

# High-Field Limit from a Stochastic BGK Model to a Scalar Conservation Law with Stochastic Forcing

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## Abstract

We study the derivation of a scalar conservation law with stochastic forcing starting from a stochastic BGK model with a high-field scaling. We prove the convergence to a new kinetic formulation where appears a modified Maxwellian. We deduce from it the existence of a weak solution to the scalar conservation law with stochastic forcing. We establish that this solution satisfies some Krushkov-like entropy relations.

**Keywords.** Stochastic conservation laws, high field limit, hydrodynamical limit, BGK model, kinetic formulation, modified Maxwellian, Krushkov-like entropies.

## 1 Introduction

In this paper, we are interested in the derivation of a solution to a scalar conservation law with stochastic forcing of the following form:

$$du + \operatorname{div}_x(B(x, u))dt = C(x, u)dW_t, \quad t \in (0, T), x \in \mathbf{T}^N, \quad (1.0.1)$$

from a stochastic BGK like model,

$$dF^\varepsilon + \operatorname{div}_x(a(x, \xi)F^\varepsilon)dt + \frac{\Lambda(x)}{\varepsilon}\partial_\xi F^\varepsilon dt = \frac{\mathbf{1}_{u_\varepsilon > \xi} - F^\varepsilon}{\varepsilon}dt - \partial_\xi F^\varepsilon \Phi dW + \frac{1}{2}\partial_\xi(G^2\partial_\xi F^\varepsilon)dt, \quad (1.0.2)$$

when  $\varepsilon$  goes to 0.

This type of problem belongs to the class of problems of hydrodynamical limits. The historical issue in that field goes back to the kinetic theory, first introduced by Maxwell and Boltzmann in order to model rarefied gas. Indeed, adopting a statistical point of view, a gas can be described by its density of particles  $f_\varepsilon$ , a function of time, position and velocity, satisfying at the mesoscopic level, the Boltzmann equation

$$\partial_t f_\varepsilon(t, x, \xi) + v \cdot \nabla_x f_\varepsilon(t, x, \xi) = \frac{Q(f_\varepsilon, f_\varepsilon)}{\varepsilon}, \quad t \in \mathbf{R}_t^+, x \in \mathbf{R}_x^d, \xi \in \mathbf{R}_\xi^n, \quad (1.0.3)$$

where  $Q(f_\varepsilon, f_\varepsilon)$  is a collision operator. Formally in that case, we obtain the following hydrodynamical limit:  $f_\varepsilon$  converges to a Maxwellian of parameter  $\rho(t, x)$ ,  $u(t, x)$  and  $T(t, x)$  such that those three functions satisfy the Euler system. In other words, starting from the Boltzmann equation at the mesoscopic level, we expect to obtain the Euler system at the macroscopic level. However, this derivation is actually still an open question and motivated the introduction of the BGK model in [4], though partial results have been established as for example in the case of the incompressible Euler system for “well prepared initial data” of the asymptotic equations or solution to the scaled Boltzmann equation with additional non uniform a priori estimates (see [?] for a more complete state of the art). The idea of the BGK model is to replace the collision kernel by an easier term which keeps some properties associated to the Boltzmann equation. Those models have been then generalized by Perthame and Tadmor [16] to derive at the macroscopic level some scalar conservation laws. More precisely, the BGK model that they investigate in [16] is the following:

$$(\partial_t + a(\xi) \cdot \partial_x) f_\varepsilon(t, x, \xi) = \frac{1}{\varepsilon} [\chi_{u_\varepsilon(t, x)}(\xi) - f_\varepsilon(t, x, \xi)], \quad (t, x, \xi) \in \mathbf{R}_t^+ \times \mathbf{R}_x^d \times \mathbf{R}_\xi \quad (1.0.4)$$

with  $u_\varepsilon(t, x) = \int_{\mathbf{R}} f_\varepsilon(t, x, \xi) d\xi$  and where  $\chi_u(\xi) = \begin{cases} \operatorname{sgn} u & \text{if } (u - \xi)\xi \geq 0, \\ 0 & \text{if } (u - \xi)\xi < 0. \end{cases}$

At the macroscopic level, they derive the following multidimensional law

$$\partial_t u(t, x) + \sum_{i=1}^d \partial_{x_i} A_i(u(t, x)) = 0, \quad (t, x) \in \mathbf{R}_t^+ \times \mathbf{R}_x^d, \quad (1.0.5)$$

with  $A_i \in C^1(\mathbf{R})$  and  $a_i = A_i'$ . Two main approaches are actually possible to get this scalar conservation law from (1.0.4). The initial one in [16] is based on a use of BV compactness arguments in the multidimensional case and compensated compactness arguments in the one dimensional case, while the second one introduced later in [14] relies on what is called a kinetic formulation. It consists in obtaining at the macroscopic level a kinetic equation involving a density-like function whose velocity distribution is the equilibrium density.

Lately, the study of scalar conservation laws starting from BGK models at the mesoscopic level have been pursued in various context: with boundary conditions in [15] and [?], with a discontinuous in the space variable flux in [3], with a high-field scaling in [2] or again in a stochastic context in [10]. In this paper, we are interested in the last two contexts mentioned. First of all, let us say a word about the stochastic one. Recently, the study of some conservation laws with stochastic forcing has been a subject of growing interest (see [1], [6],[8], [9], [11], [12] or again [?] for the case of a system). Indeed, the introduction of such terms can be justified in order to translate numerical and empirical uncertainties. Moreover, it often offers the possibility to weaken assumptions and still get results. The first result on an hydrodynamical limit starting from a BGK model in a stochastic context is due to Hofmanova [10]. The idea in this paper is to use the notion of stochastic kinetic formulation developed by Debussche and Vovelle in [6]. Certainly, it seems quite natural to adopt this approach knowing that we can not get any pathwise  $L^\infty$  a priori estimates due to the presence of the white noise term.

What we intend to do here is to extend the above result to a stochastic BGK model containing a force term with a high-field scaling. The deterministic version of this result has been established by Berthelin, Poupaud and Mauser in [2]. However, due to the presence of the white noise terms, the techniques adopted through the paper will be the ones developed by Debussche and Vovelle [6] and Hofmanová [10] by proving the convergence to a kinetic formulation associated to (1.0.1). The scaling adopted being different, some new difficulties are introduced and some additional assumptions similar to the ones of the deterministic version will be adopted. Moreover, a modified Maxwellian will be obtained in the kinetic formulation leading by its properties to get some Krushkov-like inequalities.

The organization of the paper is the following: in section 2 we will present the context and state the main result. Section 3 will be dedicated to the study of the stochastic BGK model (1.0.2). We will state its existence and prove its convergence to a kinetic formulation. Finally, in section 4 we will conclude to the existence of a weak solution to (1.0.1) which satisfies some Krushkov-like entropy relations.

## 2 Settings and Main Results

In the following, we will denote by  $\mathcal{C}^{k,\mu}$  with  $k \in \mathbf{N}$  and  $\mu > 0$  the set of  $k$  times differentiable  $\mu$ -Hölder functions. Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  be a stochastic basis with a complete, right-continuous filtration. We can assume without loss of generality that the  $\sigma$ -algebra  $\mathcal{F}$  is countably generated and  $(\mathcal{F}_t)_{t \geq 0}$  is the completed filtration generated by the Wiener process and the initial condition. We denote by  $\mathcal{P}$  the predictable  $\sigma$ -algebra on  $\Omega \times [0, T]$  associated to  $(\mathcal{F}_t)_{t \geq 0}$  and by  $\mathcal{P}_s$  the predictable  $\sigma$ -algebra on  $\Omega \times [s, T]$  associated to  $(\mathcal{F}_t)_{t \geq s}$ . We write  $L_{\mathcal{P}_s}^\infty(\Omega \times [s, T] \times \mathbf{T}^N \times \mathbf{R})$  to denote

$$L^\infty(\Omega \times [s, T] \times \mathbf{T}^N \times \mathbf{R}, \mathcal{P}_s \otimes \mathcal{B}(\mathbf{T}^N) \otimes \mathcal{B}(\mathbf{R}), d\mathbb{P} \otimes dt \otimes dx \otimes d\xi).$$

and idem for  $L^\infty(\Omega \times [s, T] \times \mathbf{T}^N \times \mathbf{R})$ . The setting of our paper has some similarities with the ones of Debussche and Vovelle [6] and Hofmanová [10]. Indeed, we will assume that we work on a finite-time interval  $[0, T]$ ,  $T > 0$  with periodic boundary conditions,  $x$  belonging to  $\mathbf{T}^N$  where  $\mathbf{T}^N$  is the  $N$ -dimensional torus. We will consider a function

$$a = (a_1, \dots, a_N) : \mathbf{T}^N \times \mathbf{R} \rightarrow \mathbf{R}^N \quad (2.0.6)$$

of class  $\mathcal{C}^{3,\mu}$  for some  $\mu > 0$  such that it satisfies for all  $x, \xi$ ,

$$0 \leq \operatorname{div}_x(a(x, \xi)) \quad (2.0.7)$$

and for all  $i = 1, \dots, N$ , for all  $x \in \mathbf{T}^N$ ,

$$\int_{\mathbf{R}} |\xi a_i(x, \xi)| d\xi < +\infty. \quad (2.0.8)$$

We assume that  $W$  is a  $d$ -dimensional  $(\mathcal{F}_t)$ -Wiener process, defined as follows

$$W(t) = \sum_{k=1}^d \beta_k(t) e_k \quad (2.0.9)$$

where  $(\beta_k)_{k=1}^d$  are mutually independent Brownian processes,  $(e_k)_{k=1}^d$  an orthonormal basis of  $H$  a finite dimensional Hilbert space.

For each  $u \in \mathbf{R}$ ,  $\Phi(u) : H \rightarrow L^2(\mathbf{T}^N)$  is defined by  $\Phi(u)e_k = g_k(u)$  where  $g_k(\cdot, u)$  is a regular function on  $\mathbf{T}^N$ . More precisely, the functions  $g_1, \dots, g_d : \mathbf{T}^N \times \mathbf{R} \rightarrow \mathbf{R}$  are of class  $\mathcal{C}^{4,\mu}$  with linear growth and bounded derivatives of all orders. In that context, the following estimate holds true

$$G^2(x, \xi) = \sum_{k=1}^d |g_k(x, \xi)|^2 \leq C(1 + |\xi|^2), \quad \forall x \in \mathbf{T}^N, \xi \in \mathbf{R}. \quad (2.0.10)$$

Moreover, we assume that

$$g_k(x, 0) = 0, \quad \forall x \in \mathbf{T}^N, k = 1, \dots, d, \quad (2.0.11)$$

and for all  $x \in \mathbf{T}^N$  and for all  $k$ ,

$$\int_{\mathbf{R}} |\xi \partial_\xi g_k(x, \xi)| d\xi < +\infty. \quad (2.0.12)$$

In addition, we assume that

$$\Lambda : \mathbf{T}^N \rightarrow \mathbf{R} \text{ is a nonpositive function of class } \mathcal{C}^{4,\mu}. \quad (2.0.13)$$

Regarding the initial data, we suppose that  $u_0 \in L^p(\Omega; L^p(\mathbf{T}^N))$  for all  $p \in [1, \infty)$  and we will consider  $F_0 = \mathbf{1}_{u_0 > \xi}$ .

Finally, quite similarly to the deterministic case with a high-field scaling [2], we will assume that

$$\left( \int_{\mathbf{R}} |\xi f_\varepsilon(\omega, t, x, \xi)| d\xi \right) \text{ is bounded in } L^\infty(\Omega \times [0, T] \times \mathbf{T}^N), \quad (2.0.14)$$

where  $f_\varepsilon := F_\varepsilon + \mathbf{1}_{0 > \xi}$ .

We finally can state our main result:

**Theorem 2.1.** *Under assumptions (2.0.7), (2.0.8), (2.0.9) (2.0.10), (2.0.11), (2.0.12), (2.0.13) and (2.0.14), for any  $\varepsilon > 0$  there exists a weak solution to the stochastic BGK model with a high field scaling (1.0.2) denoted by  $F_\varepsilon$  with initial condition  $F_0 = \mathbf{1}_{u_0 > \xi}$ . Moreover,  $F_\varepsilon$  belongs to  $L_{\mathcal{P}}^\infty(\Omega \times [0, T] \times \mathbf{T}^N \times \mathbf{R})$  and converges weak-\* to  $M_u$  a modified Maxwellian associated to  $u$  where  $u$  is a weak solution to the conservation law with stochastic forcing (1.0.1) with*

$$B(x, u) = \int_{-\infty}^u \int_0^{+\infty} a(x, \xi + v\Lambda(x))e^{-v} dv d\xi \quad (2.0.15)$$

and

$$C(x, u) = \int_{-\infty}^u \int_0^{+\infty} \partial_\xi \Phi(x, \xi + v\Lambda(x))e^{-v} dv d\xi. \quad (2.0.16)$$

Denoting  $u_\varepsilon(t, x) := \int_{\mathbf{R}} f_\varepsilon(t, x, \xi) d\xi$ , the sequence of local densities  $(u_\varepsilon)_{\varepsilon > 0}$  converges to the weak solution  $u$  in  $L^p(\Omega \times [0, T] \times \mathbf{T}^N)$  for all  $p \in [1, +\infty)$ . In addition,  $u$  satisfy the Krushkov-like inequalities (4.3.1).

### 3 Study of the stochastic BGK model with a high field scaling

#### 3.1 The stochastic kinetic model

We are interested in the following stochastic BGK model:

$$\begin{cases} dF^\varepsilon + \operatorname{div}_x(a(x, \xi)F^\varepsilon)dt + \frac{\Lambda(x)}{\varepsilon} \partial_\xi F^\varepsilon dt = \frac{\mathbf{1}_{u_\varepsilon > \xi} - F^\varepsilon}{\varepsilon} dt - \partial_\xi F^\varepsilon \Phi dW + \frac{1}{2} \partial_\xi(G^2 \partial_\xi F^\varepsilon)dt, \\ F^\varepsilon(0, x, \xi) = \mathbf{1}_{u_0(x) > \xi}(\xi). \end{cases} \quad (3.1.1)$$

The definition of a weak solution of this system is the following:

**Definition 3.1.** Let  $\varepsilon > 0$ ,  $F_\varepsilon \in L_{\mathcal{P}}^\infty(\Omega \times [0, T] \times \mathbf{T}^N \times \mathbf{R})$  is called a weak solution of (3.1.1) if for any test function  $\phi \in C_c^\infty(\mathbf{T}^N \times \mathbf{R})$ , we have a.e.  $t \in [0, T]$ ,  $\mathbb{P}$ -a.s.

$$\begin{aligned} \langle F_\varepsilon(t), \phi \rangle &= \langle F_0, \phi \rangle + \int_0^t \langle F_\varepsilon(s), a \cdot \nabla \phi \rangle ds + \frac{1}{\varepsilon} \int_0^t \langle F_\varepsilon(s), \Lambda(x) \partial_\xi \phi \rangle ds \\ &+ \frac{1}{\varepsilon} \int_0^t \langle \mathbf{1}_{u_\varepsilon(s) > \xi} - F_\varepsilon(s), \phi(s) \rangle ds + \sum_{k=1}^d \int_0^t \langle F_\varepsilon(s), \partial_\xi(g_k \phi) \rangle d\beta_k(s) \\ &+ \frac{1}{2} \int_0^t \langle F_\varepsilon(s), \partial_\xi(G^2 \partial_\xi \phi) \rangle ds. \end{aligned} \quad (3.1.2)$$

Let us state here several results. For  $\Theta$  a smooth function with compact support satisfying  $0 \leq \Theta \leq 1$  and

$$\Theta(\xi) := \begin{cases} 1 & \text{if } |\xi| \leq 1/2, \\ 0 & \text{if } |\xi| \geq 1, \end{cases} \quad (3.1.3)$$

we denote  $\Theta_R(\xi) := \Theta(\frac{\xi}{R})$ ,  $g_k^R(x, \xi) := g_k(x, \xi)\Theta_R(\xi)$  for  $k = 1, \dots, d$  and  $a^R(x, \xi) := a(x, \xi)\Theta_R(\xi)$ . The coefficients  $\Phi^R$  and  $G^{R,2}$  are defined similarly as  $\Phi$  and  $G^2$  replacing  $g_k$  by  $g_k^R$ . We introduce the intermediate problem,

$$\begin{cases} dX^\varepsilon + \operatorname{div}_x(a(x, \xi)X^\varepsilon)dt + \frac{\Lambda(x)}{\varepsilon} \partial_\xi X^\varepsilon dt = -\partial_\xi X^\varepsilon \Phi dW + \frac{1}{2} \partial_\xi(G^2 \partial_\xi X^\varepsilon)dt, \\ X^\varepsilon(s) = X_0, \end{cases} \quad (3.1.4)$$

and the truncated problem associated

$$\begin{cases} dX^\varepsilon + \operatorname{div}_x(a^R(x, \xi)X^\varepsilon)dt + \frac{\Lambda(x)}{\varepsilon} \partial_\xi X^\varepsilon dt = -\partial_\xi X^\varepsilon \Phi^R dW + \frac{1}{2} \partial_\xi(G^{R,2} \partial_\xi X^\varepsilon)dt, \\ X^\varepsilon(s) = X_0. \end{cases} \quad (3.1.5)$$

**Proposition 3.2.** *If  $X_0$  is a  $\mathcal{F}_s \otimes \mathcal{B}(\mathbf{T}^N) \otimes \mathcal{B}(\mathbf{R})$ -measurable initial data belonging to  $L^\infty(\Omega \times \mathbf{T}^N \times \mathbf{R})$ , then there exists a weak solution  $X^\varepsilon \in L^\infty_{\mathcal{P}_s}(\Omega \times [s, T] \times \mathbf{T}^N \times \mathbf{R})$  to (3.1.4). Moreover, it is represented by*

$$\mathcal{S}^\varepsilon(t, s)(\omega, x, \xi) := \lim_{R \rightarrow +\infty} [\mathcal{S}^{\varepsilon, R}(t, s)X_0](\omega, x, \xi), \quad 0 \leq s \leq t \leq T, \quad (3.1.6)$$

with  $\mathcal{S}_{\varepsilon, R}$  a solution operator of (3.1.5).

From above, we obtain the following proposition:

**Proposition 3.3.** *For any  $\varepsilon > 0$ , there exists a weak solution of the stochastic BGK model with high field scaling (1.0.2) denoted by  $F_\varepsilon$ . Moreover,  $F_\varepsilon$  is represented by*

$$F_\varepsilon(t) = e^{-t/\varepsilon} \mathcal{S}^\varepsilon(t, 0) \mathbf{1}_{u_0 > \xi} + \frac{1}{\varepsilon} \int_0^t e^{-\frac{t-s}{\varepsilon}} \mathcal{S}^\varepsilon(t, s) \mathbf{1}_{u_\varepsilon(s) > \xi} ds. \quad (3.1.7)$$

The proof of those two results is quite similar to the ones in [10] and mainly relies on a use of the stochastic characteristics method developed by Kunita in [13]. Nevertheless, there is some additional difficulties due to the presence of the term  $\frac{\Lambda(x)}{\varepsilon} \partial_\xi F_\varepsilon$  (which introduce a dependence on  $\varepsilon$  of the solution operator) and the dependence of  $a$  in  $x$ . Then, for the sake of completeness, we give a quite detailed sketch of the proof of the existence in Appendix A.

## 3.2 Convergence of the stochastic kinetic model

From now on, let us denote by  $C$  a constant which does not depend on any parameter and may change from a line to another. In this section, our purpose is to study the limit of the stochastic kinetic model (1.0.2) as  $\varepsilon$  goes to 0 in the following weak formulation satisfied by  $F_\varepsilon$ :

$$\begin{aligned} & \int_0^T \langle F_\varepsilon(t), \partial_t \varphi(t) \rangle dt + \langle F_0, \varphi(0) \rangle + \int_0^T \langle F_\varepsilon(t), a \cdot \nabla \varphi(t) \rangle dt \\ &= -\frac{1}{\varepsilon} \int_0^T \langle \mathbf{1}_{u_\varepsilon > \xi} - (F_\varepsilon(t) + \Lambda(x) \partial_\xi F_\varepsilon(t)), \varphi(t) \rangle dt + \int_0^T \langle \partial_\xi F_\varepsilon(t) \Phi dW_t, \varphi(t) \rangle \\ & \quad + \frac{1}{2} \int_0^T \langle G^2 \partial_\xi F_\varepsilon(t), \partial_\xi \varphi(t) \rangle dt. \end{aligned} \quad (3.2.1)$$

for  $\varphi \in \mathcal{C}_c^\infty([0, T] \times \mathbf{T}^N \times \mathbf{R})$ .

**Proposition 3.4.** *Up to a subsequence,  $(F_\varepsilon)_{\varepsilon > 0}$  converges weak-\* to  $F$  with  $F$  satisfying the following: for any test function  $\varphi \in \mathcal{C}_c([0, T] \times \mathbf{T}^N \times \mathbf{R})$ ,*

$$\begin{aligned} & \int_0^T \langle F(t), \partial_t \varphi(t) \rangle dt + \langle F_0, \varphi(0) \rangle + \int_0^T \langle F(t), a \cdot \nabla \varphi(t) \rangle dt \\ &= m(\partial_\xi \varphi) + \int_0^T \langle \partial_\xi F(t) \Phi dW_t, \varphi(t) \rangle + \frac{1}{2} \int_0^T \langle G^2 \partial_\xi F(t), \partial_\xi \varphi(t) \rangle dt \end{aligned} \quad (3.2.2)$$

with  $m$  a random nonnegative bounded Borel measure on  $[0, T] \times \mathbf{T}^N \times \mathbf{R}$  and where  $m(\partial_\xi \varphi)$  denotes

$$m(\partial_\xi \varphi) := \int_{\mathbf{T}^N \times [0, T] \times \mathbf{R}} \partial_\xi \varphi dm(x, t, \xi). \quad (3.2.3)$$

*Proof.* From the representation formula of  $F_\varepsilon$  (3.1.7), we deduce that the set of solutions  $\{F_\varepsilon; \varepsilon \in (0, 1)\}$  is bounded in  $L^\infty_{\mathcal{P}}(\Omega \times [0, T] \times \mathbf{T}^N \times \mathbf{R})$ . Using the Banach-Alaoglu theorem, we know that there exists  $F$  in  $L^\infty_{\mathcal{P}}(\Omega \times [0, T] \times \mathbf{T}^N \times \mathbf{R})$  such that, up to subsequences, as  $\varepsilon$  goes to 0,

$$F_\varepsilon \xrightarrow{w^*} F \text{ in } L^\infty_{\mathcal{P}}(\Omega \times [0, T] \times \mathbf{T}^N \times \mathbf{R}).$$

Then, almost surely, we have as  $\varepsilon$  goes to 0,

$$\begin{aligned} \int_0^T \langle F_\varepsilon(t), \partial_t \varphi(t) \rangle dt &\longrightarrow \int_0^T \langle F(t), \partial_t \varphi(t) \rangle dt, \\ \int_0^T \langle F_\varepsilon(t), a \cdot \nabla \varphi(t) \rangle dt &\longrightarrow \int_0^T \langle F(t), a \cdot \nabla \varphi(t) \rangle dt, \\ \frac{1}{2} \int_0^T \langle G^2 \partial_\xi F_\varepsilon(t), \partial_\xi \varphi(t) \rangle dt &\longrightarrow \frac{1}{2} \int_0^T \langle G^2 \partial_\xi F(t), \partial_\xi \varphi(t) \rangle dt. \end{aligned}$$

Moreover, regarding the stochastic term, a use of dominated convergence theorem for stochastic integrals allows us to conclude that almost surely,

$$\int_0^T \langle \partial_\xi F_\varepsilon(t) \Phi dW_t, \varphi(t) \rangle \longrightarrow \int_0^T \langle \partial_\xi F(t) \Phi dW_t, \varphi(t) \rangle.$$

Indeed,  $\langle F_\varepsilon, \partial_\xi(g_k \varphi) \rangle \longrightarrow \langle F, \partial_\xi(g_k \varphi) \rangle$  a.e.  $(\omega, t) \in \Omega \times [0, T]$ . In addition, by assumption on  $g_k$  and since  $F_\varepsilon$  is bounded,

$$|\langle F_\varepsilon, \partial_\xi(g_k \varphi) \rangle| \leq C.$$

It remains to deal with the first term of the right-hand side of (3.2.1). Let us define

$$m_\varepsilon(t, x, \xi) := \frac{1}{\varepsilon} \int_{-\infty}^{\xi} (\mathbf{1}_{u_\varepsilon(t, x) > \zeta}(\zeta) - F_\varepsilon(t, x, \zeta)) d\zeta + (-\Lambda(x) F_\varepsilon(t, x, \xi)). \quad (3.2.4)$$

We can prove similarly as in [10] that the first term of the right-hand side of (3.2.4) is a random nonnegative measure over  $[0, T] \times T^N \times \mathbf{R}$  and since by assumption (2.0.13),  $\Lambda(x) \leq 0$  for all  $x$ , so does the second term. Indeed, we recall that a positive distribution is a positive Radon measure.

Moreover,  $f_\varepsilon$  satisfies

$$\begin{aligned} df^\varepsilon + \operatorname{div}_x(a(x, \xi) f^\varepsilon) dt + \operatorname{div}_x(a(x, \xi) \mathbf{1}_{0 > \xi}) dt \\ = \partial_\xi m_\varepsilon - \partial_\xi f^\varepsilon \Phi dW_t - \partial_\xi \mathbf{1}_{0 > \xi} \Phi dW_t + \frac{1}{2} \partial_\xi (G^2 \partial_\xi F^\varepsilon) dt. \end{aligned} \quad (3.2.5)$$

since  $f_\varepsilon = F_\varepsilon - \mathbf{1}_{0 > \xi}$ .

We consider the following test functions

$$\varphi(t, x, \xi) = \Psi(t, x) \tilde{\varphi}(\xi)$$

with  $\Psi \in \mathcal{C}_c([0, T] \times \mathbf{T}^N)$ ,  $\tilde{\varphi} \in \mathcal{C}_c(\mathbf{R})$ . We denote

$$H(t, x, \xi) := \int_{-\infty}^{\xi} \varphi(t, x, \zeta) d\zeta.$$

Thus, using the fact that  $\partial_\xi \mathbf{1}_{0 > \xi} = -\delta_0$  and assumption (2.0.11), we have

$$\begin{aligned} \mathbb{E} \int_0^T \langle m_\varepsilon, \varphi \rangle dt &= -\mathbb{E} \int_0^T \langle \partial_\xi m_\varepsilon, H(t) \rangle dt \\ &= \mathbb{E} \int_0^T \langle f_\varepsilon(t), \partial_t H(t) \rangle dt + \mathbb{E} \langle f_0, \varphi(0) \rangle \\ &\quad + \mathbb{E} \int_0^T \langle f_\varepsilon(t), a \cdot \nabla H(t) \rangle dt + \mathbb{E} \int_0^T \langle \mathbf{1}_{0 > \xi}, a \cdot \nabla H(t) \rangle dt \\ &\quad - \mathbb{E} \int_0^T \langle \partial_\xi f_\varepsilon(t) \Phi dW_t, H(t) \rangle - \frac{1}{2} \mathbb{E} \int_0^T \langle G^2 \partial_\xi F_\varepsilon(t), \partial_\xi H(t) \rangle dt. \end{aligned} \quad (3.2.6)$$

At this point, our aim is to bound  $\mathbb{E} \int_0^T \langle m_\varepsilon, \varphi \rangle dt$  for all  $\varphi$  in  $\mathcal{C}_c([0, T] \times \mathbf{T}^N \times \mathbf{R})$  independently of  $\varepsilon$  in order to conclude thanks to properties of Radon measure on locally compact spaces. Let us first prove that the term  $\mathbb{E} \int_0^T \langle \partial_\xi f_\varepsilon \Phi dW_t, H(t) \rangle$  disappears. We recall that a martingale has zero expected value. Therefore, the idea is to use the stochastic Fubini theorem (see [5, Theorem 4.18]) to interchange the integrals with respect to  $x, \xi$  and the stochastic one. In order to apply this theorem, we must prove that

$$\int_{\mathbf{T}^N} \int_{\mathbf{R}} \left( \mathbb{E} \int_0^T |f_\varepsilon(t) \partial_\xi (g_k H(t))|^2 dt \right)^{1/2} d\xi dx < +\infty$$

for each  $k = 1, 2, \dots, d$ . Indeed, we have

$$\begin{aligned} & \int_{\mathbf{T}^N} \int_{\mathbf{R}} \left( \mathbb{E} \int_0^T |f_\varepsilon(t) \partial_\xi (g_k H(t))|^2 dt \right)^{1/2} d\xi dx \\ &= \int_{\mathbf{T}^N} \int_{\mathbf{R}} \left( \mathbb{E} \int_0^T |f_\varepsilon(t) \partial_\xi g_k H(t) + f_\varepsilon(t) g_k \varphi|^2 dt \right)^{1/2} d\xi dx \\ &\leq \int_{\mathbf{T}^N} \int_{\mathbf{R}} \left[ \left( \mathbb{E} \int_0^T |f_\varepsilon(t) \partial_\xi g_k H(t)|^2 dt \right)^{1/2} + \left( \mathbb{E} \int_0^T |f_\varepsilon(t) g_k \varphi|^2 dt \right)^{1/2} \right] d\xi dx \\ &\leq C \left( \int_{\mathbf{T}^N} \int_{\mathbf{R}} |\partial_\xi g_k \xi| d\xi dx + \int_{\mathbf{T}^N} \int_{\mathbf{R}} (1 + |\xi|^2)^{1/2} \tilde{\varphi} d\xi dx \right) \\ &< +\infty \end{aligned} \tag{3.2.7}$$

using the fact that  $f_\varepsilon$  is bounded and assumption (2.0.12). Then, we can interchange the integrals. It remains to prove that  $\int_0^T \partial_\xi f_\varepsilon g_k H(t) dW_t$  is a well defined martingale. For each  $k = 1, \dots, d$ , we have

$$\begin{aligned} & \mathbb{E} \int_0^T |f_\varepsilon(t) \partial_\xi (g_k H(t))|^2 dt \\ & \leq C \left( \mathbb{E} \int_0^T |f_\varepsilon(t) \partial_\xi g_k H(t)|^2 dt + \mathbb{E} \int_0^T |f_\varepsilon(t) g_k \varphi|^2 dt \right) \\ & < +\infty \end{aligned} \tag{3.2.8}$$

by the same arguments as previously. So, finally we have

$$\begin{aligned} \left| \mathbb{E} \int_0^T \langle m_\varepsilon, \varphi \rangle dt \right| & \leq \mathbb{E} \int_0^T \langle f_\varepsilon(t), |\partial_t \Psi(t) \xi| \rangle dt + |\mathbb{E} \langle f_0, \varphi(0) \rangle| \\ & \quad + \mathbb{E} \int_0^T \langle f_\varepsilon(t), |a \cdot \nabla \Psi(t) \xi| \rangle dt + \mathbb{E} \int_0^T \langle \mathbf{1}_{0 > \xi}, |a \cdot \nabla \Psi(t) \xi| \rangle dt \\ & \quad + \frac{1}{2} \left| \mathbb{E} \int_0^T \langle G^2 \partial_\xi F_\varepsilon(t), \varphi(t) \rangle dt \right|. \end{aligned} \tag{3.2.9}$$

Then by assumptions (2.0.8) and (2.0.12), we deduce that  $\mathbb{E} \int_0^T \langle m_\varepsilon, \varphi \rangle dt$  is bounded independently of  $\varepsilon$ . By density, the results hold true for all  $\varphi \in \mathcal{C}_c^\infty([0, T] \times \mathbf{T}^N \times \mathbf{R})$  and similarly for all  $\varphi \in \mathcal{C}_c([0, T] \times \mathbf{T}^N \times \mathbf{R})$ . Then, by properties of Radon measures on locally compact spaces, we can conclude that for almost every  $\omega \in \Omega$ , there exists a nonnegative measure  $m(\omega)$  such that, up to a subsequence, almost surely,

$$\int_0^T \langle m_\varepsilon, \varphi(t) \rangle dt \longrightarrow \int_0^T \langle m, \varphi(t) \rangle dt \tag{3.2.10}$$

for any test function  $\varphi \in \mathcal{C}_c^\infty([0, T] \times \mathbf{T}^N \times \mathbf{R})$ .

Finally, passing to the limit in (3.2.1) leads to the statement.  $\square$

**Remark 3.1.** *The purpose of assumptions (2.0.8) and (2.0.12) can actually be found in the above proof to bound  $\mathbb{E} \int_0^T \langle m_\varepsilon, \varphi \rangle dt$ . Indeed, due to the dependence on  $\varepsilon$  of the stochastic characteristic system because of the presence of the term  $\frac{\Lambda(x)}{\varepsilon} \partial_\xi F_\varepsilon$ , the techniques developed in [10] can not be adopted here.*

### 3.3 A kinetic formulation

As mentioned previously, a kinetic formulation is a kinetic equation at the macroscopic level satisfy by a density-like function. At this point, we already have obtained (3.2.2), the kinetic equation satisfy by  $F$  at the macroscopic level. Then, it remains to study the behavior of its velocity distribution.

In order to do so, let us recall here some notions that will be needed to get the kinetic formulation associated to (1.0.1).

**Definition 3.5** (Young measure). Let  $(X, \lambda)$  be a finite measure space. Let  $\mathcal{P}_1(\mathbf{R})$  denote the set of probability measures on  $\mathbf{R}$ . We say that a map  $\nu : X \rightarrow \mathcal{P}_1(\mathbf{R})$  is a Young measure on  $X$  if, for all  $\phi \in \mathcal{C}_b(\mathbf{R})$ , the map  $z \mapsto \nu_z(\phi)$  from  $X$  to  $\mathbf{R}$  is measurable.

We have the following compactness result:

**Theorem 3.6** (Compactness of Young measure). *Let  $(X, \lambda)$  be a finite measure space such that  $L^1(X)$  is separable. Let  $(\nu^n)$  be a sequence of Young measures on  $X$  satisfying for some  $p \geq 1$ ,*

$$\sup_n \int_X \int_{\mathbf{R}} |\xi|^p d\nu_z^n(\xi) d\lambda(z) < +\infty.$$

*Then, there exists a Young measure  $\nu$  on  $X$  and a subsequence still denoted  $(\nu^n)$  such that, for all  $h \in L^1(X)$ , for all  $\phi \in \mathcal{C}_b(\mathbf{R})$ ,*

$$\lim_{n \rightarrow +\infty} \int_X h(z) \int_{\mathbf{R}} \phi(\xi) d\nu_z^n(\xi) d\lambda(z) = \int_X h(z) \int_{\mathbf{R}} \phi(\xi) d\nu_z(\xi) d\lambda(z).$$

The proof, quite classical, can be found in [6].

The purpose of this subsection is then actually to prove the following result:

**Proposition 3.7.** *There exists  $u \in L^1(\Omega \times [0, T] \times \mathbf{T}^N)$  such that for all  $t \in [0, T]$  for almost every  $(x, \xi, \omega)$ ,  $F = M_u$  where we denote by  $M_u$  the modified Maxwellian solution to the equation*

$$M_u + \Lambda(x) \partial_\xi M_u = \mathbf{1}_{u > \xi}. \quad (3.3.1)$$

*Proof.* Multiplying (3.1.1) by  $\varepsilon$ , then we have

$$\mathbf{1}_{u_\varepsilon > \xi} - F_\varepsilon - \Lambda(x) \partial_\xi F_\varepsilon \longrightarrow 0 \quad (3.3.2)$$

in the sense of distributions over  $(0, T) \times \mathbf{T}^N \times \mathbf{R}$  almost surely. So, we deduce that

$$\partial_\xi [\mathbf{1}_{u_\varepsilon > \xi} - F_\varepsilon - \Lambda(x) \partial_\xi F_\varepsilon] \longrightarrow 0$$

or again

$$-\delta_{u_\varepsilon = \xi} - \partial_\xi F_\varepsilon - \Lambda(x) \partial_\xi^2 F_\varepsilon \longrightarrow 0 \quad (3.3.3)$$

in the sense of distributions over  $(0, T) \times \mathbf{T}^N \times \mathbf{R}$  almost surely.



We set  $\nu_{t,x}^\varepsilon := \delta_{u_\varepsilon(t,x)=\xi}$  which by Definition 3.5 is a Young measure. Then, by assumption (2.0.14), we have

$$\begin{aligned} \sup_{\varepsilon>0} \sup_{t \in [0,T]} \int_{\mathbf{T}^N} \int_{\mathbf{R}} |\xi| d\nu_{t,x}^\varepsilon(\xi) dx &= \sup_{\varepsilon>0} \sup_{t \in [0,T]} \int_{\mathbf{T}^N} |u_\varepsilon(t,x)| dx \\ &\leq \sup_{\varepsilon>0} \sup_{t \in [0,T]} \int_{\mathbf{T}^N} \int_{\mathbf{R}} |f_\varepsilon(t,x,\xi)| d\xi dx < +\infty. \end{aligned} \quad (3.3.4)$$

Using Proposition 3.6, we know that there exists a Young measure  $\nu_{t,x}$  such that  $\nu_{t,x}^\varepsilon \rightarrow \nu_{t,x}$  (up to a subsequence). Then, we deduce from (3.3.3) and Proposition 3.4 that

$$\partial_\xi F + \Lambda(x) \partial_\xi^2 F = -\nu. \quad (3.3.5)$$

Let us prove then that  $F + \Lambda(x) \partial_\xi F \in \{0, 1\}$ . Let us construct the following mollifier on  $[0, T]$ :

$$\theta(t) := \begin{cases} \tilde{C} \exp\left(\frac{1}{t^2-1}\right) & \text{if } t \leq 1, \\ 0 & \text{else} \end{cases} \quad (3.3.6)$$

with  $\tilde{C}$  the constant such that  $\int_0^T \theta(t) dt = 1$ . We denote for  $0 \leq \delta \leq 1$ ,

$$\theta_\delta(t) := \frac{1}{\delta} \theta\left(\frac{t}{\delta}\right). \quad (3.3.7)$$

Then we have

$$|\theta_\delta(t)| \leq \frac{C}{\delta} \text{ and } |\theta'_\delta(t)| \leq \frac{C}{\delta^2}, \quad (3.3.8)$$

with  $C$  a constant which does not depend on  $\delta$ . We set for  $s \in [0, T]$ ,

$$\varphi_1(t, x, \xi) = \theta_\delta(s-t) \Psi_1(x, \xi) \text{ and } \varphi_2(t, y, \zeta) = \theta_\delta(s-t) \Psi_2(y, \zeta)$$

with  $\Psi_1 \in \mathcal{C}_c^\infty(\mathbf{T}_x^N \times \mathbf{R}_\xi)$ ,  $\Psi_2 \in \mathcal{C}_c^\infty(\mathbf{T}_y^N \times \mathbf{R}_\zeta)$ .

By (3.2.1), we have that

$$\begin{aligned} &\int_0^T \langle F_\varepsilon(t) + \Lambda(x) \partial_\xi F_\varepsilon(t), \varphi_1(t) \rangle dt \times \int_0^T \langle 1 - (F_\varepsilon(t) + \Lambda(x) \partial_\xi F_\varepsilon(t)), \varphi_2(t) \rangle dt \\ &= \left[ \int_0^T \langle \mathbf{1}_{u_\varepsilon(t) > \xi}, \varphi_1(t) \rangle dt \right. \\ &\quad \left. + \varepsilon \left( \int_0^T \langle F_\varepsilon(t), \partial_t \varphi_1(t) \rangle dt + \langle F_0, \varphi_1(0) \rangle + \int_0^T \langle F_\varepsilon(t), a \cdot \nabla \varphi_1(t) \rangle dt \right. \right. \\ &\quad \left. \left. - \int_0^T \langle \partial_\xi F_\varepsilon(t) \Phi dW_t, \varphi_1(t) \rangle - \frac{1}{2} \int_0^T \langle G^2 \partial_\xi F_\varepsilon(t), \partial_\xi \varphi_1(t) \rangle dt \right) \right] \\ &\times \left[ \int_0^T \langle 1 - \mathbf{1}_{u_\varepsilon(t) > \zeta}, \varphi_2(t) \rangle dt \right. \\ &\quad \left. - \varepsilon \left( \int_0^T \langle F_\varepsilon(t), \partial_t \varphi_2(t) \rangle dt + \langle F_0, \varphi_2(0) \rangle + \int_0^T \langle F_\varepsilon(t), a \cdot \nabla \varphi_2(t) \rangle dt \right. \right. \\ &\quad \left. \left. - \int_0^T \langle \partial_\xi F_\varepsilon(t) \Phi dW_t, \varphi_2(t) \rangle - \frac{1}{2} \int_0^T \langle G^2 \partial_\xi F_\varepsilon(t), \partial_\xi \varphi_2(t) \rangle dt \right) \right] \end{aligned}$$

$$\begin{aligned}
&\leq \left( \int_0^T \langle \mathbf{1}_{u_\varepsilon(t) > \xi}, \varphi_1(t) \rangle dt \right) \times \left( \int_0^T \langle 1 - \mathbf{1}_{u_\varepsilon(t) > \zeta}, \varphi_2(t) \rangle dt \right) \\
&\quad + \varepsilon \left( 1 + \frac{C}{\delta} \int_0^T \langle 1, |\Psi_2| \rangle dt \right) \left( \left| \int_0^T \langle F(t), \partial_t \varphi_1(t) \rangle dt \right| + |\Xi_1 + r_1(\varepsilon)| \right) \\
&\quad + \varepsilon \frac{C}{\delta} \int_0^T \langle 1, |\Psi_1| \rangle dt \left( \left| \int_0^T \langle F(t), \partial_t \varphi_2(t) \rangle dt \right| + |\Xi_2 + r_2(\varepsilon)| \right) \\
&\quad + \varepsilon^2 \prod_{i=1}^2 \left( \left| \int_0^T \langle F(t), \partial_t \varphi_i(t) \rangle dt \right| + |\Xi_i + r_i(\varepsilon)| \right),
\end{aligned}$$

where for  $i = 1, 2$ ,

$$\begin{aligned}
\Xi_i := &\langle F_0, \varphi_i(0) \rangle + \int_0^T \langle F(t), a \cdot \nabla \varphi_i(t) \rangle dt \\
&- \int_0^T \langle \partial_\xi F(t) \Phi dW_t, \varphi_i(t) \rangle - \frac{1}{2} \int_0^T \langle G^2 \partial_\xi F(t), \partial_\xi \varphi_i(t) \rangle dt,
\end{aligned}$$

$r_i(\varepsilon)$  exists and is a function such that  $r_i(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0$  by Proposition 3.4. Then, we deduce from (3.3.8) that

$$\begin{aligned}
&\int_0^T \langle F_\varepsilon(t) + \Lambda(x) \partial_\xi F_\varepsilon(t), \varphi_1(t) \rangle dt \times \int_0^T \langle 1 - (F_\varepsilon(t) + \Lambda(x) \partial_\xi F_\varepsilon(t)), \varphi_2(t) \rangle dt \\
&\leq \left( \int_0^T \langle \mathbf{1}_{u_\varepsilon(t) > \xi}, \varphi_1(t) \rangle dt \right) \times \left( \int_0^T \langle 1 - \mathbf{1}_{u_\varepsilon(t) > \zeta}, \varphi_2(t) \rangle dt \right) \\
&\quad + \varepsilon \left( 1 + \frac{C}{\delta} \int_0^T \langle 1, |\Psi_2| \rangle dt \right) \left( \frac{C}{\delta^2} \int_0^T \langle F(t), |\Psi_1(t)| \rangle dt + \Xi_1 + r_1(\varepsilon) \right) \\
&\quad + \varepsilon \frac{C}{\delta} \int_0^T \langle 1, |\Psi_1| \rangle dt \left( \frac{C}{\delta^2} \int_0^T \langle F(t), |\Psi_2(t)| \rangle dt + \Xi_2 + r_2(\varepsilon) \right) \\
&\quad + \varepsilon^2 \prod_{i=1}^2 \frac{C}{\delta^2} \left( \int_0^T \langle F(t), |\Psi_i(t)| \rangle dt + |\Xi_i + r_i(\varepsilon)| \right).
\end{aligned}$$

Moreover, since

$$\begin{aligned}
&\mathbb{E} \left[ \int_0^T \langle F_\varepsilon(t) + \Lambda(x) \partial_\xi F_\varepsilon(t), \varphi_1(t) \rangle dt \times \int_0^T \langle 1 - (F_\varepsilon(t) + \Lambda(x) \partial_\xi F_\varepsilon(t)), \varphi_2(t) \rangle dt \right] \\
&\quad \xrightarrow{\delta \rightarrow 0} \mathbb{E} [\langle F_\varepsilon(s) + \Lambda(x) \partial_\xi F_\varepsilon(s), \Psi_1(s) \rangle \langle 1 - (F_\varepsilon(s) + \Lambda(x) \partial_\xi F_\varepsilon(s)), \Psi_2(s) \rangle]
\end{aligned} \tag{3.3.9}$$

and

$$\begin{aligned}
&\mathbb{E} \left[ \int_0^T \langle \mathbf{1}_{u_\varepsilon(t) > \xi}, \varphi_1(t) \rangle dt \times \int_0^T \langle 1 - \mathbf{1}_{u_\varepsilon(t) > \zeta}, \varphi_2(t) \rangle dt \right] \\
&\quad \xrightarrow{\delta \rightarrow 0} \mathbb{E} [\langle \mathbf{1}_{u_\varepsilon(s) > \xi}, \Psi_1(s) \rangle \langle 1 - \mathbf{1}_{u_\varepsilon(s) > \zeta}, \Psi_2(s) \rangle],
\end{aligned} \tag{3.3.10}$$

we deduce that

$$\begin{aligned}
&\mathbb{E} [\langle F_\varepsilon(s) + \Lambda(x) \partial_\xi F_\varepsilon(s), \Psi_1(s) \rangle \langle 1 - (F_\varepsilon(s) + \Lambda(x) \partial_\xi F_\varepsilon(s)), \Psi_2(s) \rangle + r_3(\delta)] \\
&\leq \mathbb{E} [\langle \mathbf{1}_{u_\varepsilon(s) > \xi}, \Psi_1(s) \rangle \langle 1 - \mathbf{1}_{u_\varepsilon(s) > \zeta}, \Psi_2(s) \rangle + r_4(\delta) \\
&\quad + C \frac{\varepsilon}{\delta^3} (1 + |r_1(\varepsilon)| + |r_2(\varepsilon)|) + C \frac{\varepsilon^2}{\delta^4} \prod_{i=1}^2 (1 + |r_i(\varepsilon)|)]
\end{aligned}$$

where by (3.3.9) and (3.3.10),  $r_3(\delta)$  and  $r_4(\delta)$  exist and are functions such that  $r_3(\delta), r_4(\delta) \xrightarrow{\delta \rightarrow 0} 0$ . We denote  $\alpha(x, \xi, y, \zeta) = \psi_1(x, \xi)\Psi_2(y, \zeta)$  and  $\ll \cdot, \cdot \gg$  the duality distribution over  $\mathbf{T}_x^N \times \mathbf{R}_\xi \times \mathbf{T}_y^N \times \mathbf{R}_\zeta$ . Then we can rewrite the above inequality as follows

$$\begin{aligned} \mathbb{E} \ll (F_\varepsilon(s) + \Lambda(x)\partial_\xi F_\varepsilon(s))(1 - (F_\varepsilon(s) + \Lambda(x)\partial_\xi F_\varepsilon(s))), \alpha \gg + r_3(\delta) \\ \leq \mathbb{E} \ll \mathbf{1}_{u_\varepsilon(s) > \xi} (1 - \mathbf{1}_{u_\varepsilon(s) > \zeta}), \alpha \gg + r_4(\delta) \\ + C \frac{\varepsilon}{\delta^3} (1 + |r_1(\varepsilon)| + |r_2(\varepsilon)|) + C \frac{\varepsilon^2}{\delta^4} \prod_{i=1}^2 (1 + |r_i(\varepsilon)|). \end{aligned}$$

By a density argument, it remains true for any test function  $\alpha \in \mathcal{C}_c^\infty(\mathbf{T}_x^N \times \mathbf{R}_\xi \times \mathbf{T}_y^N \times \mathbf{R}_\zeta)$ . We consider  $\varrho_{\eta_1}$  and  $\rho_{\eta_2}$  some mollifiers on respectively  $\mathbf{T}^N$  and  $\mathbf{R}$ . For any  $R > 0$ , we take

$$\alpha(x, \xi, y, \zeta) = \varrho_{\eta_1}(x - y)\rho_{\eta_2}(\xi - \zeta)\Theta_R(|x|)\Theta_R(\xi).$$

Passing to the limit  $\eta_1, \eta_2 \rightarrow 0$ , and  $\varepsilon \rightarrow 0$  with  $\delta = \varepsilon^{1/4}$ , we finally get

$$\mathbb{E} \langle F(s) + \Lambda(x)\partial_\xi F(s), (1 - (F(s) + \Lambda(x)\partial_\xi F(s))) \Theta_R(|x|)\Theta_R(\xi) \rangle \leq 0.$$

Moreover, by (3.3.2) there exists a distribution  $r_6$  such that

$$F_\varepsilon + \Lambda(x)\partial_\xi F_\varepsilon = \mathbf{1}_{u_\varepsilon > \xi} + r_6(\varepsilon)$$

with  $r_6(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0$ .

Then

$$r_6(\varepsilon) \leq F_\varepsilon + \Lambda(x)\partial_\xi F_\varepsilon \leq 1 + r_6(\varepsilon)$$

and passing to the limit, we get

$$0 \leq F + \Lambda(x)\partial_\xi F \leq 1.$$

Finally,

$$\mathbb{E} \langle F(s) + \Lambda(x)\partial_\xi F(s), (1 - (F(s) + \Lambda(x)\partial_\xi F(s))) \Theta_R(|x|)\Theta_R(\xi) \rangle = 0.$$

Thus for almost every  $x, \xi, \omega, \forall s \in [0, T]$ , we deduce from above that

$$F(s, x, \xi, \omega) + \Lambda(x)\partial_\xi F(s, x, \xi, \omega) \in \{0, 1\}.$$

Since we have

$$\begin{aligned} F + \Lambda(x)\partial_\xi F &= - \int_0^\xi -\partial_\xi (F + \Lambda(x)\partial_\xi F) d\zeta \\ &= - \int_0^\xi \nu d\zeta \end{aligned}$$

with  $\nu$  a Young measure, there exists  $u(x, t, \omega) \in \mathbf{R}$  such that

$$F + \Lambda(x)\partial_\xi F = \mathbf{1}_{u(t) > \xi}. \quad (3.3.11)$$

Moreover, we can deduce from (3.3.5) that

$$\partial_\xi \mathbf{1}_{u(t) > \xi} = -\nu$$

i.e.

$$\nu = \delta_{u=\xi}. \quad (3.3.12)$$

Therefore because of (3.3.4), we have

$$\int_{\mathbf{T}^N} |u(t, x)| dx = \int_{\mathbf{T}^N} \int_{\mathbf{R}} |\xi| d\nu_{t,x}(\xi) dx dt < +\infty \quad (3.3.13)$$

which concludes the proof.  $\square$

We can actually strengthen the convergence and get the following result:

**Proposition 3.8.**  $(u_\varepsilon)_{\varepsilon>0}$  converges in  $L^p(\Omega \times [0, T] \times \mathbf{T}^N)$  to  $u$  for all  $p \in [1, +\infty)$ .

*Proof.* Let us recall that we have (up to subsequence)

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \int_{[0, T] \times \mathbf{T}^N} h(t, x) \int_{\mathbf{R}} \phi(\xi) d\nu_{t,x}^\varepsilon dx dt = \mathbb{E} \int_{[0, T] \times \mathbf{T}^N} h(t, x) \int_{\mathbf{R}} \phi(\xi) d\nu_{t,x} dx dt$$

for all  $h \in L^1(\Omega \times [0, T] \times \mathbf{T}^N)$  i.e.

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \int_{[0, T] \times \mathbf{T}^N} h(t, x) \phi(u_\varepsilon(t, x)) dx dt = \mathbb{E} \int_{[0, T] \times \mathbf{T}^N} h(t, x) \int_{\mathbf{R}} \phi(\xi) d\nu_{t,x} dx dt. \quad (3.3.14)$$

Moreover, we have established previously that  $\nu = \delta_{u=\xi}$ . Then, we deduce from the compensated compactness theorem that since  $\nu$  is a Dirac,  $(u_\varepsilon)_{\varepsilon>0}$  converges strongly in  $L^p(\Omega \times [0, T] \times \mathbf{T}^N)$  for all  $p \in [1, +\infty)$ .  $\square$

## 4 Existence of a solution to the conservation law with stochastic forcing

### 4.1 Formal calculus and modified Maxwellian

In this section, we will show in a formal way the passage from the kinetic formulation to the conservation law with stochastic forcing (1.0.1). Indeed, by Proposition 3.7 we know that  $F = M_u$  and so satisfies

$$\Lambda(x) \partial_\xi F = \mathbf{1}_{u>\xi} - F. \quad (4.1.1)$$

For any function  $b(x, \xi)$ , we have

$$\int_{\mathbf{R}} b(x, \xi) \Lambda(x) \partial_\xi F(x, \xi) d\xi = \int_{-\infty}^u b(x, \xi) d\xi - \int_{\mathbf{R}} b(x, \xi) F(x, \xi) d\xi.$$

We denote

$$B(x, v) := \int_{-\infty}^v b(x, \xi) d\xi. \quad (4.1.2)$$

Then we get

$$\begin{aligned} B(x, u) &= \int_{\mathbf{R}} b(x, \xi) \Lambda(x) \partial_\xi F(x, \xi) d\xi + \int_{\mathbf{R}} b(x, \xi) F(x, \xi) d\xi \\ &= \int_{\mathbf{R}} [b(x, \xi) - \partial_\xi b(x, \xi) \Lambda(x)] F(x, \xi) d\xi. \end{aligned}$$

For  $b(x, \xi)$  solution of the equation

$$b(x, \xi) - \partial_\xi b(x, \xi) \Lambda(x) = a(x, \xi), \quad (4.1.3)$$

we have

$$B(x, u) = \int_{\mathbf{R}} a(x, \xi) F(x, \xi) d\xi.$$

If we impose that  $b$  is bounded, by computation of the solution of (4.1.3), we get

$$B(x, u) = \int_{-\infty}^u b(x, \xi) d\xi = \int_{-\infty}^u \int_0^{+\infty} a(x, \xi + v\Lambda(x)) e^{-v} dv d\xi. \quad (4.1.4)$$

Quite similarly, for  $c(x, \xi)$  solution of the equation

$$c(x, \xi) - \partial_\xi c(x, \xi) \Lambda(x) = \partial_\xi \Phi(x, \xi), \quad (4.1.5)$$

we have

$$C(x, u) = \int_{\mathbf{R}} \partial_{\xi} \Phi(x, \xi) F(x, \xi) d\xi$$

where

$$C(x, u) := \int_{-\infty}^u c(x, \xi) d\xi = \int_{-\infty}^u \int_0^{+\infty} \partial_{\xi} \Phi(x, \xi + v\Lambda(x)) e^{-v} dv d\xi. \quad (4.1.6)$$

Let us start from the kinetic formulation

$$dM_u + \operatorname{div}_x(a(x, \xi)M_u)dt = \partial_{\xi} m - \partial_{\xi} M_u \Phi dW_t + \frac{1}{2} \partial_{\xi} (G^2 \partial_{\xi} M_u) dt. \quad (4.1.7)$$

Since because of (4.1.1), we have

$$\int_{\mathbf{R}} (M_u - \mathbf{1}_{0 > \xi}) d\xi = u,$$

by integrating (4.1.7), we get

$$du + \operatorname{div}_x B(x, u) dt = C(x, u) dW_t.$$

In addition, the modified Maxwellian being the solution of the equation (4.1.1), we actually have an explicit expression for it:

$$M_k(x, \xi) = \int_0^{+\infty} \mathbf{1}_{k > \xi} (\xi - \Lambda(x)u) e^{-u} du. \quad (4.1.8)$$

By easy computations and Fubini's Theorem, we get the following proposition:

**Proposition 4.1.** *We get, for any  $k, k' \in \mathbf{R}$ ,*

- (i)  $\int_{\mathbf{R}} (M_k(x, \xi) - \mathbf{1}_{0 > \xi}(\xi)) d\xi = k,$
- (ii)  $\operatorname{sgn}(M_k(x, \xi) - M_{k'}(x, \xi)) = \operatorname{sgn}(k - k'),$
- (iii)  $\int_{\mathbf{R}} |M_k(x, \xi) - M_{k'}(x, \xi)| d\xi = |k - k'|,$
- (iv)  $\int_{\mathbf{R}} a(x, \xi) M_k(x, \xi) d\xi = B(x, k),$   
 $\int_{\mathbf{R}} \partial_{\xi} \Phi(x, \xi) M_k(x, \xi) d\xi = C(x, k).$

where

$$B(x, k) = \int_{-\infty}^k \int_0^{+\infty} a(x, \xi + v\Lambda(x)) e^{-v} dv d\xi, \quad (4.1.9)$$

$$C(x, k) = \int_{-\infty}^k \int_0^{+\infty} \partial_{\xi} \Phi(x, \xi + v\Lambda(x)) e^{-v} dv d\xi, \quad (4.1.10)$$

and

$$M_k(x, \xi) = \int_0^{+\infty} \mathbf{1}_{k > \xi} (\xi - \Lambda(x)u) e^{-u} du. \quad (4.1.11)$$

## 4.2 Existence of a weak solution

We are now in position to prove the following result:

**Proposition 4.2.** *There exists a weak solution  $u \in L^1(\Omega \times [0, T] \times \mathbf{T}^N)$  to the conservation law with stochastic forcing (1.0.1).*

*Proof.* Let us consider the following test function

$$\varphi(t, x, \xi) = \Theta_R(\xi)\Psi(t, x) \quad (4.2.1)$$

with  $\Theta_R$  defined in section 3.1,  $\Psi \in \mathcal{C}_c([0, T] \times \mathbf{T}^N)$ . Then, applying the kinetic formulation, we get  $\mathbb{P} - a.s.$ ,

$$\begin{aligned} & \int_0^T \langle M_u(t)\Theta_R, \partial_t \Psi \rangle dt + \langle F(0), \varphi(0) \rangle + \int_0^T \langle M_u(t)\Theta_R, a \cdot \nabla \Psi \rangle dt \\ &= m(\partial_\xi \Theta_R \Psi) - \int_0^T \langle M_u(t)\partial_\xi \Phi dW_t, \Theta_R \Psi(t) \rangle dt \\ & - \int_0^T \langle M_u(t)\Phi dW_t, \partial_\xi \Theta_R \Psi(t) \rangle + \frac{1}{2} \int_0^T \langle G^2 \partial_\xi M_u(t), \partial_\xi \Theta_R \Psi(t) \rangle dt. \end{aligned} \quad (4.2.2)$$

We denote  $\langle \cdot, \cdot \rangle_x$  the duality distribution over  $\mathbf{T}_x^N$ . Since, we have

$$0 = \langle \partial_t \mathbf{1}_{0>\xi}, \varphi \rangle = - \langle \mathbf{1}_{0>\xi}, \partial_t \varphi \rangle + \langle \mathbf{1}_{0>\xi}, \varphi(0) \rangle, \quad (4.2.3)$$

using the same arguments as previously, we can rewrite (4.2.2) as follows

$$\begin{aligned} & \int_0^T \langle \int_{\mathbf{R}} (M_u(t) - \mathbf{1}_{0>\xi}) \Theta_R d\xi, \partial_t \Psi \rangle_x dt + \langle \int_{\mathbf{R}} (F(0) - \mathbf{1}_{0>\xi}) \Theta_R(\xi) d\xi, \Psi(0, x) \rangle_x \\ & + \int_0^T \langle \int_{\mathbf{R}} M_u(t) \Theta_R(\xi) a(\xi) d\xi \cdot \nabla \Psi, \mathbf{1} \rangle_x dt \\ &= m(\partial_\xi \Theta_R \Psi) - \int_0^T \langle \int_{\mathbf{R}} M_u(t) \partial_\xi \Phi \Theta_R(\xi) d\xi dW_t, \Psi(t) \rangle_x \\ & - \int_0^T \langle M_u(t) \Phi dW_t, \partial_\xi \Theta_R \Psi(t) \rangle + \frac{1}{2} \int_0^T \langle G^2 \partial_\xi M_u(t), \partial_\xi \Theta_R \Psi(t) \rangle dt. \end{aligned} \quad (4.2.4)$$

Then, using again the dominated convergence for deterministic and stochastic integrals and the properties of the modified Maxwellian stated in Proposition 4.1, we get when  $R \rightarrow \infty$ ,

$$\begin{aligned} & \int_0^T \langle u, \partial_t \Psi \rangle dt + \langle u_0, \Psi(0) \rangle_x + \int_0^T \langle B(x, u) \cdot \nabla \Psi, \mathbf{1} \rangle dt \\ &= \int_0^T \langle -C(x, u) dW_t, \Psi \rangle \end{aligned} \quad (4.2.5)$$

which concludes the proof.  $\square$

## 4.3 Krushkov-like entropies

In this section, though we are not able to obtain the exact Krushkov entropy relations using the deterministic techniques because of the presence of the stochastic term, we still are able to establish some Krushkov-like inequalities.

**Proposition 4.3.** For all  $\varphi \in \mathcal{C}_c^\infty([0, T] \times \mathbf{T}^N \times \mathbf{R})$ , we have

$$\begin{aligned} & \int_{[0, T] \times \mathbf{T}^N \times \mathbf{R}} |u(t, x) - k| \partial_t \varphi(t, x, \xi) dt dx d\xi \\ & + \int_{[0, T] \times \mathbf{T}^N \times \mathbf{R}} [\text{sgn}(u(t, x) - k)(B(x, u(t, x)) - B(x, k)) \text{div}_x \varphi(t, x, \xi)] dt dx d\xi \\ & + \int_{[0, T] \times \mathbf{T}^N \times \mathbf{R}} \text{div}_x(B(x, k)) \text{sgn}(u(t, x) - k) \varphi(t, x, \xi) dt dx d\xi \\ & - \int_{[0, T] \times \mathbf{T}^N \times \mathbf{R}} \text{sgn}(u(t, x) - k) C(x, u(t, x)) dW_t dx d\xi \leq C_\varphi \quad \mathbb{P}\text{-a.s.} \end{aligned} \quad (4.3.1)$$

where  $C_\varphi$  is a constant which depends only on  $\varphi$ .

*Proof.* Previously, we have established that

$$dM_u + \text{div}_x(aM_u)dt = \partial_\xi m - \partial_\xi M_u \Phi dW_t + \frac{1}{2} \partial_\xi(G^2 \partial_\xi M_u)dt \quad (4.3.2)$$

in the sense of distributions  $\mathbb{P}$ -a.s.. We deduce from above that we have

$$d(M_u - M_k) + \text{div}_x(a(M_u - M_k))dt = \partial_\xi m - \partial_\xi M_u \Phi dW_t + \frac{1}{2} \partial_\xi(G^2 \partial_\xi M_u)dt - \text{div}_x(aM_k)dt \quad (4.3.3)$$

in the sense of distributions  $\mathbb{P}$ -a.s.

By a similar reasoning, we actually can extend (3.2.2) to the following test function

$$\tilde{\varphi}(\omega, t, x, \xi) = \frac{1}{\delta^{N+1}} \eta \left( \frac{y-x}{\delta}, \frac{\zeta-\xi}{\delta} \right) P'((M_u - M_k)^\delta(t, y, \zeta)) \Psi(t) \quad (4.3.4)$$

where  $P$  is a  $\mathcal{C}^2(\mathbf{R})$  function such that  $P' \in L^\infty(\mathbf{R})$ ,  $(M_u - M_k)^\delta := (M_u - M_k) * \eta^\delta$  with  $\eta, \eta^\delta$  defined similarly on  $\mathbf{T}^N \times \mathbf{R}$  to  $\theta, \theta^\delta$  on  $[0, T]$  defined on (3.3.6) and (3.3.7). So we obtain that for all  $(y, \zeta)$  in  $\mathbf{T}^N \times \mathbf{R}$ , we have

$$\begin{aligned} & d(M_u - M_k)^\delta P'((M_u - M_k)^\delta) + \text{div}_x(a(M_u - M_k)^\delta) P'((M_u - M_k)^\delta) dt \\ & = \partial_\xi m * \eta^\delta P'((M_u - M_k)^\delta) - (\partial_\xi M_u \Phi) * \eta^\delta P'((M_u - M_k)^\delta) dW_t + \frac{1}{2} \partial_\xi(G^2 \partial_\xi M_u) * \eta^\delta P'((M_u - M_k)^\delta) dt \\ & \quad - \text{div}_x(aM_k) * \eta^\delta P'((M_u - M_k)^\delta) dt - r^\delta P'((M_u - M_k)^\delta) dt \end{aligned} \quad (4.3.5)$$

where  $r^\delta := \text{div}_x(a(M_u - M_k)) * \eta^\delta - \text{div}_x(a(M_u - M_k)^\delta)$  in the sense of distributions over  $(0, T)$   $\mathbb{P}$ -a.s. In order to get the result, we need to consider the above equality against a test function depending on  $(y, \zeta)$ . Then, though we have the above equality for all  $(y, \zeta)$   $\mathbb{P}$ -a.s., we need to establish it  $\mathbb{P}$ -a.s. for all  $(y, \zeta)$ . It is first easy to deduce from above that the result holds true  $\mathbb{P}$ -a.s. for all  $(y, \zeta) \in (\mathbf{T} \cap \mathbf{Q})^N \times \mathbf{Q}$ . Extending this to all  $(y, \zeta) \in \mathbf{T}^N \times \mathbf{R}$  is direct by continuity in all the terms except  $(\partial_\xi M_u \Phi) * \eta^\delta P'((M_u - M_k)^\delta) dW_t$ . Let us now deal with this last term. We consider a test function  $\varphi \in \mathcal{C}_c^\infty(\mathbf{T}_y^N \times \mathbf{R}_\zeta)$ , and are interested in the following terms:

$$\int_0^T (M_u \partial_\xi g_k)^\delta(t, y, \zeta) \Psi(t) P'((M_u - M_k)^\delta(t, y, \zeta)) d\beta_k(t) \varphi(y, \zeta) \quad (4.3.6)$$

and

$$\int_0^T (M_u g_k) * \partial_\xi \eta^\delta(y, \zeta) \Psi(t) P'((M_u - M_k)^\delta(t, y, \zeta)) d\beta_k(t) \varphi(y, \zeta), \quad (4.3.7)$$

for  $k = 1, \dots, d$ . Let us deal with the first one. In order to use a well known result stating that we can find a continuous modification of a stochastic process, we must satisfy in our case the following

assumption, for all  $0 \leq s \leq t \leq T$ ,  $(y, \zeta), (x, \xi)$  in  $\mathbf{T}^N \times \mathbf{R}$ ,

$$\begin{aligned} & \mathbb{E} \left| \int_0^t (M_u \partial_\xi g_k)^\delta(r, y, \zeta) \Psi(r) P'((M_u - M_k)^\delta(r, y, \zeta)) d\beta_k(r) \varphi(y, \zeta) \right. \\ & \quad \left. - \int_0^s (M_u \partial_\xi g_k)^\delta(r, x, \xi) \Psi(r) P'((M_u - M_k)^\delta(r, x, \xi)) d\beta_k(r) \varphi(x, \xi) \right|^\lambda \\ & \leq C(|t - s| + |x - y|)^{N+2+\epsilon} \end{aligned} \quad (4.3.8)$$

with  $\lambda > 1$ ,  $\epsilon > 0$ . Actually, we have the following inequalities for  $n > 1$ :

$$\begin{aligned} & \mathbb{E} \left| \int_0^t (M_u \partial_\xi g_k)^\delta(r, y, \zeta) \Psi(r) P'((M_u - M_k)^\delta(r, y, \zeta)) d\beta_k(r) \varphi(y, \zeta) \right. \\ & \quad \left. - \int_0^s (M_u \partial_\xi g_k)^\delta(r, x, \xi) \Psi(r) P'((M_u - M_k)^\delta(r, x, \xi)) d\beta_k(r) \varphi(x, \xi) \right|^n \\ & \leq C \mathbb{E} \left| \int_s^t (M_u \partial_\xi g_k)^\delta(r, y, \zeta) \Psi(r) P'((M_u - M_k)^\delta(r, y, \zeta)) d\beta_k(r) \varphi(y, \zeta) \right|^n \\ & \quad + C \mathbb{E} \left| \int_0^s \Psi(r) [(M_u \partial_\xi g_k)^\delta(r, y, \zeta) P'((M_u - M_k)^\delta(r, y, \zeta)) \varphi(y, \zeta) \right. \\ & \quad \left. - (M_u \partial_\xi g_k)^\delta(r, x, \xi) P'((M_u - M_k)^\delta(r, x, \xi)) \varphi(x, \xi)] d\beta_k(r) \right|^n \\ & \leq C \mathbb{E} \left( \int_s^t |(M_u \partial_\xi g_k)^\delta(r, y, \zeta) \Psi(r) P'((M_u - M_k)^\delta(r, y, \zeta)) \varphi(y, \zeta)|^2 dr \right)^{n/2} \\ & \quad + C \mathbb{E} \left( \int_0^s |\Psi(r) [(M_u \partial_\xi g_k)^\delta(r, y, \zeta) P'((M_u - M_k)^\delta(r, y, \zeta)) \varphi(y, \zeta) \right. \\ & \quad \left. - (M_u \partial_\xi g_k)^\delta(r, x, \xi) P'((M_u - M_k)^\delta(r, x, \xi)) \varphi(x, \xi)]|^2 dr \right)^{n/2} \\ & \leq C|t - s|^{n/2} + C \left( \int_0^s |\varphi(y, \zeta) - \varphi(x, \xi)|^2 dr \right)^{n/2} \\ & \leq C(|t - s| + |(y, \zeta) - (x, \xi)|)^n \end{aligned} \quad (4.3.9)$$

by assumptions on  $g_k$ ,  $P$  and  $\varphi$  and using the fact that  $M_u$  is bounded. For  $n = N + 3$ , the assumption is satisfied. The reasoning is the same for the second term. So finally, we can conclude that (4.3.5) holds true  $\mathbb{P}$ -a.s. for all  $(y, \zeta) \in \mathbf{T}^N \times \mathbf{R}$ .

The next step consists in using the chain rule. Thus in order to do so, we must reformulate the problem in a Stratonovich version. We have  $\mathbb{P}$ -a.s. for all  $(y, \zeta) \in \mathbf{T}^N \times \mathbf{R}$ ,

$$\begin{aligned} & d(M_u - M_k)^\delta P'((M_u - M_k)^\delta) + \operatorname{div}_x(a(M_u - M_k)^\delta) P'((M_u - M_k)^\delta) dt \\ & = \partial_\xi m * \eta^\delta P'((M_u - M_k)^\delta) - (\partial_\xi M_u \Phi) * \eta^\delta P'(M_u - M_k) \circ dW_t + \frac{1}{2} \partial_\xi (G^2 \partial_\xi M_u) * \eta^\delta P'(M_u - M_k) dt \\ & \quad - \operatorname{div}_x(aM_k) * \eta^\delta P'(M_u - M_k) dt - r^\delta P'(M_u - M_k) dt + \tau^\delta. \end{aligned} \quad (4.3.10)$$

in the sense of distributions over  $(0, T)$ , with  $\tau^\delta$  the corrective terms associated to the passage from  $(\partial_\xi M_u \Phi) * \eta^\delta P'(M_u - M_k) dW_t$  to  $(\partial_\xi M_u \Phi) * \eta^\delta P'(M_u - M_k) \circ dW_t$ . We first deal with the case  $m = 0$ . We integrate (4.3.10) against  $\varphi(y, \zeta)$ . Using in a similar way as previously the dominated convergence theorem for deterministic and stochastic integrals as well as the commutation lemma of Di Perna-Lions [7] to prove the disappearance of  $r^\delta$ , we obtain

$$\begin{aligned} & dP(M_u - M_k) + \operatorname{div}_x(aP(M_u - M_k)) dt - \operatorname{div}_x(a)(P(M_u - M_k) - P'(M_u - M_k)(M_u - M_k)) dt \\ & = \partial_\xi M_u \Phi P'(M_u - M_k) \circ dW_t + \frac{1}{2} \partial_\xi (G^2 \partial_\xi M_u) P'(M_u - M_k) dt \\ & \quad - \operatorname{div}_x(aM_k) P'(M_u - M_k) dt + \lim_{\delta \rightarrow 0} \tau^\delta. \end{aligned} \quad (4.3.11)$$



in the sense of distributions against test functions of the form  $\Psi(t)\varphi(y, \zeta)$   $\mathbb{P}$ -a.s. Noticing that the limit of the corrective terms of  $(\partial_\xi M_u \Phi) * \eta^\delta P'(M_u - M_k) \circ dW_t$  is actually the corrective terms of  $\partial_\xi M_u \Phi P'(M_u - M_k) \circ dW_t$ , we finally have

$$\begin{aligned} dP(M_u - M_k) + \operatorname{div}_x(aP(M_u - M_k))dt - \operatorname{div}_x(a)(P(M_u - M_k) - P'(M_u - M_k)(M_u - M_k))dt \\ = \partial_\xi M_u \Phi P'(M_u - M_k)dW_t + \frac{1}{2}\partial_\xi(G^2\partial_\xi M_u)P'(M_u - M_k)dt \\ - \operatorname{div}_x(aM_k)P'(M_u - M_k)dt \end{aligned} \quad (4.3.12)$$

in the sense of distributions against test functions of the form  $\Psi(t)\varphi(y, \zeta)$   $\mathbb{P}$ -a.s. As previously, by density we can extend it to any test function in  $C_c^\infty([0, T] \times \mathbf{T}^N \times \mathbf{R})$ .

We conclude as in the deterministic case by taking  $P' = \operatorname{sgn}^\alpha$  and  $P(\lambda) = \int_0^\lambda \operatorname{sgn}^\alpha(s)ds$  where  $\operatorname{sgn}^\alpha$  is a smooth regularization of the sign function. By a quite similar reasoning, letting  $\alpha$  goes to 0, we finally get

$$\begin{aligned} d|M_u - M_k| + \operatorname{div}_x(a|M_u - M_k|)dt + \operatorname{div}_x(aM_k)\operatorname{sgn}(M_u - M_k)dt \\ + (\partial_\xi M_u \Phi)\operatorname{sgn}(M_u - M_k)dW_t - \frac{1}{2}\partial_\xi(G^2\partial_\xi M_u)\operatorname{sgn}(M_u - M_k)dt \\ - \operatorname{div}_x(aM_k)\operatorname{sgn}(M_u - M_k)dt = 0 \end{aligned} \quad (4.3.13)$$

in the sense of distributions over  $(0, T) \times \mathbf{T}^N \times \mathbf{R}$   $\mathbb{P}$ -a.s.

Let us finally go back to the case  $m \neq 0$ . Then we have the following inequality

$$\begin{aligned} \int_{\mathbf{T}_y^N \times \mathbf{R}_\zeta} \int_{[0, T] \times \mathbf{T}_x^N \times \mathbf{R}_\xi} \frac{1}{\delta^{N+1}} \partial_\xi \eta\left(\frac{y-x}{\delta}, \frac{\zeta-\xi}{\delta}\right) P'(M_u - M_k)(t, y, \zeta) dm(t, x, \xi) \varphi(t, y, \zeta) dy d\zeta \\ \leq \|P'\|_\infty \int_{[0, T] \times \mathbf{T}_x^N \times \mathbf{R}_\xi} \partial_\xi \varphi * \eta^\delta(x, \xi) dm(t, x, \xi) \\ \leq C \|P'\|_\infty m(K), \end{aligned} \quad (4.3.14)$$

where  $K$  is a compact containing the support of  $\varphi$ .

Finally, integrating the above inequality against a test function of the form  $\varphi(t, x, \xi) = \Theta_R(\xi)\psi(t, x)$  and using the properties associated to the modified Maxwellian stated in Proposition 4.1 together again with the use of dominated convergence theorem for  $R \rightarrow +\infty$  leads to the conclusion.  $\square$

## Appendix

### A Well posedness of the stochastic kinetic model

Our purpose is to establish the existence of a solution to the stochastic BGK model

$$\begin{cases} dF^\varepsilon + \operatorname{div}_x(a(x, \xi)F^\varepsilon)dt + \frac{\Lambda(x)}{\varepsilon}\partial_\xi F^\varepsilon dt = \frac{\mathbf{1}_{u_\varepsilon > \xi} - F^\varepsilon}{\varepsilon} dt - \partial_\xi F^\varepsilon \Phi dW + \frac{1}{2}\partial_\xi(G^2\partial_\xi F^\varepsilon)dt, \\ F_\varepsilon(0) = \mathbf{1}_{u_0 > \xi}, \end{cases} \quad (A.0.15)$$

where we recall  $u_\varepsilon(t, x) = \int_{\mathbf{R}} f_\varepsilon(t, x, \xi)d\xi$  and  $f_\varepsilon = F_\varepsilon + \mathbf{1}_{0 > \xi}$ .

The proof being quite similar to the one in the case of a classical stochastic BGK model, we will not investigate all the details in the following and we will invite the reader to consult [10] for more technical details.

In order to use Duhamel's formula, let us first show an interest in the following auxiliary problem

$$\begin{cases} dX^\varepsilon + \operatorname{div}_x(a(x, \xi)X^\varepsilon)dt + \frac{\Lambda(x)}{\varepsilon}\partial_\xi X^\varepsilon dt = -\partial_\xi X^\varepsilon \Phi dW + \frac{1}{2}\partial_\xi(G^2\partial_\xi X^\varepsilon)dt, \\ X^\varepsilon(s) = X_0. \end{cases} \quad (\text{A.0.16})$$

In order to apply the stochastic characteristic of Kunita [13], we introduce a truncated problem. Of course, we will then pass to the limit on the truncation parameter to get our original problem.

### A.1 A truncated problem

We are interested in the truncated problem described in section 3.1,

$$\begin{cases} dX^\varepsilon + \operatorname{div}_x(a^R(x, \xi)X^\varepsilon)dt + \frac{\Lambda(x)}{\varepsilon}\partial_\xi X^\varepsilon dt = -\partial_\xi X^\varepsilon \Phi^R dW + \frac{1}{2}\partial_\xi(G^{R,2}\partial_\xi X^\varepsilon)dt, \\ X^\varepsilon(s) = X_0. \end{cases} \quad (\text{A.1.1})$$

To apply the method of stochastic characteristics, we must reformulate the problem in Stratonovich form. Using the formula which links Itô and Stratonovich integrals, we are able to prove that if  $X$  is a  $\mathcal{C}^1(\mathbf{T}^N \times \mathbf{R})$ -valued continuous  $(\mathcal{F}_t)$ -semimartingale whose martingale part is given by  $-\int_0^t \partial_\xi \Phi^R dW$  then

$$-\int_0^t \partial_\xi \Phi dW + \frac{1}{2}\int_0^t \partial_\xi(G^{R,2}\partial_\xi X)dr = -\int_0^t \partial_\xi X \Phi^R \circ dW + \frac{1}{4}\int_0^t \partial_\xi X \partial_\xi G^{R,2} dr \quad (\text{A.1.2})$$

and for  $X$  being only a  $\mathcal{D}'(\mathbf{T}^N \times \mathbf{R})$ -valued continuous  $(\mathcal{F}_t)$ -semimartingale, the result is still valid in the sense of distributions (see [10] for more details). The truncated problem can then be reformulated as follows

$$\begin{cases} dX^\varepsilon + \operatorname{div}_x(a^R(x, \xi)X^\varepsilon)dt + \frac{\Lambda(x)}{\varepsilon}\partial_\xi X^\varepsilon dt = -\partial_\xi X^\varepsilon \Phi^R \circ dW + \frac{1}{4}\partial_\xi X^\varepsilon \partial_\xi G^{R,2} dt, \\ X^\varepsilon(s) = X_0. \end{cases} \quad (\text{A.1.3})$$

and applying the method, we obtain the following proposition:

**Proposition A.1.** *Let  $R > 0$ . If  $X_0 \in \mathcal{C}^{3,\mu}(\mathbf{T}^N \times \mathbf{R})$  almost surely, there exists a unique strong solution to (A.1.3) that we will denote  $X^\varepsilon(t, x, \xi; s)$ .*

*In addition,  $X^\varepsilon(t, x, \xi; s)$  is a continuous  $\mathcal{C}^{3,\nu}$ -semimartingale for some  $\nu > 0$  and is represented by*

$$X^\varepsilon(t, x, \xi; s) = \exp\left(\int_s^t -\operatorname{div}_x(a^R(\Psi_{\theta,t}^{\varepsilon,R}(x, \xi)))d\theta\right) X_0(\Psi_{s,t}^{\varepsilon,R}(x, \xi)), \quad (\text{A.1.4})$$

where  $\Psi^R$  is the inverse flow of the stochastic flow associated to the stochastic characteristic system coming from (A.1.3).

**Remark A.1.** *The stochastic characteristic system associated to (A.1.3) is the following*

$$\begin{cases} d\varphi_t^{\varepsilon,R,0} = \frac{\Lambda(\varphi_t^{\varepsilon,R,1}, \dots, \varphi_t^{\varepsilon,R,N})}{\varepsilon} dt - \frac{1}{4}\partial_\xi G^{R,2}(\varphi_t^{\varepsilon,R})dt + \sum_{k=1}^d g_k^R(\varphi_t^{\varepsilon,R}) \circ d\beta_k(t), \\ d\varphi_t^{\varepsilon,R,i} = a_i^R(\varphi_t^{\varepsilon,R})dt \text{ pour } i = 1, \dots, N, \\ d\eta_t^{\varepsilon,R} = \eta_t^{\varepsilon,R}(-\operatorname{div}_x(a^R(\varphi_t^{\varepsilon,R})))dt. \end{cases} \quad (\text{A.1.5})$$

*We notice that in our case, it still presents a dependence on  $\varepsilon$ .*

We denote by  $\mathcal{S}^{\varepsilon,R}$  the solution operator of (A.1.3). We have

$$\mathcal{S}^{\varepsilon,R}(t, s) = \exp\left(\int_s^t -\operatorname{div}_x(a^R(\Psi_{\theta,t}^{\varepsilon,R}(x, \xi)))d\theta\right) X_0(\Psi_{s,t}^{\varepsilon,R}(x, \xi)). \quad (\text{A.1.6})$$

The domain of definition of the solution operator can be extended to  $X_0$  only defined almost everywhere since diffeomorphisms preserve sets of measure zero. Nevertheless, the resulting process will no longer be a strong solution. We obtain the following properties for the operator  $\mathcal{S}^{\varepsilon,R}$ .

**Proposition A.2.** Let  $R > 0$ ,  $\varepsilon > 0$  and  $\mathcal{S}^{\varepsilon,R} = \{\mathcal{S}^{\varepsilon,R}(t,s), 0 \leq s \leq t \leq T\}$  be defined as previously. Then

(i)  $\mathcal{S}^{\varepsilon,R}$  is a family of bounded linear operators on  $L^1(\Omega \times \mathbf{T}^N \times \mathbf{R})$  with a unit operator norm, meaning for any  $X_0 \in L^1(\Omega \times \mathbf{T}^N \times \mathbf{R})$ ,  $0 \leq s \leq t \leq T$ ,

$$\|\mathcal{S}^{\varepsilon,R}(t,s)X_0\|_{L^1(\Omega \times \mathbf{T}^N \times \mathbf{R})} \leq \|X_0\|_{L^1(\Omega \times \mathbf{T}^N \times \mathbf{R})}. \quad (\text{A.1.7})$$

(ii)  $\mathcal{S}^{\varepsilon,R}$  verifies the semi-group law

$$\begin{aligned} \mathcal{S}^{\varepsilon,R}(t,s) &= \mathcal{S}^{\varepsilon,R}(t,r) \circ \mathcal{S}^{\varepsilon,R}(r,s), & 0 \leq s \leq r \leq t \leq T, \\ \mathcal{S}^{\varepsilon,R}(s,s) &= Id, & 0 \leq s \leq T. \end{aligned} \quad (\text{A.1.8})$$

*Sketch of the proof.* The proof being essentially the same as the one in [10], let us only recall here the main ideas and insist on what is a bit different in our case. The idea is the following: we first work with a regularized initial data  $X_0^\delta$  defined as follows  $X_0^\delta(\omega) = (X_0(\omega) * h_\delta)k_\delta$  where  $(k_\delta)$  is a smooth truncation on  $\mathbf{R}$  i.e.  $k_\delta(\xi) = k(\delta\xi)$  in order to build a unique strong solution  $X^{\varepsilon,\delta} = \mathcal{S}^{\varepsilon,R}(t,s)X_0^\delta$  to the following system

$$\begin{cases} dX^{\varepsilon,\delta} + \text{div}_x(a^R(x,\xi)X^{\varepsilon,\delta})dt + \frac{\Lambda(x)}{\varepsilon}\partial_\xi X^{\varepsilon,\delta}dt = -\partial_\xi X^{\varepsilon,\delta}\Phi^R dW + \frac{1}{2}\partial_\xi(G^{R,2}\partial_\xi X^{\varepsilon,\delta})dt, \\ X^{\varepsilon,\delta}(s) = X_0^\delta. \end{cases} \quad (\text{A.1.9})$$

The aim is the following: we integrate the equation (A.1.9) with respect to the variables  $\omega, x, \xi$  and get

$$\begin{aligned} &\mathbb{E} \int_{\mathbf{T}^N} \int_{\mathbf{R}} X^{\varepsilon,\delta}(t,x,\xi) d\xi dx + \mathbb{E} \int_s^t \int_{\mathbf{T}^N} \int_{\mathbf{R}} \text{div}_x(a^R(x,\xi)X^{\varepsilon,\delta}(r,x,\xi)) d\xi dx dr \\ &\quad + \mathbb{E} \int_s^t \int_{\mathbf{T}^N} \int_{\mathbf{R}} \frac{\Lambda(x)}{\varepsilon} \partial_\xi X^{\varepsilon,\delta}(r,x,\xi) d\xi dx dr \\ &= \mathbb{E} \int_{\mathbf{T}^N} \int_{\mathbf{R}} X_0^\delta(x,\xi) d\xi dx - \mathbb{E} \int_s^t \int_{\mathbf{T}^N} \int_{\mathbf{R}} \partial_\xi X^{\varepsilon,\delta}(r,x,\xi) \Phi^R dW d\xi dx dr \\ &\quad + \mathbb{E} \int_s^t \int_{\mathbf{T}^N} \int_{\mathbf{R}} \frac{1}{2} \partial_\xi(G^{R,2}(x,\xi)\partial_\xi X^{\varepsilon,\delta}(r,x,\xi)) d\xi dx dr. \end{aligned} \quad (\text{A.1.10})$$

Then, we want to show that all the terms disappears except for the first term of the left-hand side and the first term of the right-hand side. Let us first focus on the stochastic integral. All we need to do is:

- (1) Prove that this integral is a well defined martingale with zero expected value.
- (2) Use the stochastic Fubini theorem to interchange integrals with respect to  $x, \xi$  and the stochastic one.

Then we will have proven the disappearance of the second term of the right-hand side.

In order to do (1), we must prove that  $\mathbb{E} \int_s^t |\partial_\xi X^{\varepsilon,\delta}(r,x,\xi)g_k^{R,2}| dr < +\infty$ . The expression of  $X^{\varepsilon,\delta}$  being slightly more complicated in our case, we must claim that not only  $\partial_\xi \Psi_{s,r}^{\varepsilon,R}(x,\xi)$  but also  $\partial_{x_i} \Psi_{s,r}^{\varepsilon,R}$  and  $\partial_\xi \partial_{x_i} \Psi_{s,r}^{\varepsilon,R}$  for  $i = 1, \dots, N$  solves a backward stochastic differential equations with bounded coefficients (see [13, Theorem 4.6.5 and Corollary 4.6.6]) since  $g_k^R, X_0^\delta$  and all the partial derivatives of all order of  $a_i^R$  for  $i = 1, \dots, N$  are bounded. Thus, those solutions possess moments of any order which are bounded in  $0 \leq s \leq t \leq T$ ,  $x \in \mathbf{T}^N$ ,  $\xi \in \mathbf{R}$ . By assumption (2.0.7), we have  $\text{div}_x(a(x,\xi)) \geq 0$  for all  $x, \xi$  which leads to the statement (1).

To prove (2), we must establish that

$$\int_{\mathbf{T}^N} \int_{\mathbf{R}} \left( \mathbb{E} \int_s^T (|\partial_\xi X^{\varepsilon,\delta} g_k^R(x, \xi)|^2) dr \right)^{1/2} d\xi dx < +\infty. \quad (\text{A.1.11})$$

By the expression (A.1.4) of  $X^{\varepsilon,\delta}$ , we know that  $\partial_\xi X^{\varepsilon,\delta}$  only contain terms mentioned previously which all are bounded or possess moments of any order which are bounded. Then it is enough to prove that  $X_0^\varepsilon(\Psi_{s,r}^{\varepsilon,R}(x, \xi))$  and  $\nabla_{x,\xi} X_0^\varepsilon(\Psi_{s,r}^{\varepsilon,R}(x, \xi))$  are compactly supported in  $\xi$  for all  $s, r, x, \mathbb{P}$ -a.s. thanks to some growth control on the stochastic flow to conclude (for more details see [10]).

Finally, the second term of the left-hand side disappears thanks to periodic boundary conditions and the third terms of the left-hand side and the right-hand side disappears because of the support compact in  $\xi$  of respectfully  $X^{\varepsilon,\delta}$  and  $G^{R,2}$ . Therefore at the end, we obtain

$$\mathbb{E} \int_{\mathbf{T}^N} \int_{\mathbf{R}} X^{\varepsilon,\delta}(t, x, \xi) d\xi dx = \mathbb{E} \int_{\mathbf{T}^N} \int_{\mathbf{R}} X_0^\delta(x, \xi) d\xi dx. \quad (\text{A.1.12})$$

The statement (i) of Proposition A.2 is then obtained by passing to the limit in  $\delta$  and a use of the Fatou lemma. The statement (ii) of Proposition A.2 is straightforward using the flow property of  $\Psi$ .  $\square$

Moreover, a use of the regularization  $X_0^\delta$  together with the dominated convergence theorem and the dominated convergence theorem for stochastic integrals directly leads to the following result:

**Corollary A.3.** *Let  $R > 0$ . If  $X_0$  is a  $\mathcal{F}_s \otimes \mathcal{B}(\mathbf{T}^N) \otimes \mathcal{B}(\mathbf{R})$ -measurable initial data belonging to  $L^\infty(\Omega \times \mathbf{T}^N \times \mathbf{R})$ , then there exists a weak solution  $X^\varepsilon \in L_{\mathcal{P}_s}^\infty(\Omega \times [s, T] \times \mathbf{T}^N \times \mathbf{R})$  to (A.1.3) meaning for any  $\phi \in \mathcal{C}_c^\infty(\mathbf{T}^N \times \mathbf{R})$ , a.e.  $t \in [s, T]$ ,  $\mathbb{P}$  a.s.,*

$$\begin{aligned} \langle X^\varepsilon(t), \phi \rangle &= \langle X^\varepsilon(0), \phi \rangle + \int_s^t \langle X^\varepsilon(r), a^R \cdot \nabla \phi \rangle dr + \frac{1}{\varepsilon} \int_s^t \langle X^\varepsilon(r), \Lambda(x) \partial_\xi \phi \rangle dr \\ &\quad + \sum_{k=1}^d \int_s^t \langle X^\varepsilon(r), \partial_\xi (g_k^R \phi) \rangle d\beta_k(r) + \frac{1}{2} \int_s^t \langle X^\varepsilon(r), \partial_\xi (G^{R,2} \partial_\xi \phi) \rangle dr. \end{aligned} \quad (\text{A.1.13})$$

Moreover,  $X^\varepsilon$  is represented by  $X^\varepsilon = \mathcal{S}^{\varepsilon,R}(t, s) X_0$ . In particular,  $t \mapsto \langle \mathcal{S}^{\varepsilon,R}(t, s) X_0, \phi \rangle$  is a continuous  $(\mathcal{F}_t)_{t \geq s}$ -semimartingale.

**Remark A.2.** *We point out that in our situation, with the presence of the term  $\text{div}_x(a^R(x, \xi))$  which is not necessary equal to zero, the use of the commutation lemma of Di Perna-Lions ([7, Lemma II.1]) proves that the uniqueness of a weak solution no longer holds precisely in the case where  $\text{div}_x(a^R(x, \xi))$  is not equal to zero.*

## A.2 Passage to the non truncated problem

Our aim is to derive the existence of a weak solution to

$$\begin{cases} dX^\varepsilon + \text{div}_x(a(x, \xi) X^\varepsilon) dt + \frac{\Lambda(x)}{\varepsilon} \partial_\xi X^\varepsilon dt = -\partial_\xi X^\varepsilon \Phi dW + \frac{1}{2} \partial_\xi (G^2 \partial_\xi X^\varepsilon) dt, \\ X^\varepsilon(s) = X_0. \end{cases} \quad (\text{A.2.1})$$

**Proposition A.4.** *If  $X_0$  is a  $\mathcal{F}_s \otimes \mathcal{B}(\mathbf{T}^N) \otimes \mathcal{B}(\mathbf{R})$ -measurable initial data belonging to  $L^\infty(\Omega \times \mathbf{T}^N \times \mathbf{R})$ , then there exists a weak solution  $X^\varepsilon \in L_{\mathcal{P}_s}^\infty(\Omega \times [s, T] \times \mathbf{T}^N \times \mathbf{R})$  to (A.2.1). Moreover, it is represented by*

$$\mathcal{S}^\varepsilon(t, s)(\omega, x, \xi) := \lim_{R \rightarrow +\infty} [\mathcal{S}^{\varepsilon,R}(t, s) X_0](\omega, x, \xi), \quad 0 \leq s \leq t \leq T. \quad (\text{A.2.2})$$

*Proof.* We denote  $\tau^R(s, x, \xi) = \inf\{t \geq s, |\varphi_{s,t}^{\varepsilon,R,0}(x, \xi)| > \frac{R}{2}\}$  with the convention  $\inf \emptyset = T$ .  $\tau^R(s, x, \xi)$  is a stopping time with respect to the filtration  $(\mathcal{F}_t)_{t \geq s}$  for any  $s \in [0, t]$ ,  $x \in \mathbf{T}^N$ ,  $\xi \in \mathbf{R}$ . For more clarity, we do not write the dependence on  $s, x, \xi$  of  $\tau^R$  in the following.

Let us first prove that  $(\tau^R)_{R \in \mathbf{N}}$  is a non decreasing sequence such that  $\tau_\infty := \lim_{R \rightarrow \infty} \tau^R$  is equal to  $T$  almost surely. We know that for any  $R > 0$ , the process  $\varphi^{\varepsilon,R,0}$  satisfies the equation

$$d\varphi_t^{\varepsilon,R,0} = \frac{\Lambda(\varphi_t^{\varepsilon,R,1}, \dots, \varphi_t^{\varepsilon,R,N})}{\varepsilon} dt + \sum_{k=1}^d g_k^R(\varphi_t^{\varepsilon,R}) d\beta_k(t). \quad (\text{A.2.3})$$

We denote  $\tilde{\tau}_R := \inf\{\tau^R, \tau^{R+1}\}$ . The equations satisfy by  $\varphi_t^{\varepsilon,R,0}$  and  $\varphi_t^{\varepsilon,R+1,0}$  being exactly the same on  $[0, \tilde{\tau}_R]$ , then the coefficients being at least Lipschitz we know that  $\varphi_{s,t}^{\varepsilon,R,0}(x, \xi) = \varphi_{s,t}^{\varepsilon,R+1,0}(x, \xi)$  for  $t \in [0, \tilde{\tau}_R]$ . So, we get that  $\tau^R \leq \tau^{R+1}$ . Indeed, let us reason by absurd and suppose that  $\tau^R > \tau^{R+1}$ . Then for  $t \in [\tau^{R+1}, \tau^R]$ , we have

$$|\varphi_{s,t}^{\varepsilon,R,0}(x, \xi)| = |\varphi_{s,t}^{\varepsilon,R+1,0}(x, \xi)| > \frac{R+1}{2} > \frac{R}{2}$$

which is a contradiction with the fact that  $\tau^R$  is the infimum of the times realizing this condition. So the sequence  $(\tau^R)_{R \in \mathbf{N}}$  is a non decreasing and we can define the almost sure limit

$$\tau_\infty := \lim_{R \rightarrow \infty} \tau^R. \quad (\text{A.2.4})$$

We know that  $\tau_\infty$  is still a stopping time. We denote by  $\varphi_{s,t}^{\varepsilon,R,0}(x, \xi) = \varphi_{s,t}^{\varepsilon,R,0}(x, \xi)$  for  $t \in [0, \tau^R]$ . The process is defined on  $[0, \tau_\infty]$  and it is a solution of

$$d\varphi_t^{\varepsilon,R,0} = \frac{\Lambda(\varphi_t^{\varepsilon,R,1}, \dots, \varphi_t^{\varepsilon,R,N})}{\varepsilon} dt + \sum_{k=1}^d g_k^R(\varphi_t^{\varepsilon,R}) d\beta_k(t). \quad (\text{A.2.5})$$

Let us now establish some useful estimates to conclude:

$$\begin{aligned} \varphi_{s,t}^{\varepsilon,R,0} &= \varphi_{in}^{\varepsilon,R,0} + \int_s^t \frac{\Lambda(\varphi_r^{\varepsilon,R,1}, \dots, \varphi_r^{\varepsilon,R,N})}{\varepsilon} dr + \int_s^t \sum_{k=1}^d g_k^R(\varphi_r^{\varepsilon,R}) d\beta_k(r) \\ |\varphi_{s,t}^{\varepsilon,R,0}|^2 &\leq C \left( |\varphi_{in}^{\varepsilon,R,0}|^2 + \left( \int_s^t \left| \frac{\Lambda(\varphi_r^{\varepsilon,R,1}, \dots, \varphi_r^{\varepsilon,R,N})}{\varepsilon} \right| dr \right)^2 + \left| \int_s^t \sum_{k=1}^d g_k^R(\varphi_r^{\varepsilon,R}) d\beta_k(r) \right|^2 \right). \end{aligned}$$

Thus by taking the expectation, using well-known results on martingales and assumption (2.0.10), we get

$$\begin{aligned} \mathbb{E}|\varphi_{s,t}^{\varepsilon,R,0}|^2 &= C \left( \mathbb{E}|\varphi_{in}^{\varepsilon,R,0}|^2 + \frac{M^2 T^2}{\varepsilon^2} + \mathbb{E} \int_s^t \left| \sum_{k=1}^d g_k^R(\varphi_r^{\varepsilon,R}) \right|^2 dr \right) \\ &\leq C \left( \mathbb{E}|\varphi_{in}^{\varepsilon,R,0}|^2 + \frac{M^2 T^2}{\varepsilon^2} + \mathbb{E} \int_s^t (1 + |\varphi_{s,t}^{\varepsilon,R,0}|^2) dr \right) \\ &\leq C \left( \mathbb{E}|\varphi_{in}^{\varepsilon,R,0}|^2 + \frac{M^2 T^2}{\varepsilon^2} + T + \mathbb{E} \int_s^t |\varphi_{s,t}^{\varepsilon,R,0}|^2 dr \right). \end{aligned}$$

Indeed,  $\Lambda$  being continuous on  $\mathbf{T}^N$  compact, there exists a constant  $M$  such that for all  $x \in \mathbf{T}^N$ ,  $|\Lambda(x)| \leq M$ . By Gronwall lemma, we obtain

$$\mathbb{E}|\varphi_{s,t}^{\varepsilon,R,0}|^2 \leq C \left( \mathbb{E}|\varphi_{in}^{\varepsilon,R,0}|^2 + \frac{M^2 T^2}{\varepsilon^2} + T \right) \exp(CT) =: \mathcal{C}^1(\varphi_{in}^{\varepsilon,R,0}, T, \varepsilon). \quad (\text{A.2.6})$$

Moreover, we also have the following

$$\sup_{t \in [s, T]} |\varphi_{s,t}^{\varepsilon, R, 0}|^2 \leq C \left( |\varphi_{in}^{\varepsilon, R, 0}|^2 + \sup_{t \in [s, T]} \left| \int_s^t \frac{\Lambda(\varphi_r^{\varepsilon, R, 1}, \dots, \varphi_r^{\varepsilon, R, N})}{\varepsilon} dr \right|^2 + \sup_{t \in [s, T]} \left| \int_s^t \sum_{k=1}^d g_k^R(\varphi_r^{\varepsilon, R}) d\beta_k(r) \right|^2 \right). \quad (\text{A.2.7})$$

By the same arguments as previously, we deduce that

$$\begin{aligned} \mathbb{E} \sup_{t \in [s, T]} |\varphi_{s,t}^{\varepsilon, R, 0}|^2 &\leq C \left( \mathbb{E} |\varphi_{in}^{\varepsilon, R, 0}|^2 + \frac{M^2 T^2}{\varepsilon^2} + \mathbb{E} \sup_{t \in [s, T]} \left| \int_s^t \sum_{k=1}^d g_k^R(\varphi_r^{\varepsilon, R}) d\beta_k(r) \right|^2 \right) \\ &\leq C \left( \mathbb{E} |\varphi_{in}^{\varepsilon, R, 0}|^2 + \frac{M^2 T^2}{\varepsilon^2} + 4 \int_s^T \mathbb{E} \left| \sum_{k=1}^d g_k^R(\varphi_r^{\varepsilon, R}) \right|^2 dr \right) \\ &\leq C \left( \mathbb{E} |\varphi_{in}^{\varepsilon, R, 0}|^2 + \frac{M^2 T^2}{\varepsilon^2} + \int_s^T \mathbb{E} (1 + |\varphi_{s,t}^{\varepsilon, R, 0}|^2) dr \right) \\ &\leq C \left( \mathbb{E} |\varphi_{in}^{\varepsilon, R, 0}|^2 + \frac{M^2 T^2}{\varepsilon^2} + T + \int_s^T \mathbb{E} |\varphi_{s,t}^{\varepsilon, R, 0}|^2 dr \right) \\ &\leq C \left( \mathbb{E} |\varphi_{in}^{\varepsilon, R, 0}|^2 + \frac{M^2 T^2}{\varepsilon^2} + T + \int_0^T \mathcal{C}^1(\varphi_{in}^{\varepsilon, R, 0}, T, \varepsilon) dr \right) =: \mathcal{C}^2(\varphi_{in}^{\varepsilon, R, 0}, T, \varepsilon). \end{aligned} \quad (\text{A.2.8})$$

Finally, applying the Markov inequality, we get

$$\begin{aligned} \mathbb{P} \left( \sup_{t \in [s, T]} |\varphi_{s,t}^{\varepsilon, R, 0}| \geq \frac{R}{2} \right) &\leq \frac{4}{R^2} \mathbb{E} \left( \sup_{t \in [s, T]} |\varphi_{s,t}^{\varepsilon, R, 0}|^2 \right) \\ &\leq \frac{4 \mathcal{C}^2(\varphi_{in}^{\varepsilon, R, 0}, T, \varepsilon)}{R^2} \end{aligned} \quad (\text{A.2.9})$$

and then

$$\mathbb{P} \left( \sup_{t \in [s, T]} |\varphi_{s,t}^{\varepsilon, R, 0}| < \frac{R}{2} \right) \geq 1 - \frac{4 \mathcal{C}^2(\varphi_{in}^{\varepsilon, R, 0}, T, \varepsilon)}{R^2}. \quad (\text{A.2.10})$$

$(\tau^R)_{R \in \mathbf{N}}$  being a non decreasing sequence, we finally obtain that  $\mathbb{P}(\tau_\infty = T) = 1$  passing to the limit in  $R$ .

Still because of uniqueness we have  $\mathcal{S}^{\varepsilon, R+1}(t, s)X_0 = \mathcal{S}^{\varepsilon, R}(t, s)X_0$  on  $[0, \tau^R(s, x, \xi)]$ . So, we define

$$\mathcal{S}^\varepsilon(t, s)X_0(\omega, x, \xi) := \lim_{R \rightarrow +\infty} [\mathcal{S}^{\varepsilon, R}(t, s)X_0](\omega, x, \xi), \quad 0 \leq s \leq t \leq T. \quad (\text{A.2.11})$$

which is a solution to (A.2.1).  $\square$

**Corollary A.5.** *We have the following properties:*

(i)  $\mathcal{S}^\varepsilon = \{\mathcal{S}^\varepsilon(t, s), 0 \leq s \leq t \leq T\}$  is a family of bounded operators on  $L^1(\Omega \times \mathbf{T}^N \times \mathbf{R})$  with a unit operator norm meaning for any  $X_0 \in L^1(\Omega \times \mathbf{T}^N \times \mathbf{R})$ ,  $0 \leq s \leq t \leq T$ ,

$$\|\mathcal{S}^\varepsilon(t, s)X_0\|_{L^1(\Omega \times \mathbf{T}^N \times \mathbf{R})} \leq \|X_0\|_{L^1(\Omega \times \mathbf{T}^N \times \mathbf{R})}. \quad (\text{A.2.12})$$

(ii)  $\mathcal{S}^{\varepsilon, R}$  verifies the semi-group law

$$\begin{aligned} \mathcal{S}^\varepsilon(t, s) &= \mathcal{S}^\varepsilon(t, r) \circ \mathcal{S}^\varepsilon(r, s), & 0 \leq s \leq r \leq t \leq T, \\ \mathcal{S}^\varepsilon(s, s) &= Id, & 0 \leq s \leq T. \end{aligned} \quad (\text{A.2.13})$$



- [3] F. Berthelin, N. J. Mauser, and F. Poupaud. High-field limit from a kinetic equation to multidimensional scalar conservation laws. *Journal of Hyperbolic Differential Equations*, 04(01):123–145, 2007.
- [4] F. Berthelin and J. Vovelle. A Bhatnagar-Gross-Krook approximation to scalar conservation laws with discontinuous flux. *Proceedings of the Royal Society of Edinburgh, Section: A Mathematics*, 140:953–972, 10 2010.
- [5] F. Berthelin and J. Vovelle. Stochastic isentropic Euler equations. *ArXiv e-prints*, October 2013.
- [6] P. L. Bhatnagar, E. P. Gross, and M. Krook. A Model for Collision Processes in Gases. I. Small Amplitude Processes in Charged and Neutral One-Component Systems. *Phys. Rev.*, 94:511–525, May 1954.
- [7] G. Da Prato and J. Zabczyk. *Stochastic Equations in Infinite Dimensions*. Cambridge University Press, 2014.
- [8] A. Debussche and J. Vovelle. Scalar conservation laws with stochastic forcing. *Journal of Functional Analysis*, 259(4):1014 – 1042, 2010.
- [9] R.J. DiPerna and P.L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Inventiones mathematicae*, 98(3):511–547, 1989.
- [10] Weinan E, K. Khanin, A. Mazel, and Ya. Sinai. Invariant Measures for Burgers Equation with Stochastic Forcing. *Annals of Mathematics*, 151(3):877–960, 2000.
- [11] J. Feng and D. Nualart. Stochastic scalar conservation laws. *Journal of Functional Analysis*, 255(2):313 – 373, 2008.
- [12] M. Hofmanova. A Bhatnagar-Gross-Krook approximation to stochastic scalar conservation laws. *Annales de l’Institut Henri Poincaré Probabilités et Statistiques*, 51(4):1500–1528, 11 2015.
- [13] H. Holden and N.H. Risebro. Conservation laws with a random source. *Applied Mathematics and Optimization*, 36(2):229–241, 1997.
- [14] J.U. Kim. On a stochastic scalar conservation law. *Indiana Univ. Math. J.*, 2003.
- [15] H. Kunita. *Stochastic Flows and Stochastic Differential Equations*. Cambridge Studies in Advanced Mathematics, 1997.
- [16] P.-L. Lions, B. Perthame, and E. Tadmor. A kinetic formulation of multidimensional scalar conservation laws and related equations. *J. Amer. Math. Soc.*, pages 169–191, 1994.
- [17] A. Nouri, A. Omrane, and J.P. Vila. Boundary Conditions for Scalar Conservation Laws from a Kinetic Point of View. *Journal of Statistical Physics*, 1999.
- [18] B. Perthame and E. Tadmor. A kinetic equation with kinetic entropy functions for scalar conservation laws. *Communications in Mathematical Physics*, 136(3):501–517, 1991.
- [19] L. Saint-Raymond. *Hydrodynamic Limits of the Boltzmann Equation*. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 2009.

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