A kinetic model for coagulation-fragmentation

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Abstract

The aim of this paper is to show an existence theorem for a kinetic model of coagulation-fragmentation with initial data satisfying the natural physical bounds, and assumptions of finite number of particles and finite L^p-norm. We use the notion of renormalized solutions introduced dy DiPerna and Lions in [3], because of the lack of *a priori* estimates. The proof is based on weak-compactness methods in L^1, allowed by L^p-norms propagation.

1 Introduction

Coalescence and fragmentation are general phenomena which appear in dynamics of particles, in various fields (polymers chemistry, raindrops formation, aerosols, ...). We can describe them at different scales, which lead to different mathematical points of view. First, we can study the dynamics at the microscopic level, with a system of N particles which undergo successives mergers/break ups in a random way. We refer to the survey [1] for this stochastic approach. Another way to describe coalescence and fragmentation is to consider the statistical properties of the system, introducing the statistical distribution of particles $f(t, m)$ of mass $m > 0$ at time $t \geq 0$ and studying its evolution in time. This approach is rather macroscopic. But we can put in an intermediate level, by considering a density f which depends on more variables, like position x or velocity v of particles, and this description is more precise. Here, we start by discussing models with density, from the original (with $f = f(t, m)$) to the kinetic one (with $f = f(t, x, m, v)$), which is the setting of this work.

Depending on the physical context, the mass variable is discrete (polymers formation) or continuous (raindrops formation). It leads to two sorts of mathematical models, with $m \in \mathbb{N}^\star$ or $m \in (0, +\infty)$, but we focus on the continuous case. To understand the relationship between discrete and continuous equations, see [16].

1.1 The original model

The discrete equations of coagulation have been originally derived by Smoluchowski in [21, 22], by studying the Brownian motion of colloidal particles. It had been extended to the continuous setting by Müller [20], giving the following mathematical model, called the *Smoluchowski's equation of coagulation*:

$$\frac{\partial f}{\partial t}(t,m) = Q_c^+(f,f) - Q_c^-(f,f), \qquad (t,m) \in (0,+\infty)^2. \tag{1.1}$$

This equation describes the evolution of the statistical mass distribution in time. At each time $t > 0$, the term $Q_c^+(f,f)$ represents the gain of particles of mass m created by coalescence between smaller ones, by the reaction

$$\{m^\star\} + \{m - m^\star\} \to \{m\}.$$

The term $Q_c^-(f,f)$ is the depletion of particles of mass m because of coagulation with other ones, following the reaction

$$\{m\} + \{m^\star\} \to \{m + m^\star\}.$$

Namely, we have

$$\begin{cases} Q_c^+(f,f)(t,m) = \dfrac{1}{2}\displaystyle\int_0^m A(m^\star, m-m^\star) f(t,m^\star) f(t,m-m^\star) dm^\star, \\[2ex] Q_c^-(f,f)(t,m) = \displaystyle\int_0^{+\infty} A(m,m^\star) f(t,m) f(t,m^\star) dm^\star, \end{cases}$$

where $A(m,m^\star)$ is the coefficient of coagulation between two particles, which governs the frequency of coagulations, according to the mass of clusters. In his original model, Smoluchowski derived the following expression for A:

$$A(m,m^\star) = \left(m^{1/3} + m^{\star 1/3}\right)\left(m^{-1/3} + m^{\star -1/3}\right). \tag{1.2}$$

In many cases, coalescence is not the only mechanism governing the dynamics of particles, and other effects should be taken into account. A classical phenomenon which also occurs is the fragmentation of particles in two (or more) clusters, resulting from an internal dynamic (we do not deal here with fragmentation processes induced by particles collisions). This binary fragmentation is modeled by linear additional reaction terms in equation (1.1), namely

$$\begin{cases} Q_f^+(f)(t,m) = \displaystyle\int_m^{+\infty} B(m',m) f(t,m') dm', \\[2ex] Q_f^-(f)(t,m) = \dfrac{1}{2} f(t,m) B_1(m), \qquad \text{where} \qquad B_1(m') = \displaystyle\int_0^{m'} B(m',m) dm. \end{cases}$$

The function $B(m',m)$ is the fragmentation kernel, it measures the frequency of the break-up of a mass m' in two clusters m and $m'-m$, for $m < m'$. So, at each time t, the term $Q_f^+(f)$ is the gain of particles of mass m, resulting from the following reaction of fragmentation:

$$\{m'\} \to \{m\} + \{m' - m\},$$

whereas $Q_f^-(f)$ stands for the loss of particles of mass m, because of a break-up into two smaller pieces, by the following way:

$$\{m\} \to \{m^\star\} + \{m - m^\star\}, \quad \text{with} \quad m^\star < m.$$

Thus, the continuous *coagulation-fragmentation equation* writes

$$\frac{\partial f}{\partial t}(t,m) = Q_c^+(f,f) - Q_c^-(f,f) + Q_f^+(f) - Q_f^-(f), \qquad (t,m) \in (0,+\infty)^2. \tag{1.3}$$

In the 90's, many existence and uniqueness results have been proved about this problem, see for instance [23, 24], or [13] for an approach by the semigroups of operators theory. These results are true under various growth hypotheses on kernels A and B, but these assumptions often allow unbounded kernels, which is important from a physical point of view.

However, this coagulation-fragmentation model do not take the spatial distribution of particles into account. This leads to "spatially inhomogeneous" mathematical models, where the density of particles $f(t,x,m)$ depends also of a space variable $x \in \mathbb{R}^3$.

1.2 Spatially inhomogeneous models

A first example consists of diffusive models, corresponding to the situation where particles follow a Brownian motion at the microscopic scale, with a positive and mass-dependent coefficient of diffusion $d(m)$. From a physical point of view, it implies that particles are sufficiently small to undergo the interaction with the medium, i.e the shocks with the molecules of the fluid in which the particles evolve. In the statistical description, a spatial-laplacian appears, giving the *diffusive coagulation-fragmentation equation*:

$$\begin{aligned}&\frac{\partial f}{\partial t}(t,x,m) - d(m)\Delta_x f(t,x,m) = Q_c^+(f,f) - Q_c^-(f,f) + Q_f^+(f) - Q_f^-(f),\\ &(t,x,m) \in (0,+\infty) \times \mathbb{R}^3 \times (0,+\infty).\end{aligned} \tag{1.4}$$

We refer to [15] for a global existence theorem for the discrete diffusive coagulation - fragmentation equation in L^1, and to [17] for the continuous one, improved in [19] (with less restrictive conditions on the kernels), then in [2] (with uniqueness of the solution).

The second way to correct the spatially homogeneous problem is to assume that the particles are transported with a deterministic velocity v. At the statistical level, this adds a linear transport term $v.\nabla_x f$ to the equation (1.3). This velocity can be a given velocity $v = v(t,x,m)$ or the inner velocity of the particles. The first case has been studied in [6], with an existence and uniqueness theorem, and furthermore the continuous dependance on the initial data. Physically, it corresponds to the dynamics of particles with rather low mass which follow a velocity drift depending only on the surrounding fluid. In the second case, particles are also identified by their momentum $p \in \mathbb{R}^3$ in addition to their mass m (with $v = p/m$): we have a kinetic model, which is relevant to describe the dynamics of particles of varying size/mass according to coagulation/fragmentation events, like in aerosols. At the microscopic scale, the coagulation/fragmentation processes become "multi-dimensional", with mass-momentum conservation at each merger/break up according to the following scheme:

$$
\begin{array}{lccc}
Coagulation: & \{m\} + \{m^\star\} & \rightarrow & \{m'\} \\
 & \{p\} + \{p^\star\} & \rightarrow & \{p'\} \\
Fragmentation: & \{m'\} & \rightarrow & \{m\} + \{m^\star\} \\
 & \{p'\} & \rightarrow & \{p\} + \{p^\star\}
\end{array}
$$

where $m' := m + m^\star$, $m > 0$, $m^\star > 0$, and $p' := p + p^\star$.

Thus, in the statistical description, the density depends on time, position, mass and momentum: $f = f(t, x, m, p)$. But even if this kind of kinetic models provides a rather good description of phenomena, it is harder to study, so there are less results than for the diffusive ones. Moreover, it is difficult to know the exact physical form of the kernels. And finally, the numerical aspects are a real problem on these models: because of a high dimension (at least 7 plus time), it seems to be very difficult, maybe impossible, to compute the solutions on a long time.

Concerning the results, a global existence theorem for the sole coagulation has been demonstrated in [7]. The proof is based on L^p-norms dissipation for any formal solution, and on weak-compactness methods in L^1. This result has been extended to a more general class of coalescence operators in [12] (but under stronger restriction on the initial data), with a very different method of proof. For the sole fragmentation, a difficulty is due to the blow-up of kinetic energy, which grows at each microscopic event. Thus, it is reasonable to take the internal energy of particles into account, which balances the gain of kinetic energy during a break up. With that modeling, the work [11] provides global existence for a kinetic fragmentation model, with general growth assumptions on the kernel B, by using correct entropies.

The aim of this work is to combine both of these analysis. We deal with assumptions which are similar to [7], but a little bit more restrictive concerning the kernel A. The obtaining of *a priori* estimates is strongly inspired from [7], with a big difference however. The authors obtained refined estimates, including a dissipative quadratic term thanks to which coagulation bilinear terms make sense, but which is unfortunately not present here because of the balance problems between coagulation and fragmentation operators. Thus, this lack of estimates does not allows us to define well the reaction term of coagulation with only the *a priori* bounds (specifically, we can not say that the bilinear loss term $Q_c^-(f,f)$ lies in L^1_{loc}, as it is shown in subsection 3.2). That is why we use the DiPerna-Lions theory of renormalized solutions, introduced in [3] to show global existence for Boltzmann equation, which presents similar problems.

1.3 Description of the kinetic model and outline of the paper

Now, let us describe precisely the model we study. The parameters which describe the state of a particle are denoted by

$$y := (m, p, e) \in Y := (0, +\infty) \times \mathbb{R}^3 \times (0, +\infty),$$

m for the mass, p the impulsion, and e the internal energy. At the microscopic scale, coalescence and fragmentation conserve total energy (kinetic energy + internal energy), thus we can compute the internal energy of daughter(s) particle(s).

$$Coagulation: \quad \{e\} + \{e^\star\} \quad \rightarrow \quad \{e'\}$$

We have
$$\frac{|p|^2}{2m} + e + \frac{|p^\star|^2}{2m^\star} + e^\star = \frac{|p+p^\star|^2}{2(m+m^\star)} + e',$$

thus

$$e' = e + e^\star + E_-(m, m^\star, p, p^\star), \quad \text{where} \quad E_-(m, m^\star, p, p^\star) := \frac{|m^\star p - m p^\star|^2}{2mm^\star(m+m^\star)} \geq 0$$

(E_- is the loss of kinetic energy resulting from the merger).

$$Fragmentation: \quad \{e'\} \quad \rightarrow \quad \{e\} + \{e^\star\}$$

We have
$$\frac{|p'|^2}{2m'} + e' = \frac{|p|^2}{2m} + e + \frac{|p'-p|^2}{2(m'-m)} + e^\star,$$

thus

$$e^\star = e' - e - E_+(m', m, p', p), \quad \text{where} \quad E_+(m', m, p', p) := \frac{|m' p - m p'|^2}{2mm'(m'-m)} \geq 0$$

(E_+ is the gain of kinetic energy resulting from the break up).

Remark 1.1 *Let us point out the following symmetries:*

$E_-(m^\star, m, p^\star, p) = E_-(m, m^\star, p, p^\star)$ *and* $E_+(m', m, p', p) = E_+(m', m'-m, p', p'-p)$,

and the relation: $E_-(m, m^\star, p, p^\star) = E_+(m+m^\star, m, p+p^\star, p)$, *which is consistent with the two phenomena's reciprocity.*

We use the following notations:

- if $y = (m, p, e)$, $y^\star = (m^\star, p^\star, e^\star)$, then we denote
$$y' := y + y^\star := (m + m^\star, p + p^\star, e + e^\star + E_-(m, m^\star, p, p^\star)),$$

- if $y = (m, p, e)$, $y' = (m', p', e')$, with $m < m'$ and $e < e' - E_+(m', m, p', p)$, then we say that $y < y'$ and we denote
$$y^\star := y' - y := (m' - m, p' - p, e' - e - E_+(m', m, p', p)).$$

With this formalism, we naturally have $(y'-y)+y=y'$, but note carefully that $y<y'$ is not an order relation on Y.

Remark 1.2 *For all* $y' \in Y$, $\quad \{y \in Y, \quad y<y'\} \subset (0,m') \times B_{\sqrt{2m'e'+|p'|^2}} \times (0,e')$. *Denoting* $Y_R := (0,R)\times B_R \times (0,R) \subset Y$, *we have*

$$y<y', \quad y' \in Y_R \quad \Longrightarrow \quad y \in (0,R)\times B_{\sqrt{3}R} \times (0,R) \subset Y_{2R}. \tag{1.5}$$

Finally, we point out that the map $(m',m,p',p,e',e) \mapsto (m',m^\star,p',p^\star,e',e^\star)$ is a diffeomorphism with C^∞ regularity whithin the domain

$$\{0<m<m',\ p,p' \in \mathbb{R}^3,\ 0<e<e'-E_+(m',m,p',p)\} \subset Y^2$$

which preserves volume.

We denote by $f(t,x,m,p,e) = f(t,x,y)$ the particles density, which is a nonnegative function depending on time $t \geq 0$, position $x \in \mathbb{R}^3$, and the mass-momentum-energy variable y. To shorten the notations, we set for each t,x, $f(y)=f(t,x,y)$, or $f=f(t,x,y)$, $f^\star = f(t,x,y^\star)$, and $f'=f(t,x,y')$. The complete model then reads:

$$\begin{cases} \partial_t f + \dfrac{p}{m}.\nabla_x f = Q_c^+(f,f) - Q_c^-(f,f) + Q_f^+(f) - Q_f^-(f), & (ECF) \\ \\ t \in (0,+\infty), \quad x \in \mathbb{R}^3, \quad y=(m,p,e) \in Y, \end{cases}$$

with

$$\begin{cases} Q_c^+(f,f)(y) = \dfrac{1}{2}\displaystyle\int_Y A(y^\star, y-y^\star) f(y^\star) f(y-y^\star) \mathbb{1}_{\{y^\star<y\}} dy^\star, \\ \\ Q_c^-(f,f)(y) = f(y)Lf(y), \qquad Lf(y) := \displaystyle\int_Y A(y,y^\star) f(y^\star) dy^\star, \end{cases}$$

and

$$\begin{cases} Q_f^+(f)(y) = \displaystyle\int_Y B(y',y) f(y') \mathbb{1}_{\{y'>y\}} dy', \\ \\ Q_f^-(f)(y) = \dfrac{1}{2} B_1(y) f(y), \qquad B_1(y') := \displaystyle\int_Y B(y',y) \mathbb{1}_{\{y<y'\}} dy. \end{cases}$$

Functions A et B are respectively the coagulation and fragmentation kernels. They are nonnegative functions, independent of (t,x), which satisfy the natural properties of symmetry:

$$\forall (y,y^\star) \in Y^2, \quad A(y,y^\star) = A(y^\star,y), \tag{1.6}$$

$$\forall (y,y') \in Y^2, \quad y<y', \quad B(y',y) = B(y',y^\star). \tag{1.7}$$

The kernel $A(y, y^\star)$ represents the coalescence rate between two particles y and $y^\star$, whereas $B(y', y)$ is the fragmentation rate for a particle y' which breaks in two clusters y and $y^\star$.

We assume that A fulfils the following structure assumption:

$$\forall (y, y^\star) \in Y^2, \quad A(y, y^\star) \leq A(y, y') + A(y^\star, y'). \tag{1.8}$$

Remark 1.3 *We can insist on the fact that this assumption is more general than the classical Galkin-Tupchiev monotonicity condition:*

$$\forall y < y^\star, \quad A(y, y^\star - y) \leq A(y, y^\star). \tag{1.9}$$

In the "monodimensional" case, the Smoluchowski kernel given by (1.2) *do not satisfy* (1.9) *but satisfies* (1.8)*, that's why the first existence result established in [17] under Galkin-Tupchiev condition was extended in [19] to kernels which satisfy* (1.8) *only.*

We also require that A and B have a mild growth:

$$\forall R > 0, \qquad \int_{Y_R} \frac{A(y, y^\star)}{|y^\star|} dy \underset{|y^\star| \to +\infty}{\longrightarrow} 0, \tag{1.10}$$

$$\forall R > 0, \qquad \int_{Y_R} \frac{B(y', y)}{|y'|} \mathbb{1}_{\{y < y'\}} dy \underset{|y'| \to +\infty}{\longrightarrow} 0, \tag{1.11}$$

and B is truncated as:

$$\exists C_0 > 1, \quad \left\{ \begin{array}{l} m' > C_0 m \\ \text{or} \\ e' + \dfrac{|p'|^2}{2m'} > C_0 \left(e + \dfrac{|p|^2}{2m} \right) \end{array} \right. \quad \Longrightarrow \quad B(y', y) = 0. \tag{1.12}$$

Remark 1.4 *The physical interpretation of this truncature assumption is to prevent the creation of too small clusters compared to the mother particle. From a mathematical point of view, it allows the total number of particles (the L^1-norm of f) to be finite at each time $t > 0$.*

We also need to have B_1 locally bounded:

$$\forall R > 0, \qquad B_1 \in L^\infty(Y_R), \tag{1.13}$$

as well as A:

$$\forall R > 0, \qquad A \in L^\infty(Y_R^2). \tag{1.14}$$

Remark 1.5 *Unfortunately, these assumptions of growth and boundedness are more restrictive, and in the monodimensional case, the Smoluchowski kernel* (1.2) *doesn't satisfy them any more. The examples given in [7] for the sole coagulation, namely*

$$A(m, m^\star, p, p^\star) = (m^\alpha + m^{\star\alpha})^2 \left| \frac{p}{m} - \frac{p^\star}{m^\star} \right|, \quad 0 \leq \alpha < 1/2,$$

(for the dynamics of liquid droplets carried by a gaseous phase) or

$$A(m,m^\star,p,p^\star) = \left(\frac{m+m^\star}{mm^\star}\right)^\alpha \left|\frac{p}{m} - \frac{p^\star}{m^\star}\right|^\gamma, \quad 0 \le \alpha \le 1, \quad -3 < \gamma \le 0,$$

(for a stellar dynamics context) do not fit neither. Here, we need coalescence kernels which are bounded when $m, m^\star \to 0$. But it is difficult to know the exact physical form of the kernels A and B because of the complexity of this kinetic model. Nevertheless, simple kernels given by $A(m,m^\star) = m^\alpha + m^{\star\alpha}$ with $0 < \alpha < 1$ fit.

Finally, we assume that A controls B in the following sense:

$$\exists s > 1, \quad \exists 0 < \delta < \frac{1}{6s-5} < 1,$$

$$\forall y' \in Y, \quad \int_Y \frac{B(y',y)^s}{A(y,y')^{s-1}} \mathbb{1}_{\{y<y'\}} dy \le 1 + m' + \frac{|p'|^2}{2m'} + e' + \frac{1}{2} B_1(y')^\delta. \tag{1.15}$$

Remark 1.6 *This last assumption is more technical, but seems necessary to balance the contributions of the interaction terms $Q_c(f,f)$ and $Q_f(f)$, which are difficult to compare because $Q_c(f,f)$ is quadratic whereas $Q_f(f)$ is linear.*

The paper consists in the proof of the following theorem.

Theorem 1.1 *Let A and B be kernels satisfying (1.6) – (1.8) and (1.10) – (1.15) and let f^0 be a nonnegative initial data which satisfies*

$$K(f^0) := \iint_{\mathbb{R}^3 \times Y} \left(\left(1 + m + \frac{|p|^2}{2m} + e + m|x|^2\right) f^0(x,y) + f^0(x,y)^s \right) dxdy < \infty, \tag{1.16}$$

then for all $T > 0$, there exists $f \in C([0,T], L^1(\mathbb{R}^3 \times Y))$ such that $f(0) = f^0$ and f is a renormalized solution to (ECF). Moreover,

$$a.e\ t \in (0,T), \qquad \iint_{\mathbb{R}^3 \times Y} \left(1 + m + \frac{|p|^2}{2m} + e + m|x|^2\right) f(t,x,y) dxdy \le K_T, \tag{1.17}$$

$$a.e\ t \in (0,T), \qquad \iint_{\mathbb{R}^3 \times Y} f(t,x,y)^s dxdy \le K_T, \tag{1.18}$$

where the constant K_T depends only on C_0, T, $K(f^0)$, s and δ (defined in (1.12) and (1.15)).

Beyond existence problems, there are lots of others interesting subjects to explore. A first one concerns the mass conservation of the solution f, which is still an open problem for such kinetic models, even for the case of the sole coagulation. In the spatially homogeneous case, it has been shown in [5] that total mass is preserved in time under mild growth hypotheses on kernels. But we know that in case of strong coagulation (typically the case of multiplicative kernels), a phenomenon of gelation occurs, which force the total mass of the system to decay from a certain time $T_g < +\infty$. Then, problems of convergence to an equilibrium have been already studied for the spatially homogeneous equation [18], under a detailed balance condition between kernels A and B. We can also mention existence of self-similar solutions [8, 9, 14], always for the spatially homogeneous case.

In a first section, we will derive the *a priori* estimates from the equation, giving the proper setting of the problem. Then, the proof of theorem is based on a well-known stability principle which says that if we are able to pass to the limit in the equation (the set of solutions is closed in a certain sense), then it would be easy to show the existence of a solution, applying the stability result to a sequence of approached problems which we can solve. So, the aim of the last section is to prove rigorously such a stability result and in fact, we work in the context of renormalized solutions, because the reaction term can not be defined as a distribution simply using the *a priori* estimates.

1.4 Different notions of solutions

We discuss here on different notions of solutions, recalling the DiPerna-Lions results. We set $Q(f,f) = Q_c^+(f,f) - Q_c^-(f,f) + Q_f^+(f) - Q_f^-(f)$.

Definition 1.2 *Let f be a nonnegative function, such that $f \in L^1_{loc}((0,+\infty)\times\mathbb{R}^3\times Y)$. We say that f is a renormalized solution of (ECF) if*

$$\frac{Q_c^\pm(f,f)}{1+f} \in L^1_{loc}((0,+\infty)\times\mathbb{R}^3\times Y), \qquad \frac{Q_f^\pm(f)}{1+f} \in L^1_{loc}((0,+\infty)\times\mathbb{R}^3\times Y),$$

and if the function $g := \log(1+f)$ satisfies the renormalized equation

$$\partial_t g + \frac{p}{m}.\nabla_x g = \frac{Q(f,f)}{1+f} \qquad (ECFR)$$

in $\mathcal{D}'((0,+\infty)\times\mathbb{R}^3\times Y)$.

The renormalization makes passing to the limit impossible because of the quotients in the reaction term, that is why we also need another notion of solution: the mild solutions, which only require local integrability in time and provide Duhamel's integral formulations to the problem in which we are able to pass to the limit.

Definition 1.3 *Let f be a nonnegative function, such that $f \in L^1_{loc}((0,+\infty)\times\mathbb{R}^3\times Y)$. We say that f is a mild solution of (ECF) if for almost all $(x,y)\in\mathbb{R}^3\times Y$,*

$$\forall T>0, \qquad Q_c^\pm(f,f)^\sharp(t,x,y) \in L^1((0,T)), \qquad Q_f^\pm(f)^\sharp(t,x,y) \in L^1((0,T)),$$

and

$$\forall 0<s<t<\infty, \quad f^\sharp(t,x,y) - f^\sharp(s,x,y) = \int_s^t Q(f,f)^\sharp(\sigma,x,y)\,d\sigma, \tag{1.19}$$

where $h^\sharp$ denotes the restriction to the characteristic lines of the equation:

$$h^\sharp(t,x,m,p,e) := h(t, x + t\frac{p}{m}, m, p, e).$$

The following results are proved in [3]:

Lemma 1.4

(i) If $Q_c^{\pm}(f,f) \in L^1_{loc}((0,+\infty)\times\mathbb{R}^3\times Y)$ and $Q_f^{\pm}(f) \in L^1_{loc}((0,+\infty)\times\mathbb{R}^3\times Y)$, then the following assertions are equivalent:

- *f is a solution of (ECF) in the sense of distributions.*
- *f is a renormalized solution of (ECF).*
- *f is a mild solution of (ECF).*

(ii) If f is a renormalized solution of (ECF), then for all function $\beta \in C^1([0,+\infty))$ such that $|\beta'(u)| \leq \dfrac{C}{1+u}$, the composed function $\beta(f)$ is a solution of

$$\partial_t \beta(f) + \frac{p}{m}.\nabla_x \beta(f) = \beta'(f) Q(f,f).$$

in the sense of distributions (here, the right side lies in $L^1_{loc}((0,+\infty)\times\mathbb{R}^3\times Y)$).

(iii) f is a renormalized solution of (ECF) if and only if f is a mild solution of (ECF), $\dfrac{Q_c^{\pm}(f,f)}{1+f} \in L^1_{loc}((0,+\infty)\times\mathbb{R}^3\times Y)$ and $\dfrac{Q_f^{\pm}(f)}{1+f} \in L^1_{loc}((0,+\infty)\times\mathbb{R}^3\times Y)$.

2 *A priori* estimates

We consider the Cauchy problem

$$\begin{cases} (ECF) \\ f(0,x,y) = f^0(x,y). \end{cases} \tag{2.20}$$

We suppose in this section that (2.20) admit a sufficiently smooth solution f in order to handle some formal quantities which are conserved or propagated by the equation (ECF). More precisely, we will show the propagation of L^q bounds for the solution along time:

Proposition 2.1 *If the initial data f^0 satisfies*

$$K(f^0) := \iint_{\mathbb{R}^3\times Y} \left(\left(1 + m + \frac{|p|^2}{2m} + e + m|x|^2\right) f^0(x,y) + f^0(x,y)^s \right) dxdy < \infty, \tag{2.21}$$

then for all $T>0$, any classical solution of the Cauchy problem (2.20) satisfies

$$\sup_{t\in[0,T]} \iint_{\mathbb{R}^3\times Y} \left(\left(1 + m + \frac{|p|^2}{2m} + e + m|x|^2\right) f(t,x,y) + f(t,x,y)^q \right) dxdy \leq K_T, \tag{2.22}$$

for all the exponents $q \in (5/6\,,\, s]$, *and also*

$$\int_0^T\int_{\mathbb{R}^3} \left(D_1(f(t,x)) + D_2(f(t,x))\right) dxdt \le K_T, \tag{2.23}$$

where

$$D_1(f(t,x)) := \frac{1}{2}\iint_{Y\times Y} A(y,y^\star)\sup(f,f^\star)\inf(f,f^\star)^s dy^\star dy \ge 0, \tag{2.24}$$

$$D_2(f(t,x)) := \frac{s-\delta}{2}\iint_{Y\times Y} B(y',y) f(t,x,y')^s \mathbb{1}_{\{y<y'\}} dy' dy \ge 0, \tag{2.25}$$

and the constant K_T *depends only on* C_0, T, $K(f^0)$, s *and* δ.

2.1 Basic physical estimates

We start with a fundamental formula, which gives the variation in time of some integral quantities involving the solution f.

Lemma 2.2 *Let* $H(u)$ *be a function with* C^1 *regularity on* $[0,+\infty)$ *and* $\Phi(y)$ *a real or vectorial function. We have*

$$\begin{aligned}
&\frac{d}{dt}\iint_{\mathbb{R}^3\times Y} \Phi(y) H(f(t,x,y)) dxdy \\
= \quad & \frac{1}{2}\iiint_{\mathbb{R}^3\times Y\times Y} Aff^\star \left(\Phi' d_u H(f') - \Phi d_u H(f) - \Phi^\star d_u H(f^\star)\right) dy^\star dy dx \\
+ \quad & \frac{1}{2}\iiint_{\mathbb{R}^3\times Y\times Y} Bf' \left(\Phi d_u H(f) + \Phi^\star d_u H(f^\star) - \Phi' d_u H(f')\right) \mathbb{1}_{\{y<y'\}} dy dy' dx,
\end{aligned} \tag{2.26}$$

where $d_u H = \dfrac{dH}{du}$.

<u>Proof</u>: Using (ECF), we have

$$\begin{aligned}
\frac{d}{dt}\iint_{\mathbb{R}^3\times Y} \Phi(y) H(f) dxdy &= \iint_{\mathbb{R}^3\times Y} \Phi(y)\, d_u H(f)\, \partial_t f dy dx \\
&= \iint_{\mathbb{R}^3\times Y} \Phi(y)\, d_u H(f)\, \left(Q_c^+(f,f) - Q_c^-(f,f)\right) dy dx \\
&\quad + \iint_{\mathbb{R}^3\times Y} \Phi(y)\, d_u H(f)\, \left(Q_f^+(f) - Q_f^-(f)\right) dy dx \\
&\quad - \iint_{\mathbb{R}^3\times Y} div_x(-\Phi(y) H(f)\frac{p}{m}) dy dx.
\end{aligned}$$

The integral with divergence vanishes thanks to Stokes' formula. Whence

$$\begin{aligned}\frac{d}{dt}\iint_{\mathbb{R}^3\times Y}\Phi(y)H(f)dxdy &= \iint_{\mathbb{R}^3\times Y}\Phi(y)\, d_u H(f)\, \left(Q_c^+(f,f)-Q_c^-(f,f)\right) dydx\\ &+ \iint_{\mathbb{R}^3\times Y}\Phi(y)\, d_u H(f)\, \left(Q_f^+(f)-Q_f^-(f)\right) dydx.\end{aligned}$$

Using Fubini's theorem (formally), we can write

$$\begin{aligned}&\frac{d}{dt}\iint_{\mathbb{R}^3\times Y}\Phi(y)H(f)dxdy\\ &= \frac{1}{2}\iiint_{\mathbb{R}^3\times Y\times Y}\Phi(y)\, d_u H(f)\, A(y^\star, y-y^\star)f(y^\star)f(y-y^\star)\mathbb{1}_{\{y^\star<y\}}dy^\star dydx\\ &- \iiint_{\mathbb{R}^3\times Y\times Y}\Phi(y)\, d_u H(f)A(y,y^\star)f(y)f(y^\star)dy^\star dydx\\ &+ \iiint_{\mathbb{R}^3\times Y\times Y}\Phi(y)\, d_u H(f)\, B(y',y)f(y')\mathbb{1}_{\{y'>y\}}dy'dydx\\ &- \frac{1}{2}\iint_{\mathbb{R}^3\times Y}\Phi(y')\, d_u H(f')B_1(y')f(y')dy'dx.\end{aligned}$$

If we change variables $(y^\star, y-y^\star)\to(y^\star,y)$ in the first integral, we obtain

$$\begin{aligned}&\frac{d}{dt}\iint_{\mathbb{R}^3\times Y}\Phi(y)H(f)dxdy\\ &= \frac{1}{2}\iiint_{\mathbb{R}^3\times Y\times Y}\Phi(y+y^\star)\, d_u H(f(y+y^\star))\, A(y^\star,y)f(y^\star)f(y)dy^\star dydx\\ &- \iiint_{\mathbb{R}^3\times Y\times Y}\Phi(y)\, d_u H(f)A(y,y^\star)f(y)f(y^\star)dy^\star dydx\\ &+ \iiint_{\mathbb{R}^3\times Y\times Y}\Phi(y)\, d_u H(f)\, B(y',y)f(y')\mathbb{1}_{\{y'>y\}}dy'dydx\\ &- \frac{1}{2}\iiint_{\mathbb{R}^3\times Y\times Y}\Phi(y')\, d_u H(f')\, B(y',y)f(y')\mathbb{1}_{\{y<y'\}}dydy'dx.\end{aligned}$$

The symmetry of A allows us to write

$$\iiint_{\mathbb{R}^3\times Y\times Y} \Phi(y)\, d_uH(f)A(y,y^\star)f(y)f(y^\star)dy^\star dydx$$
$$= \frac{1}{2}\iiint_{\mathbb{R}^3\times Y\times Y} \Phi(y)\, d_uH(f)A(y,y^\star)f(y)f(y^\star)dy^\star dydx$$
$$+ \frac{1}{2}\iiint_{\mathbb{R}^3\times Y\times Y} \Phi(y^\star)\, d_uH(f^\star)A(y,y^\star)f(y)f(y^\star)dy^\star dydx,$$

using the change of variables $(y, y^\star) \to (y^\star, y)$.
The same applies to B with $(y', y) \to (y', y' - y)$:

$$\iiint_{\mathbb{R}^3\times Y\times Y} \Phi(y)\, d_uH(f)\, B(y',y)f(y')\mathbb{1}_{\{y'>y\}}dy'dydx$$
$$= \frac{1}{2}\iiint_{\mathbb{R}^3\times Y\times Y} \Phi(y)\, d_uH(f)\, B(y',y)f(y')\mathbb{1}_{\{y'>y\}}dy'dydx$$
$$+ \frac{1}{2}\iiint_{\mathbb{R}^3\times Y\times Y} \Phi(y'-y)\, d_uH(f(y'-y))\, B(y',y)f(y')\mathbb{1}_{\{y'>y\}}dy'dydx.$$

□

Applying this lemma with $H(u) = u$, it gives

$$\frac{d}{dt}\iint_{\mathbb{R}^3\times Y} \Phi(y) f dxdy = \frac{1}{2}\iiint_{\mathbb{R}^3\times Y\times Y} Aff^\star\left(\Phi' - \Phi - \Phi^\star\right) dy^\star dydx$$
$$+ \frac{1}{2}\iiint_{\mathbb{R}^3\times Y\times Y} Bf'\left(\Phi + \Phi^\star - \Phi'\right)\mathbb{1}_{\{y<y'\}}dy'dydx. \tag{2.27}$$

Choosing $\Phi(y) = m$, we obtain mass conservation:

$$\frac{d}{dt}\iint_{\mathbb{R}^3\times Y} mf(t,x,y)dxdy = 0. \tag{2.28}$$

With $\Phi(y) = p$, we also get the momentum conservation:

$$\frac{d}{dt}\iint_{\mathbb{R}^3\times Y} pf(t,x,y)dxdy = 0. \tag{2.29}$$

Then, choosing $\Phi(y) = \dfrac{|p|^2}{2m} + e$, we recover the total energy conservation:

$$\frac{d}{dt}\iint_{\mathbb{R}^3\times Y} \left(\frac{|p|^2}{2m} + e\right) f(t,x,y)dxdy = 0. \tag{2.30}$$

Moreover, we can control space momenta:

Lemma 2.3 *For all $T>0$, there exists a constant $C_T>0$ that*

$$\forall t\in[0,T],\quad \iint_{\mathbb{R}^3\times Y} m|x|^2 f(t,x,y)dxdy \le C_T. \tag{2.31}$$

Proof: In view of the equation (ECF) and the Stokes formula, we have

$$\begin{aligned}
\frac{d}{dt}\iint_{\mathbb{R}^3\times Y} m|x|^2 f dxdy &= -\iint_{\mathbb{R}^3\times Y} |x|^2 p.\nabla_x f dxdy \\
&= 2\iint_{\mathbb{R}^3\times Y} x.p\ f(t,x,y)dxdy \\
&\le 2\left(\iint_{\mathbb{R}^3\times Y} m|x|^2 f dxdy\right)^{1/2}\left(\iint_{\mathbb{R}^3\times Y} \frac{|p|^2}{m} f dxdy\right)^{1/2},
\end{aligned}$$

and we conclude with (2.30) and Gronwall's lemma.

$\square$

Finally, we can control the number of particles in finite time:

Lemma 2.4 *We set*

$$N_0 := \iint_{\mathbb{R}^3\times Y} f^0(x,y)dxdy,\quad M_0 := \iint_{\mathbb{R}^3\times Y} m f^0(x,y)dxdy,$$

$$E_0 := \iint_{\mathbb{R}^3\times Y} \left(\frac{|p|^2}{2m}+e\right) f^0(x,y)dxdy.$$

Then, there exists a constant $C>0$ depending only on C_0 that

$$\forall T>0,\quad \forall t\in[0,T],\quad \iint_{\mathbb{R}^3\times Y} f(t,x,y)dxdy \le (N_0+CT(M_0+E_0))e^{CT}+M_0+E_0. \tag{2.32}$$

Proof: We use formula (2.27) with $\Phi(y)=\mathbb{1}_{\{m\le 1,\ e+\frac{|p|^2}{2m}\le 1\}}$. Since Φ is nonnegative and subadditive in the sense of coalescence (ie $\Phi'\le\Phi+\Phi^\star$), we have

$$\begin{aligned}
\frac{d}{dt}\iint_{\mathbb{R}^3\times Y} \mathbb{1}_{\{m\le 1,\ e+\frac{|p|^2}{2m}\le 1\}} f dy dx &\le \frac{1}{2}\iiint_{\mathbb{R}^3\times Y\times Y} B f' \left(\Phi+\Phi^\star-\Phi'\right)\mathbb{1}_{\{y<y'\}} dy dy' dx \\
&= \iiint_{\mathbb{R}^3\times Y\times Y} B f' \left(\Phi-\frac{\Phi'}{2}\right)\mathbb{1}_{\{y<y'\}} dy dy' dx \\
&\le \iiint_{\mathbb{R}^3\times Y\times Y} B f' \Phi \mathbb{1}_{\{y<y'\}} dy dy' dx \\
&= \iiint_{\mathbb{R}^3\times Y\times Y} B f' \mathbb{1}_{\{y<y',\ m\le 1,\ e+\frac{|p|^2}{2m}\le 1\}} dy dy' dx.
\end{aligned}$$

In the last integral, if $m' > C_0$, then, since $m \le 1$, we have $B(y', y) = 0$ according to assumption (1.12). The same applies if $e' + \dfrac{|p'|^2}{2m'} > C_0$. Thus,

$$\begin{aligned}
&\frac{d}{dt}\iint_{\mathbb{R}^3\times Y} f(t,x,y)\mathbb{1}_{\{m\le 1,\ e+\frac{|p|^2}{2m}\le 1\}} dy dx \\
&\qquad \le \iint_{\mathbb{R}^3\times Y}\left(\int_Y B(y',y)\mathbb{1}_{\{y<y'\}} dy\right) f(t,x,y')\mathbb{1}_{\{m'\le C_0,\ e'+\frac{|p'|^2}{2m'}\le C_0\}} dy' dx \\
&\qquad = \iint_{\mathbb{R}^3\times Y} B_1(y') f(t,x,y')\mathbb{1}_{\{m'\le C_0,\ e'+\frac{|p'|^2}{2m'}\le C_0\}} dy' dx.
\end{aligned}$$

Denoting $C := \sup\limits_{y'\in Y_{2C_0}} B_1(y')$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\iint_{\mathbb{R}^3\times Y} f(t,x,y)\mathbb{1}_{\{m\le 1,\ e+\frac{|p|^2}{2m}\le 1\}} dy dx \\
&\qquad \le C\iint_{\mathbb{R}^3\times Y} f(t,x,y) dy dx \\
&\qquad \le C\left(\iint_{\mathbb{R}^3\times Y} f(t,x,y)\mathbb{1}_{\{m\le 1,\ e+\frac{|p|^2}{2m}\le 1\}} dy dx + \iint_{\mathbb{R}^3\times Y} m f(t,x,y) dy dx\right. \\
&\qquad\qquad \left. + \iint_{\mathbb{R}^3\times Y}\left(e+\frac{|p|^2}{2m}\right) f(t,x,y) dy dx\right).
\end{aligned}$$

Using (2.28) and (2.30), we have

$$\frac{d}{dt}\iint_{\mathbb{R}^3\times Y} f(t,x,y)\mathbb{1}_{\{m\le 1,\ e+\frac{|p|^2}{2m}\le 1\}}dydx$$
$$\le\ C\iint_{\mathbb{R}^3\times Y} f(t,x,y)\mathbb{1}_{\{m\le 1,\ e+\frac{|p|^2}{2m}\le 1\}}dydx + C(M_0+E_0).$$

We integrate this inequality in time. Then, Gronwall's lemma provides

$$\forall T>0,\quad \forall t\in[0,T],\quad \iint_{\mathbb{R}^3\times Y} f(t,x,y)\mathbb{1}_{\{m\le 1,\ e+\frac{|p|^2}{2m}\le 1\}}dydx \le (N_0+CT(M_0+E_0))e^{CT}.$$

We conclude noting that

$$\iint_{\mathbb{R}^3\times Y} f(t,x,y)dydx \ \le\ \iint_{\mathbb{R}^3\times Y} f(t,x,y)\mathbb{1}_{\{m\le 1,\ e+\frac{|p|^2}{2m}\le 1\}}dydx + \iint_{\mathbb{R}^3\times Y}\left(m+e+\frac{|p|^2}{2m}\right)f(t,x,y)dydx,$$

and using (2.28) and (2.30) again.

$\square$

To summarize, if we set $E(x,y)=1+m+\dfrac{|p|^2}{2m}+e+m|x|^2$, and if we suppose that the initial data satisfies

$$K(f^0) := \iint_{\mathbb{R}^3\times Y} E(x,y)f^0(x,y)dxdy < +\infty,$$

then, for all $T>0$, there exists a constant K_T (depending on T, C_0 and $K(f^0)$) such that

$$\sup_{t\in[0,T]}\iint_{\mathbb{R}^3\times Y} E(x,y)f(t,x,y)dxdy \le K_T. \tag{2.33}$$

Remark 2.1 *For $\gamma>5$, we have*

$$\iint_{\mathbb{R}^3\times Y}\frac{1}{E^\gamma(x,y)}dxdy < +\infty. \tag{2.34}$$

It will be very useful to show that some L^q bounds of f (for $5/6<q<1$ and $q=s>1$) also propagate in time.

2.2 L^q bounds

Obtaining L^q bounds propagation is more technical, that is why we split the proof in several lemmas.

Lemma 2.5 *Let $\beta \in (5/6\,,1)$. Then, for all $T>0$, there exists a constant K_T (depending on T, C_0 and $K(f^0)$) such that*

$$\sup_{t\in[0,T]} \iint_{\mathbb{R}^3\times Y} f^\beta(t,x,y)dxdy \le K_T. \tag{2.35}$$

<u>Proof</u>: Writing

$$\iint_{\mathbb{R}^3\times Y} f^\beta(t,x,y)dxdy = \iint_{\mathbb{R}^3\times Y} f^\beta(t,x,y)\frac{E^\beta(x,y)}{E^\beta(x,y)}dxdy,$$

we use Young inequality

$$\forall\alpha>1,\quad \forall u\ge 0,\quad \forall v\ge 0,\qquad uv \le \frac{u^\alpha}{\alpha}+\frac{v^{\alpha^\star}}{\alpha^\star} \tag{2.36}$$

with $u=f^\beta(t,x,y)E^\beta(x,y)$, $v=\dfrac{1}{E^\beta(x,y)}$, and $\alpha=\dfrac{1}{\beta}>1$,
and obtain

$$\begin{aligned}\iint_{\mathbb{R}^3\times Y} f^\beta(t,x,y)dxdy \quad &\le \quad \beta\iint_{\mathbb{R}^3\times Y} E(x,y)f(t,x,y)dxdy\\ &+\quad (1-\beta)\iint_{\mathbb{R}^3\times Y}\frac{1}{E^{\frac{\beta}{1-\beta}}(x,y)}dxdy.\end{aligned}$$

We conclude with (2.33) and (2.34).

□

Lemma 2.6 *For any convex and nonnegative function $H\in C^1([0,+\infty))$ such that $H(0)=0$, and for all $t>0$, we have*

$$\begin{aligned}\iint_{\mathbb{R}^3\times Y} H(f(t,x,y))dxdy \quad \le \quad & \iint_{\mathbb{R}^3\times Y} H(f^0(x,y))dxdy\\ -\quad & \frac{1}{2}\int_0^t\iiint_{\mathbb{R}^3\times Y\times Y} A\sup(f,f^\star)H(\inf(f,f^\star))dy^\star dydxd\tau\\ +\quad & \int_0^t\iiint_{\mathbb{R}^3\times Y\times Y} A'H\left(\frac{B}{A'}\right)f'\mathbb{1}_{\{y<y'\}}dydy'dxd\tau\\ -\quad & \frac{1}{2}\int_0^t\iiint_{\mathbb{R}^3\times Y\times Y} Bf'd_uH(f')\mathbb{1}_{\{y<y'\}}dydy'dxd\tau,\end{aligned} \tag{2.37}$$

where $A=A(y,y^\star)$, $A'=A(y,y')$, $B=B(y',y)$.

Proof: The formula (2.26) yields

$$\frac{d}{dt}\iint_{\mathbb{R}^3\times Y} H(f)dxdy \quad = \quad \iiint_{\mathbb{R}^3\times Y\times Y} Aff^\star\left(\frac{d_uH(f')}{2} - d_uH(f)\right)dy^\star dydx$$
$$+ \quad \iiint_{\mathbb{R}^3\times Y\times Y} Bf'\left(d_uH(f) - \frac{d_uH(f')}{2}\right)\mathbb{1}_{\{y<y'\}}dydy'dx. \tag{2.38}$$

Let us rewrite the term $I_1 := \displaystyle\iiint_{\mathbb{R}^3\times Y\times Y} Aff^\star d_uH(f')dy^\star dydx$, by the following way:

$$I_1 = \iiint_{\mathbb{R}^3\times Y\times Y} A\inf(f,f^\star)\sup(f,f^\star)d_uH(f')dy^\star dydx.$$

We use the Young inequality:

$$\forall u>0, \quad \forall v>0, \quad uv \le H(u) + H^\star(v) \tag{2.39}$$

with $u = \sup(f,f^\star)$ and $v = d_uH(f')$, where $H^\star$ stands for the convex conjugate function of H. A simple calculus shows that

$$H^\star(d_uH(u)) = ud_uH(u) - H(u),$$

and this quantity is nonnegative, by the assumptions on H. We denote

$$\Theta(u) := H^\star(d_uH(u)) \ge 0.$$

It leads to the inequality:

$$I_1 \quad \le \quad \iiint_{\mathbb{R}^3\times Y\times Y} A\inf(f,f^\star)H(\sup(f,f^\star))dy^\star dydx$$
$$+ \quad \iiint_{\mathbb{R}^3\times Y\times Y} A\inf(f,f^\star)\Theta(f')dy^\star dydx.$$

We can dominate the second term of the right member using the hypothesis (1.8) by the following way:

$$
\begin{aligned}
&\iiint_{\mathbb{R}^3\times Y\times Y} A\inf(f,f^\star)\Theta(f')dy^\star dydx\\
&\le \iiint_{\mathbb{R}^3\times Y\times Y} (A(y,y+y^\star)+A(y^\star,y+y^\star))\inf(f,f^\star)\Theta(f')dy^\star dydx\\
&\le \iiint_{\mathbb{R}^3\times Y\times Y} A(y,y+y^\star)f\Theta(f')dy^\star dydx + \iiint_{\mathbb{R}^3\times Y\times Y} A(y^\star,y+y^\star)f^\star\Theta(f')dy^\star dydx\\
&= 2\iiint_{\mathbb{R}^3\times Y\times Y} A(y^\star,y+y^\star)f(y^\star)\Theta(f(y+y^\star))dy^\star dydx\\
&= 2\iiint_{\mathbb{R}^3\times Y\times Y} A(y^\star,y)f(y^\star)\Theta(f(y))\mathbb{1}_{\{y^\star<y\}}dy^\star dydx
\end{aligned}
$$

(the last identity resulting from the change of variables $(y^\star, y+y^\star)\to(y^\star,y)$).
This yields

$$
\begin{aligned}
I_1 \;\le\; & \iiint_{\mathbb{R}^3\times Y\times Y} A\inf(f,f^\star)H(\sup(f,f^\star))dy^\star dydx\\
& + \; 2\iiint_{\mathbb{R}^3\times Y\times Y} Af^\star\Theta(f)\mathbb{1}_{\{y^\star<y\}}dy^\star dydx.
\end{aligned}
$$

Thus, we have the following control of the coagulation contribution in (2.38):

$$
\begin{aligned}
&\iiint_{\mathbb{R}^3\times Y\times Y} Aff^\star\left(\frac{d_uH(f')}{2}-d_uH(f)\right)dy^\star dydx\\
\le\; & \frac{1}{2}\iiint_{\mathbb{R}^3\times Y\times Y} A\inf(f,f^\star)H(\sup(f,f^\star))dy^\star dydx\\
& + \iiint_{\mathbb{R}^3\times Y\times Y} Af^\star\Theta(f)\mathbb{1}_{\{y^\star<y\}}dy^\star dydx - \iiint_{\mathbb{R}^3\times Y\times Y} Aff^\star d_uH(f)dy^\star dydx.
\end{aligned}
$$

Now, we can write the right member of this inequality by the following way:

$$
\begin{aligned}
= \quad & - \frac{1}{2}\iiint_{\mathbb{R}^3\times Y\times Y} A\sup(f,f^\star)H(\inf(f,f^\star))dy^\star dydx \\
& + \frac{1}{2}\iiint_{\mathbb{R}^3\times Y\times Y} AfH(f^\star)dy^\star dydx + \frac{1}{2}\iiint_{\mathbb{R}^3\times Y\times Y} Af^\star H(f)dy^\star dydx \\
& + \iiint_{\mathbb{R}^3\times Y\times Y} Af^\star\Theta(f)\mathbb{1}_{\{y^\star<y\}}dy^\star dydx - \iiint_{\mathbb{R}^3\times Y\times Y} Aff^\star d_uH(f)dy^\star dydx \\
= \quad & - \frac{1}{2}\iiint_{\mathbb{R}^3\times Y\times Y} A\sup(f,f^\star)H(\inf(f,f^\star))dy^\star dydx \\
& + \iiint_{\mathbb{R}^3\times Y\times Y} Af^\star\Theta(f)\mathbb{1}_{\{y^\star<y\}}dy^\star dydx - \iiint_{\mathbb{R}^3\times Y\times Y} Af^\star\Theta(f)dy^\star dydx.
\end{aligned}
$$

We deduce

$$
\begin{aligned}
& \iiint_{\mathbb{R}^3\times Y\times Y} Aff^\star\left(\frac{d_uH(f')}{2} - d_uH(f)\right)dy^\star dydx \\
& \quad \leq \quad - \frac{1}{2}\iiint_{\mathbb{R}^3\times Y\times Y} A\sup(f,f^\star)H(\inf(f,f^\star))dy^\star dydx \\
& \quad\quad + \iiint_{\mathbb{R}^3\times Y\times Y} Af^\star\Theta(f)\mathbb{1}_{\{y^\star<y\}}dy^\star dydx - \iiint_{\mathbb{R}^3\times Y\times Y} Af^\star\Theta(f)dy^\star dydx.
\end{aligned}
\tag{2.40}
$$

Then, we can also control the fragmentation contribution:

$$
\iiint_{\mathbb{R}^3\times Y\times Y} Bf'd_uH(f)\mathbb{1}_{\{y<y'\}}dydy'dx - \iiint_{\mathbb{R}^3\times Y\times Y} Bf'\frac{d_uH(f')}{2}\mathbb{1}_{\{y<y'\}}dydy'dx.
$$

We rewrite the first term by the following way:

$$
\iiint_{\mathbb{R}^3\times Y\times Y} Bf'd_uH(f)\mathbb{1}_{\{y<y'\}}dydy'dx = \iiint_{\mathbb{R}^3\times Y\times Y} \frac{B}{A'}A'f'd_uH(f)\mathbb{1}_{\{y<y'\}}dydy'dx
$$

and we use (2.39) again, with $u = \dfrac{B}{A'}$ and $v = d_uH(f)$, whence

$$
\begin{aligned}
&\iiint_{\mathbb{R}^3\times Y\times Y} Bf'\left(d_uH(f)-\frac{d_uH(f')}{2}\right)\mathbb{1}_{\{y<y'\}}dydy'dx\\
\leq &\qquad \iiint_{\mathbb{R}^3\times Y\times Y} H\left(\frac{B}{A'}\right)A'f'\mathbb{1}_{\{y<y'\}}dydy'dx+\iiint_{\mathbb{R}^3\times Y\times Y}A'f'\Theta(f)\mathbb{1}_{\{y<y'\}}dydy'dx\\
&- \iiint_{\mathbb{R}^3\times Y\times Y} Bf'\frac{d_uH(f')}{2}\mathbb{1}_{\{y<y'\}}dydy'dx.
\end{aligned}
\tag{2.41}
$$

Eventually, using (2.38), (2.40) and (2.41), we infer

$$
\begin{aligned}
\frac{d}{dt}\iint_{\mathbb{R}^3\times Y} H(f(t,x,y))dxdy \;\leq\; &- \frac{1}{2}\iiint_{\mathbb{R}^3\times Y\times Y} A\sup(f,f^\star)H(\inf(f,f^\star))dy^\star dydx\\
&- \iiint_{\mathbb{R}^3\times (Y^2-(\{y<y^\star\}\cup\{y^\star<y\}))} Af^\star\Theta(f)dydy^\star dx\\
&+ \iiint_{\mathbb{R}^3\times Y\times Y} A'H\left(\frac{B}{A'}\right)f'\mathbb{1}_{\{y<y'\}}dydy'dx\\
&- \iiint_{\mathbb{R}^3\times Y\times Y} Bf'\frac{d_uH(f')}{2}\mathbb{1}_{\{y<y'\}}dydy'dx.
\end{aligned}
\tag{2.42}
$$

□

Lemma 2.7 *Forall $T>0$, there exists a constant $C_T>0$ depending only on T, the initial values N_0,M_0,E_0 and the truncature parameter C_0 such that for all $t\in[0,T]$,*

$$
\begin{aligned}
\iint_{\mathbb{R}^3\times Y} f(t,x,y)^s dxdy \;\leq\; &\iint_{\mathbb{R}^3\times Y} f^0(x,y)^s dxdy + C_T\\
&-\frac{1}{2}\int_0^T\iiint_{\mathbb{R}^3\times Y\times Y} A\sup(f,f^\star)\inf(f,f^\star)^s dy^\star dydxd\tau\\
&-\left|\frac{s-\delta}{2}\right|\int_0^T\iiint_{\mathbb{R}^3\times Y\times Y} B\,(f')^s\,\mathbb{1}_{\{y<y'\}}dy'dydxd\tau,
\end{aligned}
\tag{2.43}
$$

where s and δ are given by (1.15).

Proof: We use the previous lemma with $H(u) = u^s$. We obtain

$$\begin{aligned}
\iint_{\mathbb{R}^3\times Y} f(t,x,y)^s dxdy \quad &\le \quad \iint_{\mathbb{R}^3\times Y} f^0(x,y)^s dxdy \\
&- \quad \frac{1}{2}\int_0^t\iiint_{\mathbb{R}^3\times Y\times Y} A\sup(f,f^\star)\inf(f,f^\star)^s dy^\star dydxd\tau \\
&+ \quad \int_0^t\iiint_{\mathbb{R}^3\times Y\times Y}\left(\frac{B^s}{A'^{s-1}}\right) f'\mathbb{1}_{\{y<y'\}}dydy'dxd\tau \\
&- \quad \frac{s}{2}\int_0^t\iint_{\mathbb{R}^3\times Y} B_1(y')(f')^s dy'dxd\tau.
\end{aligned}$$

According to (1.15) and (2.33),

$$\begin{aligned}
&\int_0^t\iiint_{\mathbb{R}^3\times Y\times Y}\left(\frac{B^s}{A'^{s-1}}\right) f'\mathbb{1}_{\{y<y'\}}dydy'dxd\tau \\
&\quad\le \quad \int_0^t\iint_{\mathbb{R}^3\times Y}\left(1+m'+\frac{|p'|^2}{2m'}+e'\right) f'dy'dxd\tau + \frac{1}{2}\int_0^t\iint_{\mathbb{R}^3\times Y} B_1(y')^\delta f'dy'dxd\tau \\
&\quad\le \quad TK_T + \frac{1}{2}\int_0^t\iint_{\mathbb{R}^3\times Y} B_1(y')^\delta f'^{s\delta} f'^{1-s\delta}dy'dxd\tau.
\end{aligned}$$

We apply the Young inequality again with the exponent $1/\delta > 1$:

$$\begin{aligned}
B_1(y')^\delta f'^{s\delta} f'^{1-s\delta} \quad &\le \quad \frac{\left(B_1(y')^\delta f'^{s\delta}\right)^{1/\delta}}{1/\delta} + \frac{\left(f'^{1-s\delta}\right)^{(1/\delta)^\star}}{(1/\delta)^\star} \\
&= \quad \delta B_1(y')f'^s + (1-\delta)f'^{\frac{1-s\delta}{1-\delta}}.
\end{aligned}$$

Thus we deduce

$$\begin{aligned}
\iint_{\mathbb{R}^3\times Y} f(t,x,y)^s dxdy \quad &\le \quad \iint_{\mathbb{R}^3\times Y} f^0(x,y)^s dxdy \\
&- \quad \frac{1}{2}\int_0^t\iiint_{\mathbb{R}^3\times Y\times Y} A\sup(f,f^\star)\inf(f,f^\star)^s dy^\star dydxd\tau \\
&+ \quad TK_T + \frac{1-\delta}{2}\int_0^t\iint_{\mathbb{R}^3\times Y} f'^{\frac{1-s\delta}{1-\delta}}dy'dxd\tau \\
&+ \quad \frac{\delta-s}{2}\int_0^t\iint_{\mathbb{R}^3\times Y} B_1(y')(f')^s dy'dxd\tau.
\end{aligned}$$

We can use (2.35) since $\dfrac{1-s\delta}{1-\delta} \in (\,5/6\,,\,1\,)$.

$$\begin{aligned}\iint_{\mathbb{R}^3\times Y} f(t,x,y)^s dxdy \;\; &\leq \;\; \iint_{\mathbb{R}^3\times Y} f^0(x,y)^s dxdy + C_T \\ &- \;\; \frac{1}{2}\int_0^t\iiint_{\mathbb{R}^3\times Y\times Y} A\sup(f,f^\star)\inf(f,f^\star)^s dy^\star dydxd\tau \\ &+ \;\; \frac{\delta-s}{2}\int_0^t\iint_{\mathbb{R}^3\times Y} B_1(y')(f')^s dy'dxd\tau.\end{aligned}$$

We conclude noting that $\delta < \dfrac{1}{6s-5} < 1 < s$.

$\square$

3 A stability result

The proof of theorem 1.1 relies on a stability theorem, which claims that we can pass to the limit in the equation (ECF) in a certain sense, namely in an integral formulation.

Definition 3.1 *Let $T>0$ and let f^0 be a nonnegative initial data which satisfies (2.21). A weak solution of (2.20) is a nonnegative function $f \in C([0,T], L^1(\mathbb{R}^3\times Y))$, verifying the estimates (2.22) and (2.23), satisfying (ECF) in $\mathcal{D}'((0,+\infty)\times\mathbb{R}^3\times Y)$, and such that $f(0)=f^0$.*

Now let us state the result we will prove in this section:

Theorem 3.2 *Let $(f_n)_{n\geq 1}$ be a sequence of weak solutions of (2.20), with initial data f_n^0, and such that*

$$\forall n\in\mathbb{N}, \qquad f_n \in W^{1,1}((0,+\infty)\times\mathbb{R}^3\times Y), \tag{3.44}$$

$$\sup_{n\geq 1}\sup_{t\in[0,T]}\iint_{\mathbb{R}^3\times Y}\left(\left(1+m+\frac{|p|^2}{2m}+e+m|x|^2\right)f_n(t,x,y)+f_n(t,x,y)^q\right)dxdy \leq K_T, \tag{3.45}$$

for all the exponents $q\in(\,5/6\,,\,s\,]$, and also

$$\sup_{n\geq 1}\int_0^T\int_{\mathbb{R}^3}\left(D_1(f_n(t,x))+D_2(f_n(t,x))\right)dxdt \leq K_T. \tag{3.46}$$

(the a priori estimates hold uniformly in n).
Then, up to a subsequence, $f_n \rightharpoonup f$ weakly in $L^1((0,T)\times\mathbb{R}^3_{loc}\times Y)$, where f is a renormalized solution of (ECF). Furthermore, $f\in C([0,T],L^1(\mathbb{R}^3\times Y))$.

3.1 Weak compactness of (f_n)

Let $0 < T < \infty$. The bounds on f_n provides some weak compactness, and thus the existence of a limit f after extraction.

Lemma 3.3 *For all $R > 0$, the sequence $(f_n)_{n\geq 1}$ is weakly compact in $L^1((0,T) \times B_R \times Y)$.*

Proof: We set $\Phi(\xi) := \xi^s$ and $\Psi(m,p,e) := m + \dfrac{|p|^2}{2m} + e$. The function Φ is nondecreasing, nonnegative and $\Phi(\xi)/\xi \underset{\xi\to+\infty}{\longrightarrow} +\infty$, Ψ is nonnegative and $\Psi(y) \underset{|y|\to+\infty}{\longrightarrow} +\infty$. The estimate (3.45) gives

$$\sup_{n\geq 1} \int_0^T \int_{B_R} \int_Y \left((1+\Psi(y)) f_n + \Phi(f_n) \right) dt dx dy < +\infty,$$

and we conclude by Dunford-Pettis theorem.

□

Thus, there exists a nonnegative function f such that for all $R > 0$, $f_n \rightharpoonup f$ in $L^1((0,T) \times B_R \times Y)$ for a subsequence (not relabeled). Moreover, we can show easily (diagonal extraction) that the subsequence is not depending on R. Then we notice that in fact, $f \in L^\infty((0,T), L^1(\mathbb{R}^3 \times Y))$ and

$$a.e\ t \in (0,T), \qquad \iint_{\mathbb{R}^3\times Y} \left(1 + m + \frac{|p|^2}{2m} + e + m|x|^2\right) f(t,x,y) dx dy \leq K_T. \quad (3.47)$$

Moreover, since the function $\xi \mapsto |\xi|^s$ is convex, we have

$$a.e\ t \in (0,T), \qquad \iint_{\mathbb{R}^3\times Y} f(t,x,y)^s dx dy \leq K_T. \quad (3.48)$$

3.2 Weak compactness of the renormalized coalescence term

The bounds on f_n are not enough to define the term $Q_c^-(f_n,f_n)$ as a distribution, unlike the term $Q_c^+(f_n,f_n)$. In fact, since for all $R,T > 0$,

$$\int_0^T \int_{B_R} \int_{Y_R} Q_c^-(f_n,f_n) dy dx dt = \int_0^T \int_{B_R} \int_{Y_R} \int_Y A(y,y^\star) f_n(y) f_n(y^\star) dy^\star dy dx dt,$$

we have a bound on the contribution corresponding to $\inf(f_n, f_n^\star) > 1$ due to the estimate (2.23). But the other contribution, where $\inf(f_n, f_n^\star) \leq 1$, requires strong integrability assumptions on A to be finite, which are not reasonable. So, it seems that renormalization is necessary to obtain well-defined and weakly compact coalescence terms.

Lemma 3.4 *For all $R > 0$, the sequence $(Q_c^+(f_n,f_n))_{n\geq 1}$ is weakly compact in $L^1((0,T) \times B_R \times Y_R)$, where $Y_R := (0,R) \times B_R \times (0,R)$.*

Proof: Let E be a measurable subset of $(0,T)\times B_R\times Y_R$. We set $\varphi(t,x,y) := \mathbb{1}_E(t,x,y)$. Performing the change of variables $(y,y^\star)\to(y^\star,y-y^\star)$, we obtain

$$\int_{Y_R} Q_c^+(f_n,f_n)(y)\varphi(y)dy = \frac{1}{2}\int_{Y_R}\int_Y A(y^\star,y-y^\star)f_n(y^\star)f_n(y-y^\star)\varphi(y)\mathbb{1}_{\{y^\star<y\}}dy^\star dy$$

$$= \frac{1}{2}\iiint_{\substack{0<m<R\\ p\in\mathbb{R}^3\\ 0<e<R}}\iiint_{\substack{0<m^\star<R-m\\ p^\star\in B(-p,R)\\ 0<e^\star<R-e-E_-(m,m^\star,p,p^\star)}} A(y,y^\star)f_n(y)f_n(y^\star)\varphi(y+y^\star)dy^\star dy.$$

In fact, we only integrate over $p\in B_{2R}$ because

$$\frac{|p|^2}{2m}\le\frac{|p|^2}{2m}+\frac{|p^\star|^2}{2m^\star}=\frac{|p+p^\star|^2}{2(m+m^\star)}+E_-(m,m^\star,p,p^\star)\le\frac{R^2}{2(m+m^\star)}+R,$$

which yields

$$|p|^2\le R^2\frac{m}{m+m^\star}+2mR\le 3R^2$$

(and the same applies to $p^\star$ because of the symmetry in the previous computation). Thus we have

$$\int_{Y_R} Q_c^+(f_n,f_n)(y)\varphi(y)dy = \frac{1}{2}\iiint_{\substack{0<m<R\\ p\in B_{2R}\\ 0<e<R}}\iiint_{\substack{0<m^\star<R-m\\ p^\star\in B(-p,R)\\ 0<e^\star<R-e-E_-(m,m^\star,p,p^\star)}} Af_nf_n^\star\varphi' dy^\star dy.$$

Using the inequality

$$Af_nf_n^\star\le\frac{1}{M^{s-1}}A\sup(f_n,f_n^\star)\inf(f_n,f_n^\star)^s\mathbb{1}_{\{\inf(f_n,f_n^\star)>M\}}+MA\sup(f_n,f_n^\star)\mathbb{1}_{\{\inf(f_n,f_n^\star)\le M\}},\tag{3.49}$$

we obtain

$$\begin{aligned}\int_{Y_R}Q_c^+(f_n,f_n)(y)\varphi(y)dy &\le \frac{D_1(f_n(t,x))}{M^{s-1}}+\frac{M}{2}\int_{Y_{2R}}\int_{Y_{2R}}A\sup(f_n,f_n^\star)\varphi' dy^\star dy\\ &\le \frac{D_1(f_n(t,x))}{M^{s-1}}+M\int_{Y_{2R}}\int_{Y_{2R}}Af_n\varphi' dy^\star dy\\ &\le \frac{D_1(f_n(t,x))}{M^{s-1}}+M\|A\|_{\infty,Y_{2R}^2}\int_{Y_{2R}}\int_{Y_{2R}}f_n\varphi' dy^\star dy.\end{aligned}$$

We fix $\varepsilon > 0$ and choose M such that $1/M^{s-1} \leq \varepsilon$.
So, we can write

$$\begin{aligned} \int_{Y_R} Q_c^+(f_n, f_n)(y)\varphi(y)dy &\leq \varepsilon D_1(f_n(t,x)) + M\|A\|_{\infty, Y_{2R}^2} \int_{Y_{2R}} \int_{Y_{2R}} f_n \varphi' dy^\star dy \\ &\leq \varepsilon D_1(f_n(t,x)) + M\|A\|_{\infty, Y_{2R}^2} \int_Y \int_Y f_n \varphi^\star dy^\star dy. \end{aligned}$$

Eventually, in view of (3.45) and (3.46), we obtain

$$\begin{aligned} \int_0^T \int_{B_R} \int_{Y_R} Q_c^+(f_n, f_n)\varphi(t,x,y)dydxdt &\leq M\|A\|_{\infty, Y_{2R}^2} \int_0^T \int_{B_R} \int_Y \int_Y f_n \varphi^\star dy^\star dydxdt \\ &+ \varepsilon K_T. \end{aligned}$$

We conclude by letting $mes(E) \to 0$ and using the weak compactness of (f_n).

$\square$

Corollary 3.5 *For all $R > 0$, the sequence $\left(\dfrac{Q_c^+(f_n, f_n)}{1 + f_n}\right)_{n \geq 1}$ is weakly compact in $L^1((0,T) \times B_R \times Y_R)$, where $Y_R := (0,R) \times B_R \times (0,R)$.*

Proof: It's obvious by the previous lemma and Dunford-Pettis theorem since $\dfrac{Q_c^+(f_n, f_n)}{1 + f_n} \leq Q_c^+(f_n, f_n)$.

$\square$

Lemma 3.6 *For all $R > 0$, $Lf_n \rightharpoonup Lf$ weakly in $L^1((0,T) \times B_R \times Y_R)$, where $Y_R := (0,R) \times B_R \times (0,R)$.*

Proof: Let $\varphi(t,x,y) \in L^\infty((0,T) \times B_R \times Y_R)$. We have

$$\int_{Y_R} Lf_n(y)\varphi(y)dy = \int_{Y_R} \int_Y A(y, y^\star) f_n(y^\star)\varphi(y)dy^\star dy.$$

We fix $\varepsilon > 0$ and, in view of the assumptions (1.6) and (1.10), we choose $R^\star > 0$ such that

$$\forall |y^\star| > R^\star, \qquad \int_{Y_R} \frac{A(y, y^\star)}{|y^\star|} dy \leq \varepsilon.$$

We can write

$$\begin{aligned} \int_0^T \int_{B_R} \int_{Y_R} Lf_n \, \varphi dydxdt &= \int_0^T \int_{B_R} \int_{Y_R} \int_{Y_{R^\star}} A(y, y^\star) f_n(y^\star)\varphi(y)dy^\star dydxdt \\ &+ \int_0^T \int_{B_R} \int_{Y_R} \int_{Y - Y_{R^\star}} A(y, y^\star) f_n(y^\star)\varphi(y)dy^\star dydxdt. \end{aligned}$$

First,

$$\int_0^T \int_{B_R} \int_{Y_R} \int_{Y_{R^\star}} A f_n(y^\star)\varphi(y) dy^\star dy dx dt \underset{n}{\longrightarrow} \int_0^T \int_{B_R} \int_{Y_R} \int_{Y_{R^\star}} A f(y^\star)\varphi(y) dy^\star dy dx dt.$$

Indeed, setting $\theta(t,x,y^\star) = \displaystyle\int_{Y_R} A(y,y^\star)\varphi(t,x,y)dy$, we have

$$\int_0^T \int_{B_R} \int_{Y_R} \int_{Y_{R^\star}} A f_n(y^\star)\varphi(y) dy^\star dy dx dt = \int_0^T \int_{B_R} \int_{Y_{R^\star}} \theta(t,x,y^\star) f_n(t,x,y^\star) dy^\star dx dt$$

and we conclude by lemma 3.3, because the assumption (1.14) implies
$\theta \in L^\infty((0,T) \times B_R \times Y_{R^\star})$.
Moreover,

$$\left| \int_0^T \int_{B_R} \int_{Y_R} \int_{Y-Y_{R^\star}} A f_n(y^\star)\varphi(y) dy^\star dy dx dt \right| \leq \varepsilon \left\| \varphi \right\|_\infty \int_0^T \int_{B_R} \int_{Y-Y_{R^\star}} |y^\star| f_n(y^\star) dy^\star dx dt$$

$$\leq \varepsilon T \left\| \varphi \right\|_\infty K_T,$$

and the inequality (3.47) yields

$$\left| \int_0^T \int_{B_R} \int_{Y_R} \int_{Y-Y_{R^\star}} A f(y^\star)\varphi(y) dy^\star dy dx dt \right| \leq \ \varepsilon T \left\| \varphi \right\|_\infty K_T.$$

Finally, we infer

$$\left| \int_0^T \int_{B_R} \int_{Y_R} L f_n \, \varphi dy dx dt - \int_0^T \int_{B_R} \int_{Y_R} L f \, \varphi dy dx dt \right| \leq \underset{n \to +\infty}{o(1)} + C(T,R,\varphi)\varepsilon.$$

□

Corollary 3.7 *For all $R > 0$, the sequence $\left(\dfrac{Q_c^-(f_n,f_n)}{1+f_n} \right)_{n \geq 1}$ is weakly compact in $L^1((0,T) \times B_R \times Y_R)$, where $Y_R := (0,R) \times B_R \times (0,R)$.*

Proof: It's obvious because $\dfrac{Q_c^-(f_n,f_n)}{1+f_n} = \dfrac{f_n}{1+f_n} L f_n \leq L f_n$.

□

3.3 Weak convergence of the fragmentation term

Since they are linear, the fragmentation terms easily pass to the limit, and we have the following lemma.

Lemma 3.8 *For all $R > 0$, we have* *(i)* $Q_f^+(f_n) \rightharpoonup Q_f^+(f)$ *weakly in* $L^1((0,T) \times B_R \times Y_R)$,

(ii) $Q_f^-(f_n) \rightharpoonup Q_f^-(f)$ *weakly in* $L^1((0,T) \times B_R \times Y_R)$,

where $Y_R := (0,R) \times B_R \times (0,R)$.

Proof: The part (ii) results immediately from (1.13), and the proof of (i) is the same as lemma 3.6.

□

3.4 Strong compactness of y-averages

Strong compactness is needed to pass to the limit in coalescence terms (because they are quadratic), that's why we use the following averaging lemma, inspired by [7], [3], and [4]:

Theorem 3.9 *Let* (g_n) *be a bounded sequence in* $L^1((0,T) \times \mathbb{R}^3 \times Y)$ *and weakly compact in* $L^1((0,T) \times B_R \times Y_R)$*, for all* $R > 0$*. Let* (G_n) *be a bounded sequence in* $L^1((0,T) \times B_R \times Y_R)$ *for all* $R > 0$*. We assume that*

$$\partial_t g_n + \frac{p}{m}.\nabla_x g_n = G_n \quad in \quad \mathcal{D}'((0,+\infty) \times \mathbb{R}^3 \times Y).$$

Then, for any function $\Psi \in L^\infty(Y^2)$*, with compact support, the sequence*

$$\left(\int_Y g_n(t,x,y)\Psi(y,y^\star)dy \right)_{n\in\mathbb{N}}$$

is strongly compact in $L^1((0,T) \times B_R \times Y_R)$*, for all* $R > 0$*.*

This result can be improved:

Corollary 3.10 *With the assumptions of theorem 3.9, we also have:*
for all $R > 0$ *and for any function* $\Psi \in L^\infty((0,T) \times B_R \times Y_R^2)$*, the sequence*

$$\left(\int_Y g_n(t,x,y)\Psi(t,x,y,y^\star)dy \right)_{n\in\mathbb{N}}$$

is strongly compact in $L^1((0,T) \times B_R \times Y_R)$*.*

Proof: The case of separated variables is obvious. Then, we proceed by a density argument as in [3].

□

Corollary 3.11 *With the assumptions of theorem 3.9, we also have: for all $R > 0$ and for any sequence (Ψ_n) bounded in $L^\infty((0,T) \times B_R \times Y_R)$ which converges a.e to $\Psi \in L^\infty((0,T) \times B_R \times Y_R)$, the sequence*

$$\left(\int_Y g_n(t,x,y)\Psi_n(t,x,y)dy \right)_{n\in\mathbb{N}}$$

is strongly compact in $L^1((0,T) \times B_R)$.

Proof: Let $\varepsilon > 0$. The sequence (g_n) being weakly compact in $L^1((0,T) \times B_R \times Y_R)$, there exists $\delta > 0$ such that

$$\forall E \in \mathcal{B}((0,T) \times B_R \times Y_R), \quad |E| < \delta, \qquad \sup_n \iiint_E |g_n| dtdxdy \leq \varepsilon.$$

Then, by Egoroff theorem, there exists $E_0 \in \mathcal{B}((0,T) \times B_R \times Y_R)$ such that $|E_0| < \delta$ and Ψ_n converge uniformly to Ψ on $E_1 := ((0,T) \times B_R \times Y_R) \setminus E_0$. Whence

$$\begin{aligned} \left\| \int_{Y_R} g_n \Psi_n dy - \int_{Y_R} g_n \Psi dy \right\|_{L^1((0,T)\times B_R)} &\leq \int_0^T \int_{B_R} \int_{Y_R} |g_n||\Psi_n - \Psi| dydxdt \\ &\leq 2C\ \varepsilon + \sup_{E_1} |\Psi_n - \Psi| \iiint_{E_1} |g_n| dydxdt \\ &= 2C\ \varepsilon + \underset{n\to+\infty}{o(1)}\ . \end{aligned}$$

We infer

$$\left\| \int_{Y_R} g_n \Psi_n dy - \int_{Y_R} g_n \Psi dy \right\|_{L^1((0,T)\times B_R)} \xrightarrow{n} 0.$$

The sequence $\left(\int_{Y_R} g_n \Psi dy \right)$ being compact in $L^1((0,T) \times B_R)$ in view of corollary 3.10, the results follows.

□

Now, we are able to establish the strong compactness of the sequence of f_n y-averages, and also the (Lf_n) one.

Lemma 3.12 *For all $R > 0$, and for all function $\Psi \in L^\infty(Y)$ with compact support,*

$$\int_Y f_n(t,x,y)\Psi(y)dy \xrightarrow{n} \int_Y f(t,x,y)\Psi(y)dy \quad \textit{in} \quad L^1((0,T) \times B_R).$$

Proof: Since it is not clear that $\left(Q_c^-(f_n,f_n)\right)_n$ is bounded in $L^1((0,T)\times B_R\times Y_R)$, we can not directly apply theorem 3.9 to the sequence (f_n). For $\nu>0$, we consider the sequence $g_n^\nu := \dfrac{1}{\nu}\log(1+\nu f_n)$ and we set

$$G_n^\nu := \frac{Q_c^+(f_n,f_n)}{1+\nu f_n} - \frac{Q_c^-(f_n,f_n)}{1+\nu f_n} + \frac{Q_f^+(f_n)}{1+\nu f_n} - \frac{Q_f^-(f_n)}{1+\nu f_n}.$$

By the assumptions on (f_n), we have

$$\partial_t g_n^\nu + \frac{p}{m}.\nabla_x g_n^\nu = G_n^\nu \quad \text{in} \quad \mathcal{D}'((0,+\infty)\times\mathbb{R}^3\times Y). \tag{3.50}$$

Since $0\le g_n^\nu \le f_n$, the weak compactness of (f_n) established in the lemma 3.3 implies that (g_n^ν) is also weakly compact. Similarly, the sequence (g_n^ν) is bounded in $L^1((0,T)\times\mathbb{R}^3\times Y)$. Then, by corollaries 3.5, 3.7 and lemma 3.8, (G_n^ν) is bounded in $L^1((0,T)\times B_R\times Y_R)$. Therefore, theorem 3.9 applies to (g_n^ν) for all $\nu>0$. In particular, for all function $\Psi\in L^\infty(Y)$ with compact support and for all $\nu>0$, the sequence
$\left(\displaystyle\int_Y g_n^\nu(t,x,y)\Psi(y)dy\right)_n$ is compact in $L^1((0,T)\times B_R)$, thus, by the uniqueness of weak limit,

$$\int_Y g_n^\nu(t,x,y)\Psi(y)dy \xrightarrow{n} \int_Y g^\nu(t,x,y)\Psi(y)dy \quad \text{in} \quad L^1((0,T)\times B_R), \tag{3.51}$$

where g^ν is the weak limit of (g_n^ν) (up to an extraction).
The result follows because

$$\sup_n \sup_{t\in[0,T]} \iint_{\mathbb{R}^3\times Y} |g_n^\nu - f_n| dydx \underset{\nu\to 0}{\longrightarrow} 0, \tag{3.52}$$

which implies the strong compactness in $L^1((0,T)\times B_R)$ of the sequence
$\left(\displaystyle\int_Y f_n(t,x,y)\Psi(y)dy\right)_n$.
To show (3.52), we can use the inequality

$$\forall M>0, \qquad 0\le u-\frac{1}{\nu}\log(1+\nu u) = \frac{\nu M}{2}u\mathbb{1}_{\{u\le M\}} + u\mathbb{1}_{\{u>M\}}. \tag{3.53}$$

Then we obtain, for all n and for all $t\in[0,T]$,

$$\begin{aligned}
\iint_{\mathbb{R}^3\times Y} |g_n^\nu - f_n|dydx &\le \frac{\nu M}{2}\iint_{\mathbb{R}^3\times Y} f_n dydx + \iint_{\mathbb{R}^3\times Y} f_n \mathbb{1}_{\{f_n>M\}}dydx \\
&\le \frac{\nu M}{2}K_T + \frac{1}{M^{s-1}}\iint_{\mathbb{R}^3\times Y} f_n^s dydx \le \left(\frac{\nu M}{2}+\frac{1}{M^{s-1}}\right)K_T.
\end{aligned}$$

We conclude by letting $\nu\to 0$, and $M\to+\infty$.

$\square$

Proposition 3.13 *We set* $\rho_n(t,x) := \int_Y f_n(t,x,y)dy$ *and* $\rho(t,x) := \int_Y f(t,x,y)dy$. *Then, up to a subsequence, we have, for all* $R > 0$,

$$\rho_n \longrightarrow \rho \quad in \quad L^1((0,T) \times B_R) \quad and \quad a.e. \tag{3.54}$$

<u>Proof</u>: We have

$$\rho_n = \rho_n^M + \sigma_n^M, \quad \text{where} \quad \rho_n^M := \int_{Y_M} f_n(t,x,y)dy.$$

By the preceding lemma, $\rho_n^M \underset{n}{\longrightarrow} \rho^M := \int_{Y_M} f(t,x,y)dy$ in $L^1((0,T) \times B_R)$ for all $M > 0$, and

$$\begin{aligned} \sigma_n^M := \int_{Y - Y_M} f_n(t,x,y)dy &\leq \frac{1}{M} \int_{Y-Y_M} |y| f_n(t,x,y)dy \\ &\leq \frac{Cte}{M} \int_Y \left(m + \frac{|p|^2}{2m} + e \right) f_n(t,x,y)dy, \end{aligned}$$

whence $\sigma_n^M \underset{M \to +\infty}{\longrightarrow} 0$ in $L^1((0,T) \times B_R)$, uniformly in n.

$\square$

Lemma 3.14 *For all* $R > 0$, *we have, up to a subsequence,*

$$Lf_n \longrightarrow Lf \quad in \quad L^1((0,T) \times B_R \times Y_R) \quad and \quad a.e. \tag{3.55}$$

<u>Proof</u>: Applying the corollary 3.10 with $\Psi(y, y^\star) = A(y, y^\star) 1\!\!1_{\{y \in Y_R\}} 1\!\!1_{\{y^\star \in Y_{R^\star}\}}$, we infer that the sequence $\left(\int_{Y_{R^\star}} g_n^\nu(t,x,y^\star) A(y,y^\star) dy^\star \right)_n$ is compact in $L^1((0,T) \times B_R \times Y_R)$ for all $R^\star > 0$. Using (3.52) again, we obtain, for all $R^\star > 0$, the compactness of $\left(\int_{Y_{R^\star}} f_n(t,x,y^\star) A(y,y^\star) dy^\star \right)_n$ in $L^1((0,T) \times B_R \times Y_R)$. We conclude similarly as for the proof of lemma 3.6, establishing

$$\lim_{R^\star \to +\infty} \sup_n \left\| \int_{Y_{R^\star}} A f_n^\star dy^\star - \int_Y A f_n^\star dy^\star \right\|_{L^1((0,T) \times B_R \times Y_R)} = 0,$$

and identifying the weak limits.

$\square$

3.5 Regularity in time of the limit f

In this subsection, we show the continuity in time of the limit f, which gives a sense to the Cauchy data $f(0) = f^0$.

Proposition 3.15 *In fact, we have* $f \in C([0,T], L^1(\mathbb{R}^3 \times Y))$.

Proof: We use the integral formulation. Each g_n^ν is a distributional solution of the renormalized equation, by (3.50), so a mild solution. Therefore we have, for $a.e$ $(x,y) \in \mathbb{R}^3 \times Y$, and for all $t, t+h \in [0,T]$,

$$g_n^{\nu\sharp}(t+h,x,y) - g_n^{\nu\sharp}(t,x,y) = \int_t^{t+h} G_n^\nu(\sigma,x,y)d\sigma,$$

thus

$$\|g_n^{\nu\sharp}(t+h) - g_n^{\nu\sharp}(t)\|_{L^1(B_R\times Y_R)} \leq \iint_{B_R\times Y_R} \int_t^{t+h} |G_n^\nu(\sigma,x,y)|\, d\sigma.$$

Moreover, by the subsections 3.2 and 3.3, the sequence $(G_n^\nu)_n$ is weakly compact in $L^1((0,T)\times B_R \times Y_R)$, thus for all $t \in [0,T]$,

$$\lim_{h\to 0} \sup_n \|g_n^{\nu\sharp}(t+h) - g_n^{\nu\sharp}(t)\|_{L^1(B_R\times Y_R)} = 0.$$

Therefore the sequence $(g_n^{\nu\sharp})$ is equicontinuous in $C([0,T], L^1(B_R \times Y_R))$. By the compactness of $[0,T]$, this sequence is in fact uniformly equicontinuous, thus

$$\lim_{h\to 0} \sup_n \sup_{t\in[0,T]} \|g_n^{\nu\sharp}(t+h) - g_n^{\nu\sharp}(t)\|_{L^1(B_R\times Y_R)} = 0.$$

Then, (3.52) and the estimate (3.45) yield

$$\lim_{h\to 0} \sup_n \sup_{t\in[0,T]} \|f_n^\sharp(t+h) - f_n^\sharp(t)\|_{L^1(\mathbb{R}^3\times Y)} = 0.$$

Ascoli theorem entails that the sequence $(f_n^\sharp)$ is compact in $C([0,T], L^1(\mathbb{R}^3 \times Y))$. The uniqueness of the limit in $\mathcal{D}'((0,+\infty)\times\mathbb{R}^3\times Y)$ yields $f^\sharp \in C([0,T], L^1(\mathbb{R}^3\times Y))$, and so $f \in C([0,T], L^1(\mathbb{R}^3 \times Y))$ by change of variables.

□

3.6 Passing to the limit in a new integral equation

Even if the renormalization provides weak compactness, a new problem appears: we will not be able to pass to the weak limit in $(ECFR)$, because of the non-linearity of the factor $f_n/(1+f_n)$. That's why we need another formulation to our problem, which avoids the renormalization. But, remember that the term $Q_c^-(f,f)$ can not be defined as a distribution, so we will use an integral equation which doesn't involve this term. We proceed as in [10].

We denote by T the linear transport operator

$$T = \partial_t + \frac{p}{m}.\nabla_x.$$

Let T^{-1} be the resolvant of transport operator, defined by: for $g(t,x,y)$, we set $u = T^{-1}g$ if $u|_{t=0} = 0$ and $Tu = g$. So, we have

$$T^{-1}g(t,x,m,p,e) := \int_0^t g(s, x-(t-s)p/m, m, p, e)ds.$$

T^{-1} satisfies the following properties:

(i) Forall $R > 0$, $T^{-1}\big(L^1((0,T)\times B_R\times Y_R)\big) \subset C([0,T], L^1(B_R\times Y_R))$ continuously and weakly continuously.

(ii) T^{-1} is nonnegative ($\forall g \geq 0, T^{-1}g \geq 0$).

Forall $F \in C([0,T], L^1(B_R\times Y_R))$ such that $TF \geq 0$, we set

$$T_F^{-1} = e^{-F}T^{-1}e^F.$$

This operator is well defined from $L^1((0,T)\times B_R\times Y_R)$ to $C([0,T], L^1(B_R\times Y_R))$ and has the same continuity properties as T^{-1}.
Moreover, if (F_n) is a bounded sequence in $C([0,T], L^1(B_R\times Y_R))$ such that $TF_n \geq 0$, if $F_n(t,x,y) \to F(t,x,y)$ for all t and $a.e\,(x,y)$, and if $g_n \rightharpoonup g$ weakly in $L^1((0,T)\times B_R\times Y_R)$, then

$$\forall t \in [0,T], \qquad T_{F_n}^{-1}g_n(t) \rightharpoonup T_F^{-1}g(t) \quad \text{weakly in} \quad L^1(B_R\times Y_R).$$

The operator T_F^{-1} allows us to build an new formulation of our problem, which is better because it only involves $Q_c^+(f_n,f_n)$, $Q_f^+(f_n)$, $Q_c^-(f_n)$:

Lemma 3.16 *$f \in C([0,T], L^1(\mathbb{R}^3\times Y))$ is a mild solution of (ECF) with initial data $f(0) = f^0$ if and only if*

$$f = e^{-F}f^0(x-tp/m,y) + T_F^{-1}(Q_c^+(f,f)) + T_F^{-1}(Q_f^+(f)) - T_F^{-1}(Q_c^-(f)), \qquad (3.56)$$

where $F := T^{-1}(Lf)$.

Proof: The result is deduced from the following fact: if f is a distributional solution of (ECF), then

$$T(e^F f) = TFe^F f + e^F Tf = e^F(f\,Lf + Q_c^+(f,f) - Q_c^-(f,f) + Q_f^+(f) - Q_f^-(f))$$
$$= e^F(Q_c^+(f,f) + Q_f^+(f) - Q_f^-(f)).$$

$\square$

Now, we can finish the proof of theorem 3.2.

End of the proof of theorem 3.2: We will pass to the weak limit in the following equation, satisfied by each (f_n):

$$f_n = e^{-F_n} f_n^0(x - tp/m, y) + T_{F_n}^{-1}(Q_c^+(f_n, f_n)) + T_{F_n}^{-1}(Q_f^+(f_n)) - T_{F_n}^{-1}(Q_c^-(f_n)), \quad (3.57)$$

where $F_n := T^{-1}(Lf_n)$.
Notice that in view of (3.55) and the continuity properties of T^{-1}, the sequence (F_n) is bounded in $C([0,T], L^1(B_R \times Y_R))$, and $F_n(t,x,y) \to F(t,x,y)$ for all t and $a.e\,(x,y)$. Thus, we can pass to the weak limit in the terms $T_{F_n}^{-1}(Q_f^+(f_n))$ and $T_{F_n}^{-1}(Q_c^-(f_n))$ thanks to the lemma 3.8. The term $e^{-F_n} f_n^0(x - tp/m, y)$ can be treated with the continuity in $t = 0$ (for the L^1-norm) of each f_n and f, established in the previous section. Eventually, the last term $T_{F_n}^{-1}(Q_c^+(f_n, f_n))$ also pass to the weak limit thanks to the following lemma and proposition, which use the $a.e$ convergence of the y-averages obtained in the previous subsection.

Lemma 3.17 *For all $R > 0$ and for all function $\varphi \in L^\infty((0,T) \times B_R \times Y_R)$, we have, up to a subsequence,*

$$\frac{\int_Y Q_c^+(f_n, f_n)(t,x,y)\varphi(t,x,y)dy}{1 + \rho_n(t,x)} \xrightarrow{n} \frac{\int_Y Q_c^+(f, f)(t,x,y)\varphi(t,x,y)dy}{1 + \rho(t,x)}$$

in $L^1((0,T) \times B_R)$ and a.e.

Proof: We have

$$\frac{\int_Y Q_c^+(f_n, f_n)\varphi dy}{1 + \rho_n} = \frac{1}{2} \int_Y f_n(t,x,y^\star) \left(\int_Y \frac{f_n(t,x,y) A(y,y^\star) \varphi(t,x,y+y^\star) dy}{1 + \rho_n(t,x)} \right) dy^\star.$$

Now, we apply the corollary 3.10 with $\Psi(t,x,y,y^\star) = A(y,y^\star)\varphi(t,x,y+y^\star)$ (notice that $\Psi \in L^\infty((0,T) \times B_R \times Y_{2R}^2)$ thanks to (1.5)).

Therefore, the sequence $\left(\int_Y g_n^\nu(t,x,y) A(y,y^\star)\varphi(t,x,y')dy \right)_n$ is compact in $L^1((0,T) \times B_R \times Y_R)$, and we have

$$\int_Y g_n^\nu(t,x,y) A(y,y^\star)\varphi(t,x,y')dy \xrightarrow{n} \int_Y g^\nu(t,x,y) A(y,y^\star)\varphi(t,x,y')dy \quad (3.58)$$

in $L^1((0,T) \times B_R \times Y_R)$, for all $\nu > 0$.

Using (3.52) again, we obtain, up to a subsequence, that

$$\int_Y f_n(t,x,y) A(y,y^\star)\varphi(t,x,y')dy \xrightarrow{n} \int_Y f(t,x,y) A(y,y^\star)\varphi(t,x,y')dy \quad (3.59)$$

in $L^1((0,T)\times B_R\times Y_R)$ and a.e.

Up to another extraction, we infer, by (3.59) and (3.54),

$$\int_Y \frac{f_n(t,x,y)A(y,y^\star)\varphi(t,x,y')dy}{1+\rho_n(t,x)} \xrightarrow{n} \int_Y \frac{f(t,x,y)A(y,y^\star)\varphi(t,x,y')dy}{1+\rho(t,x)} \tag{3.60}$$

a.e in $(t,x,y)\in(0,T)\times B_R\times Y_R$.

Applying the corollary 3.11 with $\Psi_n(t,x,y^\star) := \displaystyle\int_Y \frac{f_n(t,x,y)A(y,y^\star)\varphi(t,x,y')dy}{1+\rho_n(t,x)}$ (which satisfies the required assumptions because φ is compact supported and A is locally bounded), we obtain the compactness of the sequence $\left(\displaystyle\int_Y g_n^\nu(t,x,y^\star)\Psi_n(t,x,y^\star)dy^\star\right)_{n\in\mathbb{N}}$ in $L^1((0,T)\times B_R)$, and so, by (3.52), we deduce that $\left(\displaystyle\int_Y f_n(t,x,y^\star)\Psi_n(t,x,y^\star)dy^\star\right)_{n\in\mathbb{N}}$ is compact.

Finally, we conclude that

$$\int_Y f_n(t,x,y^\star)\Psi_n(t,x,y^\star)dy^\star \xrightarrow{n} \int_Y f(t,x,y^\star)\Psi(t,x,y^\star)dy^\star \quad \text{in} \quad L^1((0,T)\times B_R).$$

□

Proposition 3.18 *Up to a subsequence, we have, for all $R>0$,*

$$Q_c^+(f_n,f_n) \rightharpoonup Q_c^+(f,f) \quad \textit{weakly in} \quad L^1((0,T)\times B_R\times Y_R).$$

Proof: We know by the lemma 3.4 that there exists $\overline{Q}(t,x,y)$ such that for all $R>0$,

$$Q_c^+(f_n,f_n) \rightharpoonup \overline{Q} \quad \text{weakly in} \quad L^1((0,T)\times B_R\times Y_R).$$

By (3.54) and a standard integration argument (we can refer to [17] for a proof), it leads to

$$\frac{Q_c^+(f_n,f_n)}{1+\rho_n} \rightharpoonup \frac{\overline{Q}}{1+\rho} \quad \text{weakly in} \quad L^1((0,T)\times B_R\times Y_R).$$

Moreover, the previous lemma shows that

$$\frac{Q_c^+(f_n,f_n)}{1+\rho_n} \rightharpoonup \frac{Q_c^+(f,f)}{1+\rho} \quad \text{weakly in} \quad L^1((0,T)\times B_R\times Y_R).$$

We conclude identifying weak limits.

□

We have shown that f is a mild solution of (ECF). Since $Q_c^+(f_n,f_n)$, $Q_f^+(f_n)$ and $Q_f^-(f_n)$ converge weakly to $Q_c^+(f,f)$, $Q_f^+(f)$ and $Q_f^-(f)$ respectively, these three terms lie in L^1_{loc}, and *a fortiori*,

$$\frac{Q_c^+(f,f)}{1+f}, \frac{Q_f^+(f)}{1+f}, \frac{Q_f^-(f)}{1+f} \in L^1_{loc}((0,+\infty)\times\mathbb{R}^3\times Y).$$

The term $\dfrac{Q_c^-(f,f)}{1+f}$ is automatically in L^1_{loc} because $Lf \in L^1((0,T)\times B_R \times Y_R)$ for all $R > 0$ and $\dfrac{Q_c^-(f,f)}{1+f} \leq Lf$.

Thus, f is indeed a renormalized solution of (ECF).

$\square$

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