset \mathbb{R}^n$ be an open set with a Riemannian metric g. Then any strictly short immersion $u : (\Omega, g) \hookrightarrow \mathbb{R}^{n+1}$ can be uniformly approximated by $C^{1,\alpha}$ isometric immersions for $\alpha < 1/(1 + n + n^2)$.*

In this final lecture I will explain the main steps of the proof and its relation to Onsager's 1/3. If time permits, I will also talk about how - for the case of $S^2 \hookrightarrow \mathbb{R}^3$ - one can lower the regularity assumption from C^2 to $C^{1,2/3+}$ for rigidity.

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