Non destructive testing using non linear vibroacoustic

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1 Introduction

Several recent experimental studies show that it is possible to detect defects in a structure by considering its vibro- acoustic response to an external actuation.

1.1 Some previous papers

On this topic there is a vast literature in applied physics. We recall some papers related to the use of the frequency response for non destructive testing; in particular generation of higher harmonics, cross-modulation of a high frequency by a low frequency:

- In Ekimov-Didenkulov-Kasakov (1999), [2], the authors report experiments with torsional waves in a rod with a crack: they use HF torsional wave (20kHz) and a LF flexural wave (12 Hz).
- In Zaitsev-Sas (1999), [9], the authors report experiments with plate vibration submitted to LF (20-60Hz) vibration by a shaker and HF (15-30 kHz) oscillations by a piezo-actuator. They notice that weak modulation side-lobes are observed for the undamaged sample but drastic increase in nonlinear vibro-acoustic of the damaged sample. Some theoretical explanations are provided.
- Other results may be found in Sedunov-Tsionsky-Donskoy(2002) [3],Sutin-Donskoy (1998), [1], Moussatov-Castagnede-Gusev(2002), [5] ...
- GDR 2501 (Etude de la propagation ultrasonore en milieux inhomognes en vue du controle non destructif)

In Vanderborck-Lagier-Groby (2003) [8], "a vibro-acoustic method, based on frequency modulation, is developed in order to detect defects on aluminum and concrete beams. Flexural waves are generated at two very separated frequencies by the way of two piezoelectric transducers. The low one corresponds to the first resonance f_m, the second one to a high non modal frequency f_p. The nonlinear response, due to the defects inside the structure, is detected by non-zero flexural waves at $f_p \pm n f_m$ frequencies.

see Vanderborck-Lagier(2004) $\longmapsto$ beam experimentation

Very recent experiments

- have been performed on a real bridge by G. Vanderborck with four prestressed cables: two undamaged cables, a damaged cable and a safe one but damaged at the anchor;
- these experiments have been performed in the frame of the European program "Promoting competitive and sustainable growth "of 15/12/99.
- The cables are roughly 100 m long, 4 tones weight, 15cm in diameters.
- The experiments have proved the presence of the damaged cable but also the safe one damaged at the anchor.
- Routine experimental checking with the lower eigenfrequencies had **only** proved only the presence of the very damaged cable by comparison with data collected 15 years ago.
- See Vanderborck-Lagier(2004) [10] for a presentation of the results of the experiment with a new post processing graphic presentation of experimental results.

Difficulties of the experiments:

- non linearities of the shakers (including piezoelectric actuators)
- Natural non linearities: supports, links of complex multi structures as air planes, bridges etc

Orientation

We intend to present simple spring mass models, simple bar and beam models with damage and use **asymptotic expansions and numerical methods** to try to get results which show some similarity with the experiments of [8]. Asymptotic expansions have been used for at least a century and for example has been used recently for numerical approximation of bifurcation of structures in PotierFerry-Cochelin and coworkers (1993) [4].

The key idea is to look at the solution in the frequency domain for the experiments and consequently for the numerics.

In a paper to be submitted (Lagier-Vandeborck) [7] several types of nonlinearities of defects are considered: contact elasticity, threshold contact model, nonlinear filling material. This last case will be considered for bar models: it may happen in case of corrosion: the voided crack is filled by a new dusty material: then the elastic crack response is related to the elastic properties of the filler. In this case it seems reasonable to consider a nonlinear elastic relation for the filler.

2 Background of Fourier transform

2.1 Basic formulas

2.1.1 Fourier serie

For a detailed presentation, see for example Gasquet-Witomski [11] and for an engineering view point Lathy [6]; for a function f of period T, its expansion in fourier serie is:

$$f(t) = \sum_{n=-\infty}^{+\infty} c_n e^{\frac{2\pi i n t}{T}} \text{ with } c_n = \frac{1}{T}\int_0^T f(t) e^{\frac{-2\pi i n t}{T}} dt \tag{2.1}$$

Discrete Fourier transform: D.F.T. which may be computed quickly by the algorihm of F.F.T. To a sequence $(y_k), k = 0, \ldots, N-1$, is associated an other sequence $(Y_n), n = 0, \ldots, N-1$ with the formulas:

$$Y_n = \frac{\mathrm{T}}{\mathrm{NT}} \sum_{k=0}^{N-1} y_k e^{\frac{-2\pi i n}{T}\frac{kT}{N}} \qquad y_k = \sum_{n=0}^{N-1} Y_n e^{\frac{2\pi i n k}{N}} \tag{2.2}$$

An approximation of Fourier coefficients (caution to indixes) may be obtained with :

$$c_n^N = \frac{\mathrm{T}}{\mathrm{NT}} \sum_{k=0}^{N-1} f(\frac{kT}{N}) e^{\frac{2\pi in}{T}\frac{kT}{N}} \quad \text{or with the DFT using} \tag{2.3}$$

$$y_k = f(\frac{kT}{N}) \quad c_n \simeq c_n^N = \begin{cases} Y_n \text{ for } 0 \leq n < \frac{N}{2} \\ Y_{n+N} \text{ for } -\frac{N}{2} \leq n < 0 \end{cases} \tag{2.4}$$

with the error approximation:

$$c_n^N = \sum_{q \neq 0} c_{n+qN} \tag{2.5}$$

Beware to Scilab FFT which provides: $X_n = \sum_{k=0}^{N-1} y_k e^{\frac{2\pi in\ k}{N}}$ ie the Fourier coefficients are computed from: $Y_n = \frac{1}{N} X_n$.

2.1.2 Fourier transform

$$(\mathcal{F}f)(\nu) = \hat{f}(\nu) = \int_{\mathbf{R}} f(t) e^{-2\pi i\nu t} dt \tag{2.6}$$

$$(\mathcal{F}^{-1}g)(t) = (\bar{\mathcal{F}}g)(t) = \int_{\mathbf{R}} g(\nu) e^{+2\pi i\nu t} d\nu \tag{2.7}$$

$$\widehat{f^m} = (2\pi i\nu)^m \hat{f} \quad \mathcal{F}(-(2\pi it)^{(m)} f(t)) = \hat{f}^m(\nu) \tag{2.8}$$

$$\tag{2.9}$$

$$\widehat{\chi_{[-A,A]}} = \frac{sin(2\pi\nu A)}{\pi\nu} = 2A.\mathbf{sinc}(2\pi\nu A) \quad \widehat{\chi_{[0,A]}} = e^{-i\pi A\nu}\frac{sin(\pi A\nu)}{\pi\nu} = A\mathbf{sinc}(\pi A\nu) \tag{2.10}$$

$$\text{with } \textit{the sampling function (“sinus cardinal”)} \quad \boldsymbol{sinc}(t) = \frac{sin(t)}{t} \tag{2.11}$$

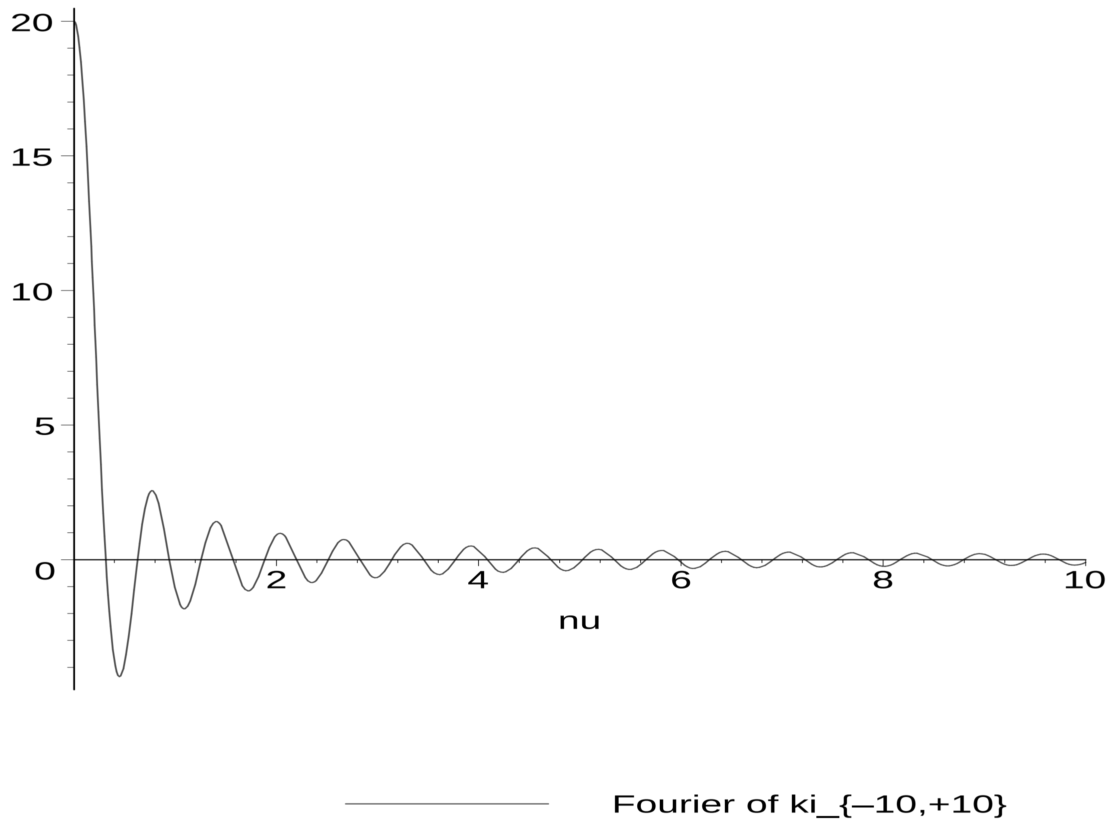
$$\mathcal{F}(e^{2\pi iat}) = \delta_a, \ \text{ and } \ \mathcal{F}(e^{-2\pi iat}T) = \tau_a\hat{T} = \delta_a * \hat{T} \tag{2.12}$$

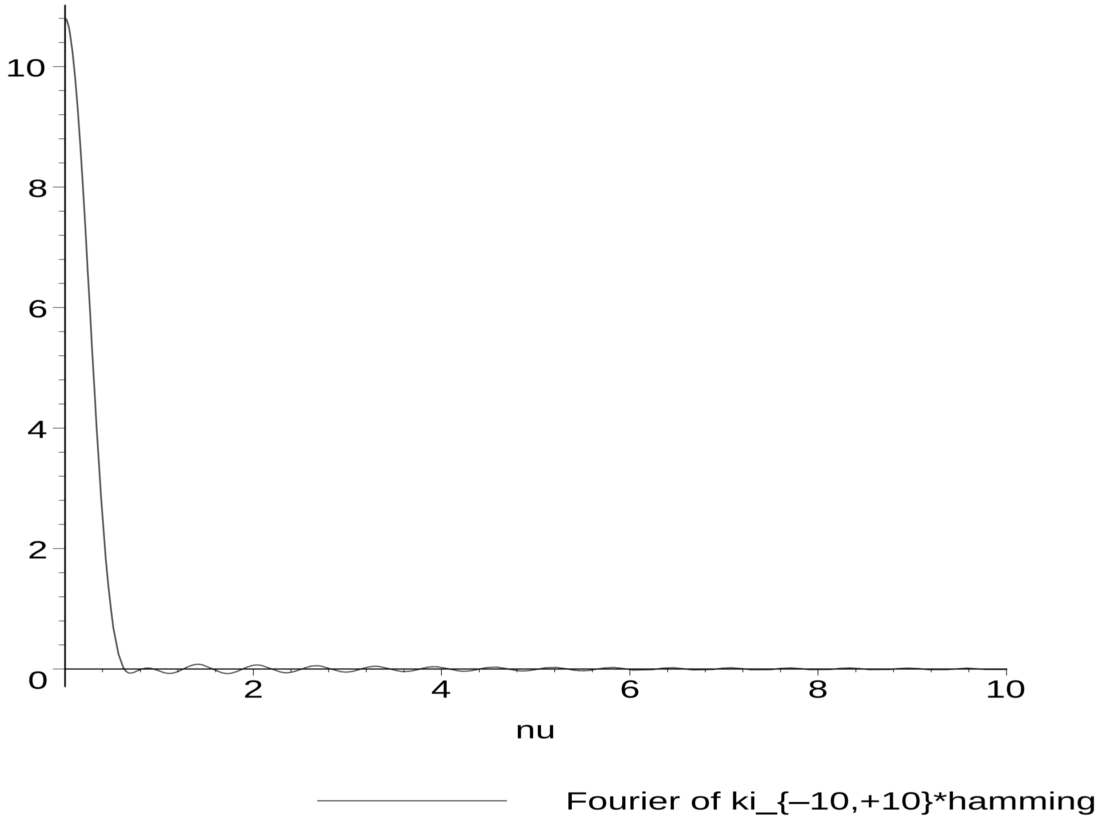
$$(\mathcal{F}(cos(2\pi iat)T) = \frac{1}{2}(\tau_a\hat{T} + \tau_{-a}\hat{T}) = \frac{1}{2}(\delta_a * \hat{T} + \delta_{-a} * \hat{T}) \tag{2.13}$$

$$\mathcal{F}(cos(2\pi iat)\chi_{[-A,A]}) = A(\tau_a.\mathbf{sinc}(\mathbf{2}\pi\nu\mathbf{A}) + \tau_{-a}.\mathbf{sinc}(\mathbf{2}\pi\nu\mathbf{A})) \tag{2.14}$$

$$\mathcal{F}(sin(2\pi iat)\chi_{[-A,A]}) = iA(-\tau_a\mathbf{sinc}(\mathbf{2}\pi\nu\mathbf{A}) + \tau_{-a}\mathbf{sinc}(\mathbf{2}\pi\nu\mathbf{A})) \tag{2.15}$$

$$\mathcal{F}(cos(2\pi iat)\chi_{[0,A]}) = \frac{A}{2}(\tau_a e^{-i\pi A\nu}\mathbf{sinc}(\pi\nu\mathbf{A}) + \tau_{-a}e^{-i\pi A\nu}\mathbf{sinc}(\mathbf{2}\pi\nu\mathbf{A})) \tag{2.16}$$





2.2 Sampling

$$\Delta_a = \sum_{n\in\mathbf{Z}} \delta_{na} \quad \widehat{\Delta_a} = \sum_{n\in\mathbf{Z}} e^{-2\pi i\, n\, a\nu} = \frac{1}{a}\Delta_{\frac{1}{a}} \quad \text{le “peigne”} \tag{2.17}$$

$$\text{also} \quad \Delta_a = \frac{1}{a}\sum_{n\in\mathbf{Z}} e^{2\pi i n \frac{t}{a}} \tag{2.18}$$

a **sampling** of f is: $af\Delta_a = a\sum_{\mathbf{Z}} f(na)\delta_{na}$ with a the sampling period.

Poisson formula:

$$\sum_{\mathbf{Z}} f(t-na) = \frac{1}{a}\sum_{\mathbf{Z}} \hat{f}(\frac{n}{a})e^{2\pi i n \frac{t}{a}} \text{ (f distribution à support compact)} \tag{2.19}$$

$$\sum_{\mathbf{Z}} \hat{g}(\nu - \frac{n}{a}) = a\sum_{\mathbf{Z}} g(na)e^{-2\pi i n\nu a} \text{ ($\hat{g}$ distribution à support compact)} \tag{2.20}$$

$$a(\widehat{f\Delta_a})(\nu) = \sum_{n\in\mathbf{Z}} \hat{f}(\nu - \frac{n}{a}) = a\sum_{n\in\mathbf{Z}} f(na)e^{-2\pi i\nu n a} \tag{2.21}$$

(f tempered and $\hat{f}$ and $\hat{f}$ with support in $[-\nu_c, \nu_c]$)

no overlap of the spectrum for $a \leq \frac{1}{2\nu_c}$.

Shanon For

$$\hat{f} \in L^2(\mathbf{R}) \text{ et } Supp(\hat{f}) \subset [-\nu_c, \nu_c] \quad (2.22)$$

$$\forall a \leq \frac{1}{2\nu_c} \quad f(t) = \sum_{\mathbf{Z}} f(na)\mathbf{sinc}(\frac{\pi}{a}(t - na)) \quad (2.23)$$

$$\textbf{with the sampling function } \mathbf{sinc}(t) = \frac{sin(t)}{t} \quad (2.24)$$

Use a low-pass filter before sampling.

2.3 Numerical computation of Fourier transform

$$\hat{f}(\nu) \simeq \int_{-T/2}^{T/2} f(t)e^{-2\pi i\nu t}dt = Tc_{\nu T} \quad \text{or with } \nu = \frac{n}{T} \quad (2.25)$$

$$\hat{f}(\frac{n}{T}) \simeq Tc_n \quad \text{for } \frac{-N}{2} \leq n \leq \frac{N}{2} \quad (2.26)$$

Fourier coefficients c_n are numerically computed with FFT where $y_k = f(\frac{kT}{N})$:

$$\hat{f}(\nu) \simeq \sum_k f(\frac{kT}{N})exp(-2\pi i\nu\frac{kT}{N}) \text{ with } \nu = \frac{n}{T} \quad (2.27)$$

$$\simeq \sum_k f(\frac{kT}{N})exp(-2\pi in\frac{k}{N}) \text{ with } n = \nu T \quad (2.28)$$

with sampling period $\frac{T}{N} \leq \frac{1}{2\nu_c}$ (no overlap of the spectrum) but ... a function of compact support in time is not of compact support in frequency...

2.4 Exemples

Gate function Commenons par un exemple classique: $f = \chi_{[0,b]}$, sa transformée de Fourier est:

$$\hat{f}(\nu) = \frac{sin(\pi b\nu)}{\pi\nu} exp(-i\pi b\nu) \tag{2.29}$$

Avec $b = 1/2$ et 1000 points utilisés dans l'intervalle $[0, 1]$, on trouve les transformées sur les figures ci jointe. On pourra remarquer que le maximum est correct

Cosinus Pour la fonction $cos(2\pi t)$, il est bien connu que la transformée de Fourier est $\delta_1 + \delta_{-1}$. la transformée discrète est elle mme une approximation numérique de $\int_0^T cos(2\pi t)exp(-2\pi i\nu t)dt$ et l'on trouve un pic de hauteur la moitié de l'intervalle d'intégration. On trouve dans les figures 3 et 4, les transformée de fourier discrète calculée dans $[0, 1]$ puis $[0, 20]$

Exponentielle-cosinus On constate que la transformée de Fourier de $exp(10^{-3}t) * cos(2\pi t)$ est sensiblement égale à celle du cosinus tandis que

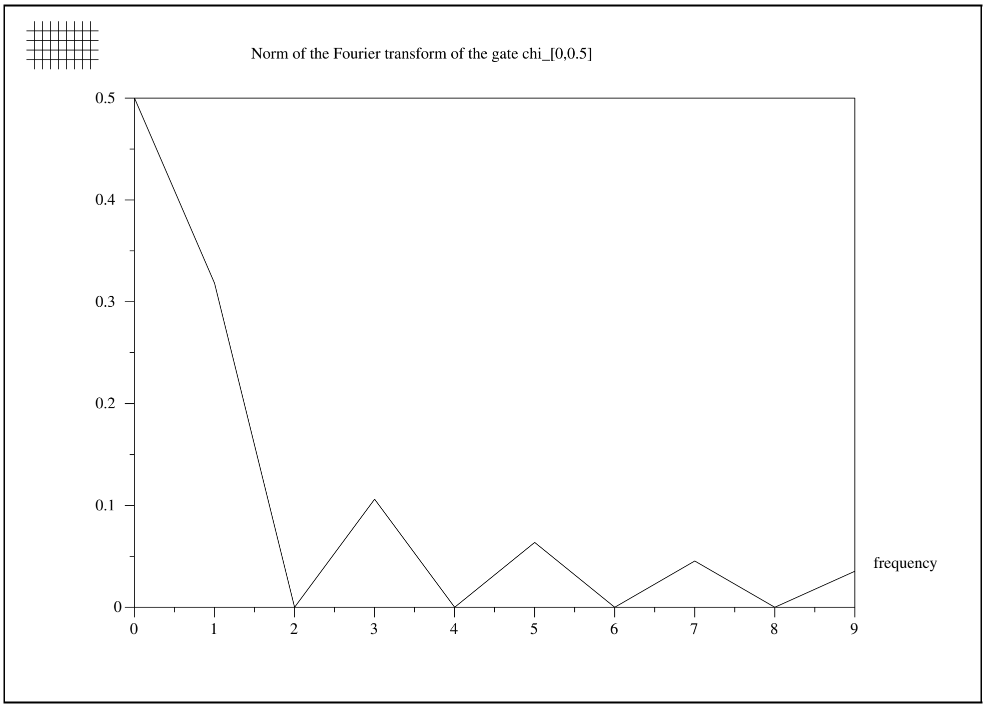


Figure 1: Norm of the Fourier transform of the gate $\chi_{[0,0.5]}$ in $[0, 9]$ herz

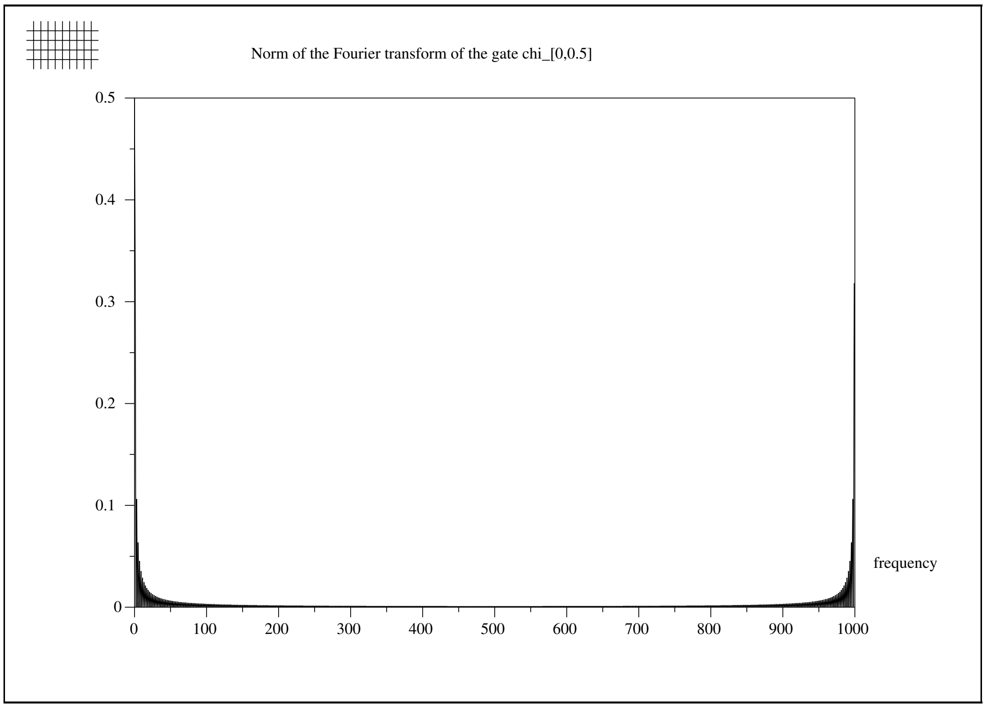


Figure 2: Norm of the Fourier transform of the gate $\chi_{[0,0.5]}$ in $[0, 1000]$ herz

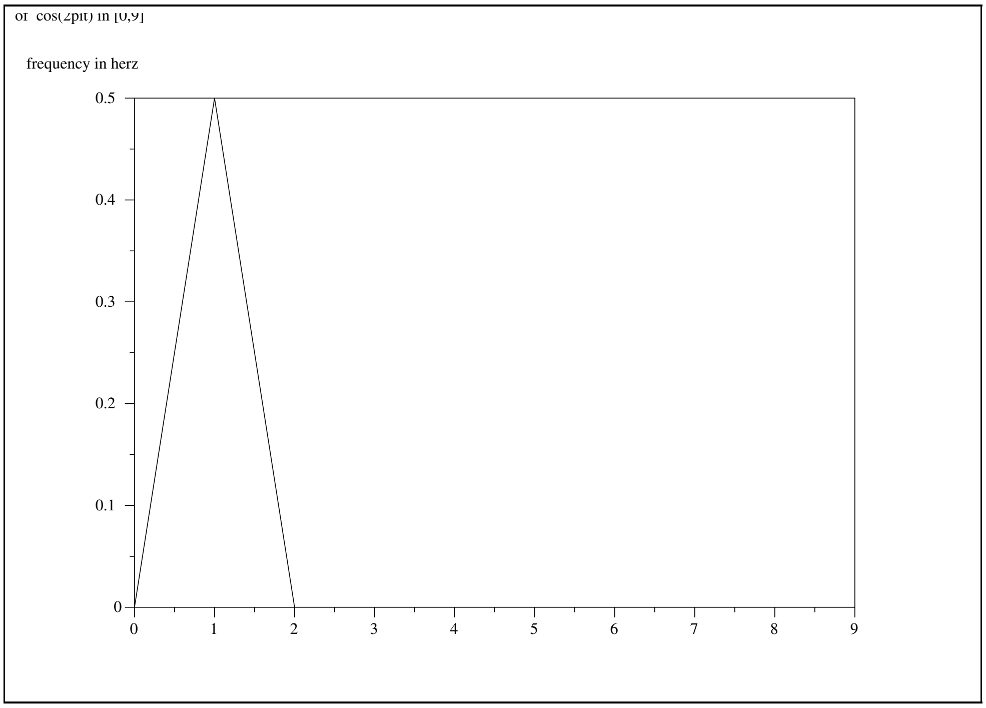


Figure 3: Norm of the Fourier transform of $cos(2\pi t)$ in $[0, 1000]$ herz

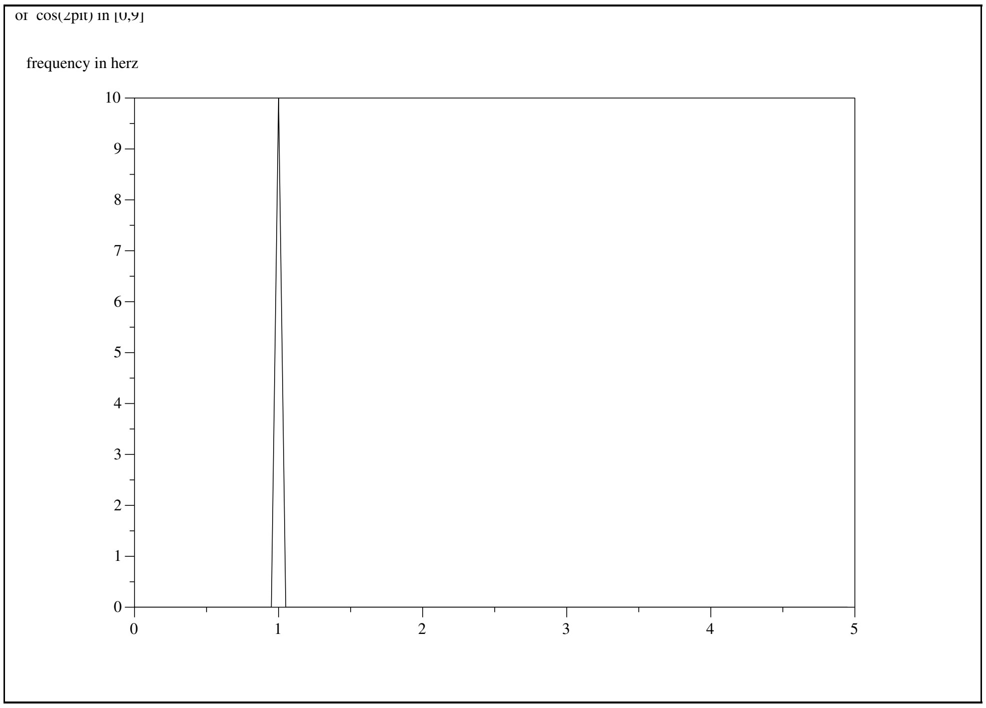


Figure 4: Norm of the Fourier transform of $cos(2\pi t)$ in $[0, 1000]$ herz

celle de $exp(10^{-2}t) * cos(2\pi t)$ est un peu différente, voir les figures 5 et 6

somme de deux sinus

`sin_p_sin.sci`

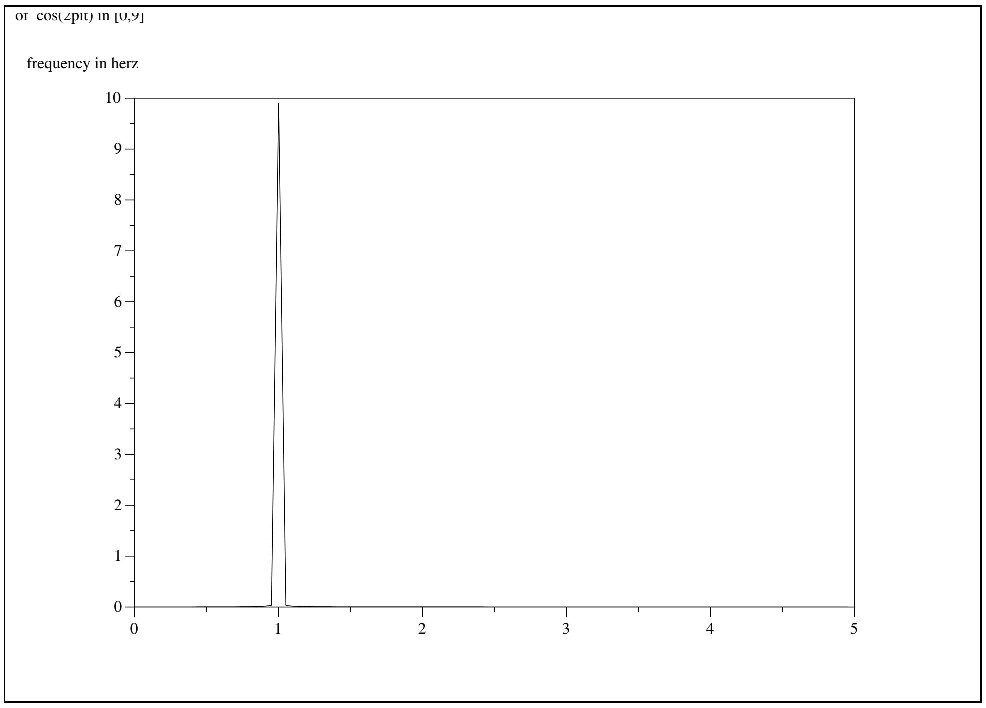


Figure 5: Norm of the Fourier transform of $exp(10^{-3}t) * cos(2\pi t)$ in $[0, 1000]$ herz

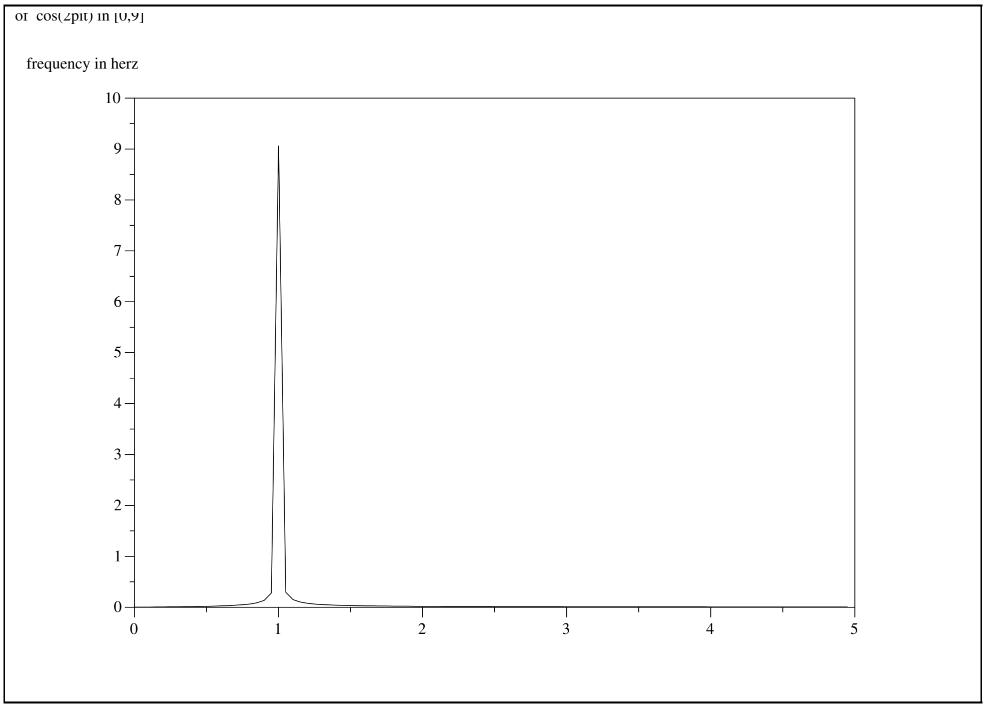


Figure 6: Norm of the Fourier transform of $exp(10^{-2}t) * cos(2\pi t)$ in $[0, 1000]$ herz

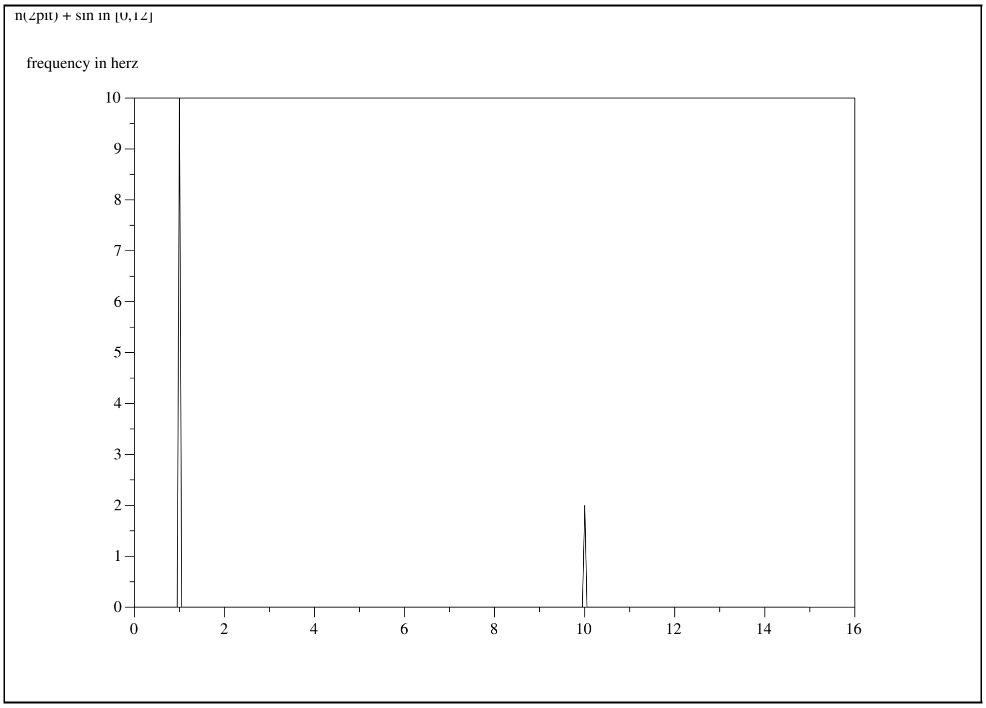


Figure 7: Norm of the Fourier transform of $sin(2\pi t) + .2sin(20\pi t)$ in $[0, 50]$ herz

16-9

3 Simplest mechanical example

in which we can exhibit intermodulations.

We consider a 1 d.o.f example of a spring mass system with a non linear spring.

$$m\ddot{y} = -ky - k_3 y^3 + \epsilon F sin(\alpha t) \quad \text{or} \tag{3.1}$$

$$\ddot{y} = -\omega_0^2 y - \frac{k_3}{m} y^3 + \epsilon \frac{F}{m} sin(\alpha t) \quad \text{with } \omega_0^2 = \frac{k}{m} \tag{3.2}$$

$$\text{with initial conditions} \quad y(0) = \epsilon \eta_1, \quad \dot{y} = \epsilon v_1 \tag{3.3}$$

We are going to solve this equation **symbolically** with an asymptotic expansion with respect to ϵ: $y = \epsilon y_1 + \epsilon^2 y_2 + \epsilon^3 y_3 + \ldots$; then **numerically...**

3.1 The linear case

The first term is solution of:

$$\ddot{y_1} = -\omega_0^2 y_1 + \frac{F}{m} sin(\alpha t) \quad \text{with } y(0) = \eta_1, \quad \dot{y} = v_1 \text{ which gives:} \tag{3.4}$$

$$y_1 = Ae^{i\omega_0 t} + \overline{A}e^{-i\omega_0 t} + De^{i\alpha t} + \overline{D}e^{-i\alpha t} \quad \text{with} \tag{3.5}$$

$$A = \frac{\eta_1}{2} - \frac{iF\alpha}{2m\omega_0(\alpha^2 - \omega_0^2)} - \frac{iv_1}{2\omega_0} \quad D = -\overline{D} = -\frac{F}{2im(\alpha^2 - \omega_0^2)} \quad \text{or} \tag{3.6}$$

$$\text{with } \phi = \frac{F}{m(\alpha^2 - \omega_0^2)},\ y_1 = \eta_1 cos(\omega_0 t) + (\frac{v_1}{\omega_0} + \frac{\alpha}{\omega_0}\phi) sin(\omega_0 t) - \phi sin(\alpha t) \tag{3.7}$$

Remarque 1. If we set $\eta_1 = 0,\ v_1 = 0$, then the term of pulsation ω_0 has magnitude $\frac{\alpha}{\omega_0}$ times the magnitude of the term of pulsation α; this is not a good choice for the non linear case in which $\frac{\alpha}{\omega_0}$ is of order 100; *it seems good choice*
$\eta_1 = 0,\ v_1 = \omega_0(-\frac{\alpha}{\omega_0} + 1)\frac{F}{m(\alpha^2 - \omega_0^2)} = (-\alpha + \omega_0)\phi$

3.2 Other terms

The term y_2 is zero but the **third term** satisfies:

$$\ddot{y_3} = -\omega_0^2 y_3 + \frac{k_1}{m} y_1^3 \tag{3.8}$$

to simplify, we assume $\eta_1 = 0$ and set $\phi = \frac{F}{m(\alpha^2 - \omega_0^2)}, \psi = \frac{v_1}{\omega_0} + \frac{\alpha\phi}{\omega_0}$ so that:

$$y_1^3 = (\psi sin(\omega_0 t) - \phi sin(\alpha t))^3 = \tag{3.9}$$

$$-1/4\psi^3 sin(3\omega_0 t) + 3/2\psi(1/4\psi^2 + \phi^2) sin(\omega_0 t) \tag{3.10}$$

$$-3/4\psi\phi^2 sin((\omega_0 + 2\alpha)t) - 3/4\psi\phi^2 sin((\omega_0 - 2\alpha)t) \tag{3.11}$$

$$+3/4\psi^2\phi sin((\alpha + 2\omega_0)t) + 3/4\psi^2\phi sin((\alpha - 2\omega_0)t) + \tag{3.12}$$

$$-3/2(\psi^2\phi + 1/2\phi^3) sin(\alpha t) + 1/4\phi^3 sin(3\alpha t) \tag{3.13}$$

$$\tag{3.14}$$

If we *go on in the expansion,* we get terms of pulsation $\alpha + 4\omega_0,\ \alpha + 6\omega_0$ etc

3.2.1 Numerical issues

$$\text{For } \alpha = 40\pi, \omega_0 = 2\pi, F = 100,\ \phi = .6348445087e - 2,\ \alpha\phi = .7977691380, \tag{3.15}$$

$$v_1 = -.7578806812, \psi = -5.366518580,\ 3/4\psi^2\phi = -.1371241364 \tag{3.16}$$

$$\phi^3 \ll \psi^3 \tag{3.17}$$

General tendency:

- The pick of $3\omega_0$ is much larger than the pick in $\alpha \pm 2\omega_0$ which are the most natural picks in the experiments;
- it is delicate to find datas such that the secondary picks at $\alpha \pm 2\omega_0$ actually appear when the differential equation is solved numerically.
- Question: algorithm and software for detecting the secondary picks?
- then find (by optimization) datas such that the secondary picks are important: criteria for damage.

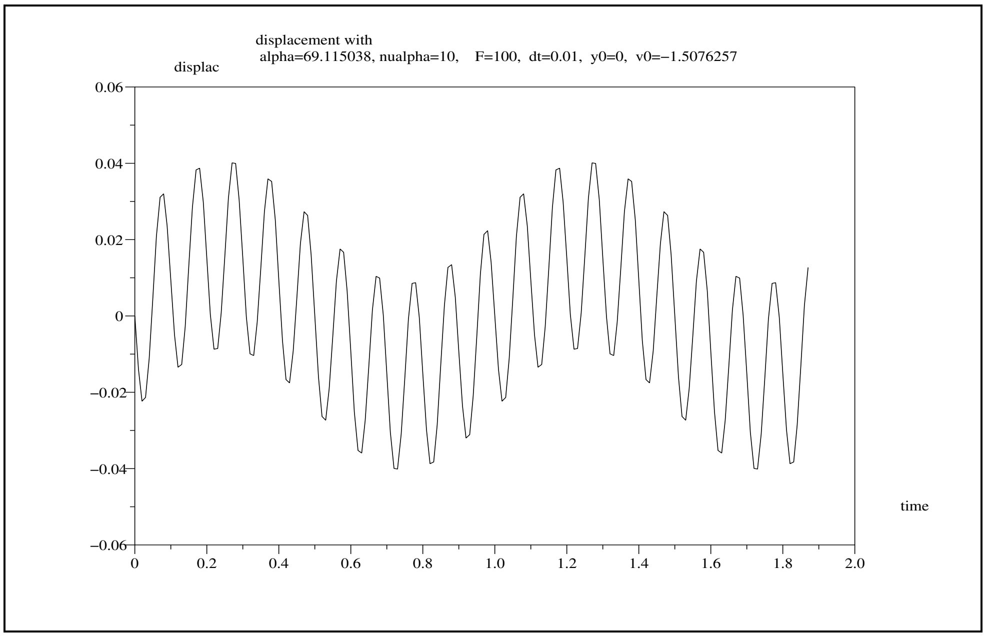


Figure 8: Linear response y, $\nu_{\omega_0} = 1$

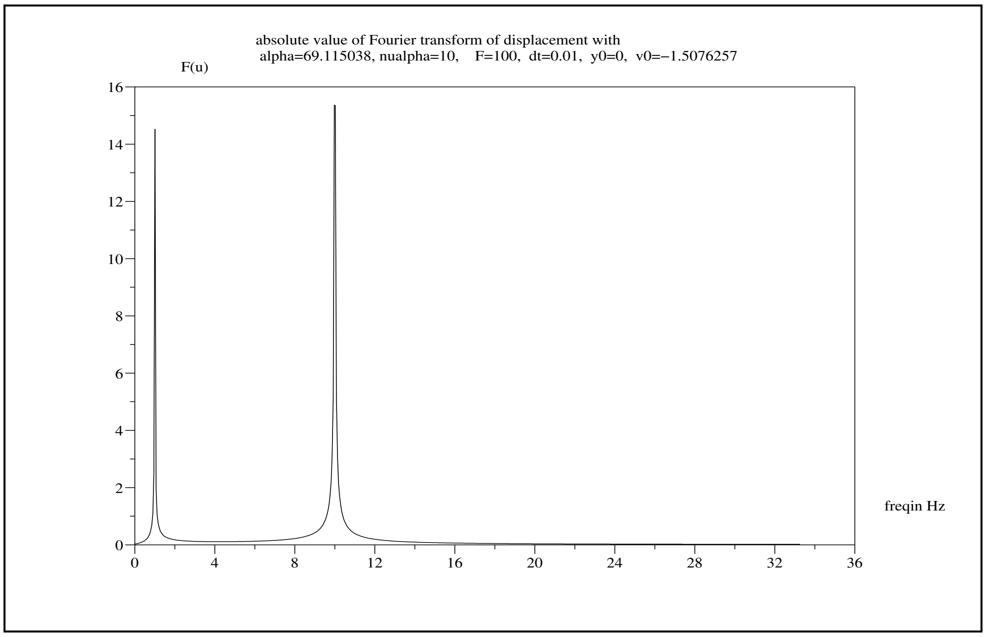


Figure 9: Fourier of linear response, $\nu_{\omega_0} = 1$


```
fig y1.ps etc      :
k=(2*%pi)^2; m=1; nua=10; alfa=2*%pi*nua; F=1450; lam=.2;
A=[0 1;-k/m 0]; dt=.01; tmax=6*%pi;
k1=950; sk1=" ,k1="+string(k1);
function [xdot]=f3(t,x)
xdot=A*x+ [0;1 ]*( ( -k1*([1,0]*x)^3  )+
 F*sin(alfa*t)  - lam*[0,1]*x)
endfunction
function [Jf3]=jacf3(t,y)
Jf3=A+  [0,0; ( -3*k1*([1,0]*y)^2  ) , - lam ]
endfunction
```

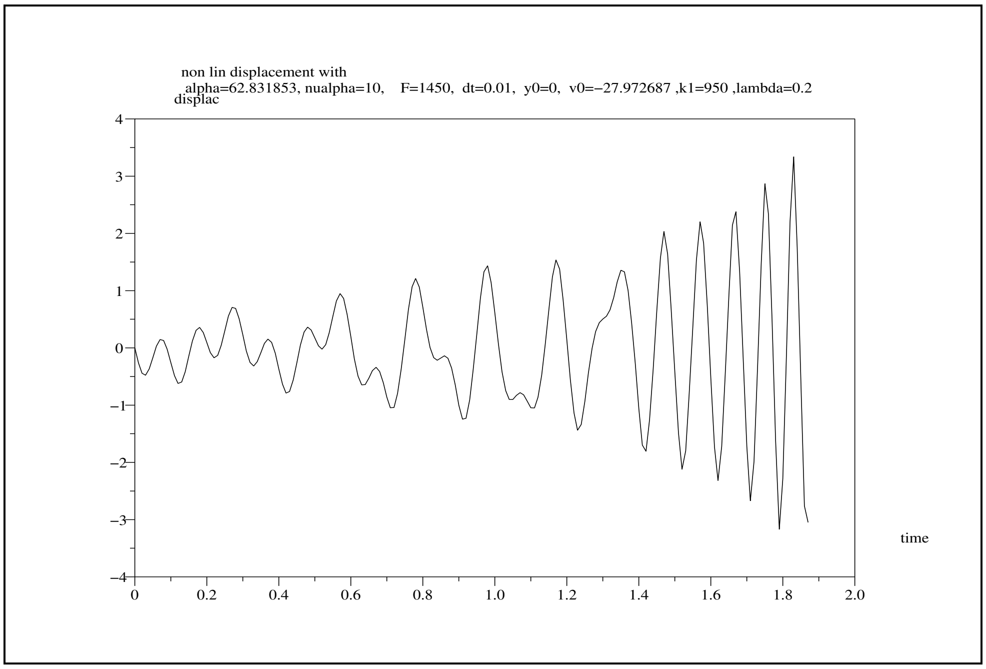


Figure 10: zoom of non linear response y, $\nu_{\omega_0} = 1$

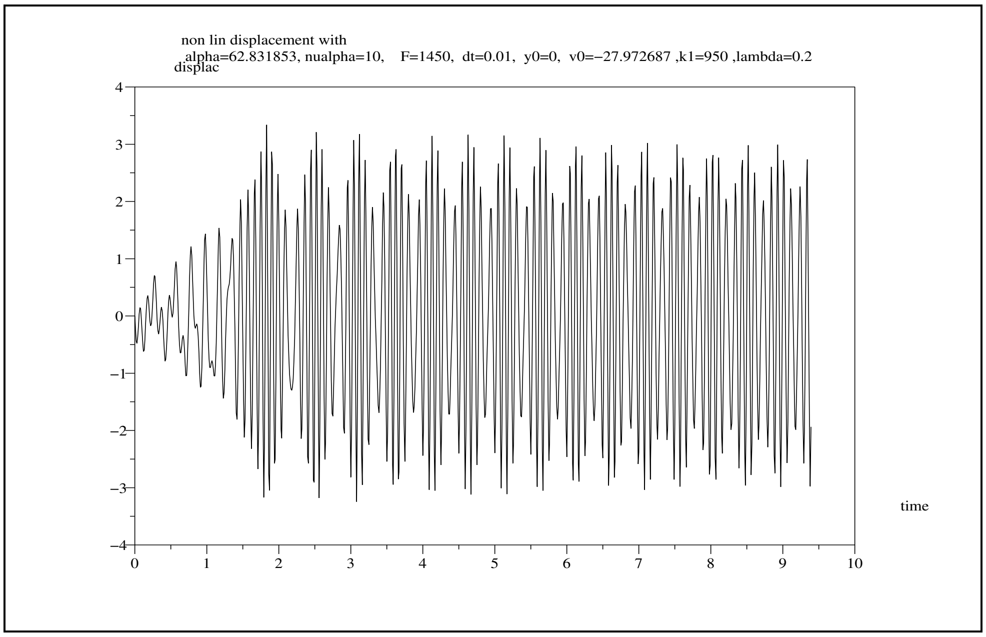


Figure 11: non linear response y, $\nu_{\omega_0} = 1$

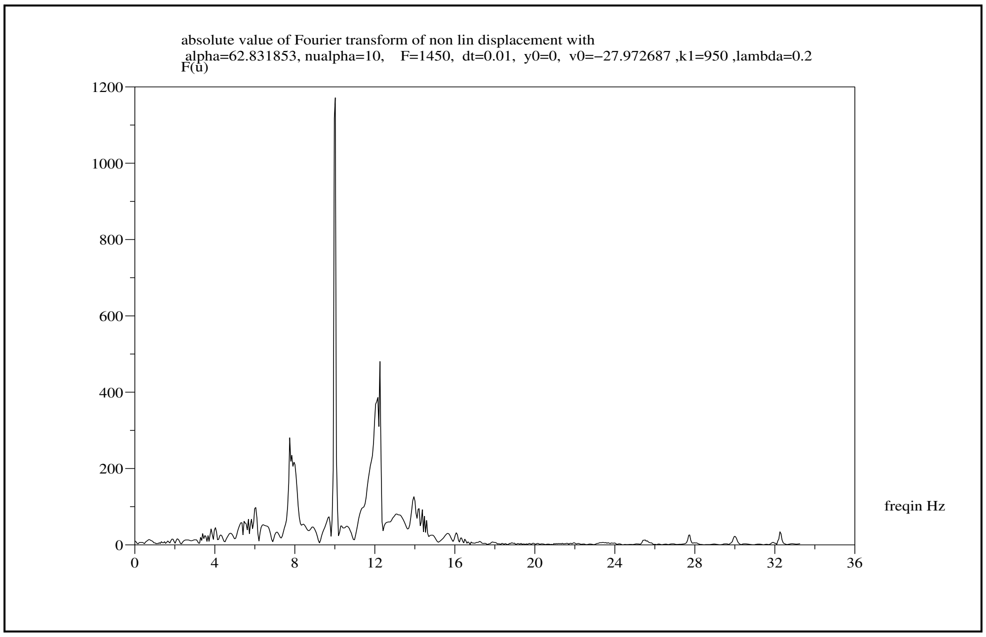


Figure 12: Fourier of non linear response y, $\nu_{\omega_0} = 1$

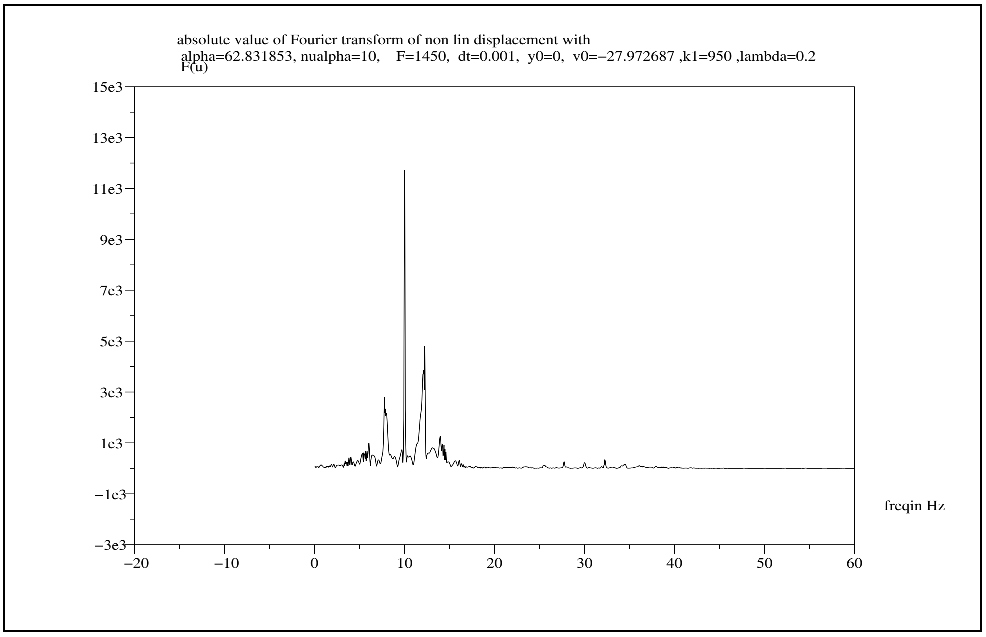
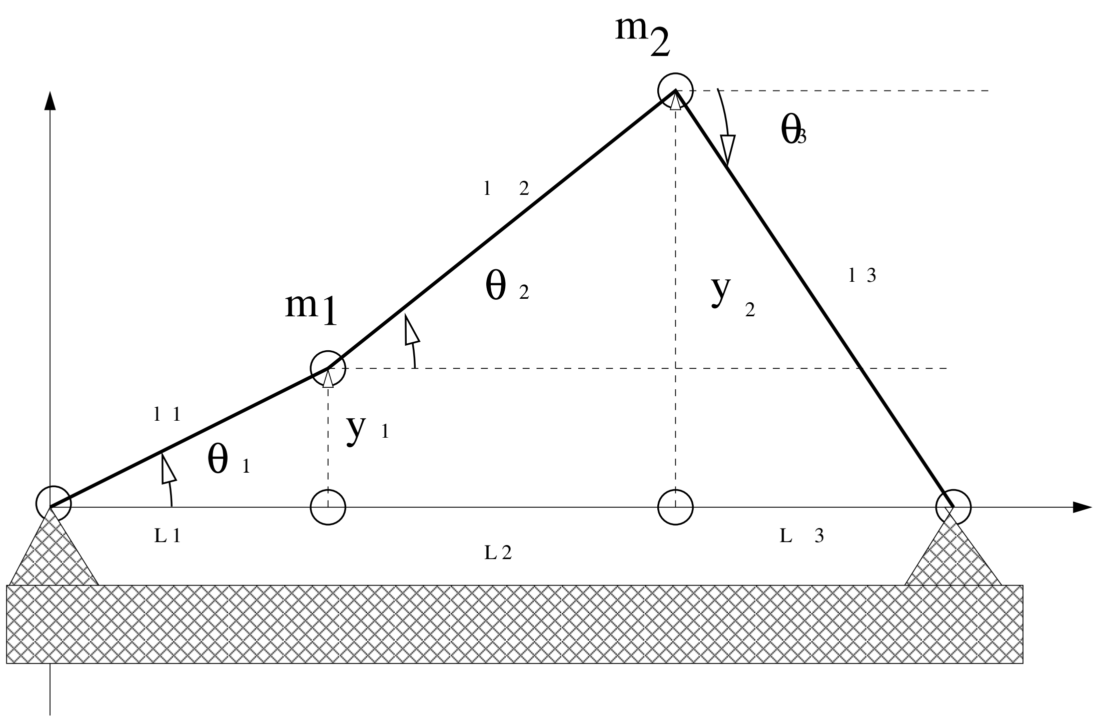


Figure 13: Fourier of non linear response y $\nu_{\omega_0} = 1$



Two masses on stretched cables

4 Transverse vibrations: vibrating masses on streched cables in large displacement

Work performed with Theissen (doctoral student of U. Muenster); Erasmus students N. Goris and I. Altrogge worked on this topic during their stay in UNSA (2004-2005). We consider n masses attached to horizontal springs (or cables) which are in tension T_0, at rest ; the tension is positive when the cable is in traction which is assumed; at rest the mass m_i is submited to the force T the masses are moving (vertically) transversely to the springs; we denote by uper case letters quantities in the rest position and lower case in the current configuration.

4.1 Masses in vertical displacement

Here we assume that the masses can move only vertically.

- L_i lenth at rest; l_i lenth at time t; as the masses are moving vertically: $l_i^2 = L_i^2 + (y_i - y_{i-1})^2$
- and the change of tension of the linear elastic spring due to the change of of lenth $T_i = T_0 + k_i[l_i(y) - L_i] = T_0 + k_i(\sqrt{L_i^2 + (y_i - y_{i-1})^2} - L_i)$. this tension is directed

along the axis of the spring.

- Denote by θ_i, the angle of the spring with the horizontal axis, we have
- $y_1 = L_1 tan(\theta_1), \quad y_i - y_{i-1} = L_i tan(\theta_i) \quad\quad y_n - y_{n-1} = L_n tan(\theta_n).$

We enforce $y_n = 0$. See the picture with two masses and 3 cables.

The equation of the dynamics:

$$m_i y_i" = -T_i sin(\theta_i) + T_{i+1} sin(\theta_{i+1}) + u_i \quad i = 1 \dots n \tag{4.1}$$

where $-T_i sin(\theta_i) + T_{i+1} sin(\theta_{i+1})$ is the vertical component of the force acting on mass i; we assume that there is no horizontal movement so the horizontal component of the force does not work. The applied load on mass i is denoted by u_i; it is the control to be determined.

Set

$$\zeta_i = \frac{(y_i - y_{i-1})}{L_i}, \quad \text{and note that } sin(arctan(\zeta_i)) = \frac{\zeta_i}{\sqrt{1+\zeta_i^2}} \text{ so that} \tag{4.2}$$

$$T_i sin(\theta_i) = \left(T_0 + k_i(L_i\sqrt{1+\zeta_i^2} - L_i)\right)\frac{\zeta_i}{\sqrt{1+\zeta_i^2}} = \tag{4.3}$$

$$(T_0 - k_iL_i)\frac{\zeta_i}{\sqrt{1+\zeta_i^2}} + k_iL_i\zeta_i \tag{4.4}$$

Possible approximations:

$$T_i sin(\theta_i) = (T_0 - k_iL_i)(\zeta_i - \frac{1}{2}{\zeta_i}^3 + \frac{3}{8}{\zeta_i}^5 + O\left({\zeta_i}^7\right)) + k_iL_i\zeta_i = \tag{4.5}$$

$$T_0\zeta_i + (T_0 - k_iL_i)(-\frac{1}{2}{\zeta_i}^3 + \frac{3}{8}{\zeta_i}^5 + O\left({\zeta_i}^7\right)) \tag{4.6}$$

Same expansion for $T_{i+1}sin(\theta_{i+1})$ with $\zeta_{i+1} = \frac{(y_{i+1}-y_i)}{L_{i+1}}$

4.1.1 Linearized equation

$$m_iy_i'' = -T_0\left(\frac{(y_i - y_{i-1})}{L_i} + \frac{(y_{i+1} - y_i)}{L_{i+1}}\right) + u_i$$

corrector equations may be obtained; details for 1 d.o.f below.

$$l_i(y) - L_i = \frac{(y_i - y_{i-1})^2}{2L_i} - \frac{(y_i - y_{i-1})^4}{8L_i^3} + O((y_i - y_{i-1})^6) \tag{4.7}$$

$$sin(\theta_i) = sin(atan(\frac{y_i - y_{i-1}}{L_i}) = \tag{4.8}$$

$$\frac{y_i - y_{i-1}}{L_i} - \frac{(y_i - y_{i-1})^3}{2L_i} + \frac{3(y_i - y_{i-1})^5}{8L_i} + O((\frac{y_i - y_{i-1}}{L_i}))^7 \tag{4.9}$$

4.2 Case with 1 d.o.f

4.2.1 Model with 1 d.o.f

In this case, with $y_0 = 0,\ y_2 = 0$ we have

$$m_1 y_1" = -T_1 sin(\theta_1) + T_2 sin(\theta_2) + u_1 \tag{4.10}$$

with $\theta_1 = atan(y_1/L_1), \quad \theta_2 = -atan(y_1/L_2)$

$$m_1 y_1" = -T_1 sin(atan(\frac{y_1}{L_1})) - T_2 sin(atan(\frac{y_1}{L_2})) + u_1 \tag{4.11}$$

Linearized equation

$$m_1 y_1" = -T_0(\frac{1}{L_1} + \frac{1}{L_2}) y_1 + u_1 \tag{4.12}$$

The numerical solution of this model may be performed without stiff hypothesis with `scilab` routine `ode`; (sin(tan) is Lipshitz) but

it is not obvious to prescribe the right mechanical constants

to obtain clear intermodulation peaks;

also trouble of the experiments!

4.2.2 Approximation

Here set $\zeta_1 = \frac{y_1}{L_1}$, $\zeta_2 = -\frac{y_1}{L_2}$. Start from previous approximation

$$-T_1 sin(\theta_1) + T_2 sin(\theta_2) = \tag{4.13}$$

$$T_0(\zeta_2 - \zeta_1) - (T_0 - k_1 L_1)(-\frac{\zeta_1^3}{2} + \frac{3\zeta_1^5}{8}) + (T_0 - k_2 L_2)(-\frac{\zeta_2^3}{2} + \frac{3\zeta_2^5}{8}) + O(\zeta_1^7 + \zeta_2^7), \tag{4.14}$$

$$\text{expand } y_1 = \epsilon\eta_1 + \epsilon^2\eta_2 + \epsilon^3\eta_3 + O(\epsilon^4) \text{ to get} \tag{4.15}$$

$$-T_1 sin(\theta_1) + T_2 sin(\theta_2) = \tag{4.16}$$

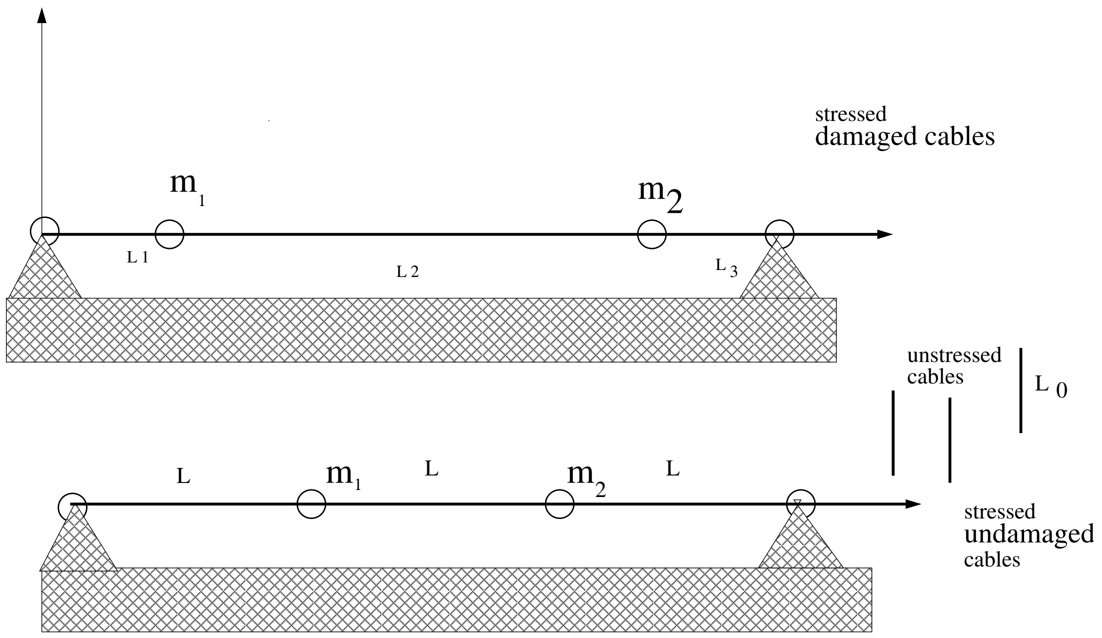
$$-\epsilon T_0(\frac{1}{L_1} + \frac{1}{L_2})\eta_1 - \epsilon^2 T_0(\frac{1}{L_1} + \frac{1}{L_2})\eta_2 - \epsilon^3 T_0(\frac{1}{L_1} + \frac{1}{L_2})\eta_3 + \tag{4.17}$$

$$\frac{\epsilon^3}{2}\left(\frac{T_0 - k_1 L_1}{L_1^3} + \frac{T_0 - k_2 L_2}{L_2^3}\right)\eta_1^3 + O(\epsilon^4) \tag{4.18}$$

- The term in ϵ provides the linearised equation,
- the second equation provides $\eta_2 = 0$
- and the term in ϵ^3,

$$m\eta_3'' = T_0(\frac{1}{L_1} + \frac{1}{L_2})\eta_3 + \frac{1}{2}\left(\frac{T_0 - k_1 L_1}{L_1^3} + \frac{T_0 - k_2 L_2}{L_2^3}\right)\eta_1^3 \tag{4.19}$$

equation similar to what is obtained for the simplest mechanical example!



Two masses on stretched cables

4.2.3 A possible damage of a cable

is breakage of several fibers, this will cause decrease of rigidity k_1 say for cable 1.

- Let us start with undamaged cables of same rigidity k. If we note L_0, the common length of the unstressed cables, and L their common stressed lenth, their tension is $T_0 = k(L - L_0)$;
- now, after damage, $k_1 < k = k_2$, cable 1 becomes longer and cable 2 shorter, $L_1 > L_2$, the tension goes down to $T_{00} = k_1(L_1 - L_0) = k_2(L_2 - L_0)$;
- note the limit case of cable 1 broken is $k_1 = 0$ so that the cable 2 gets lenth L_0 but the system is no longer working properly!
- Before such a breakdown, if the change of tension is substantial, this causes a substantial change of the fundamental frequency; indeed, this is the routine monitoring of cable bridges!
- The nonlinear vibroacoustic testing aims at monitoring the cables before such a substantial change.

4.2.4 Datas

- L_0 unstressed length,
- L half of the lenth of the span, or lenth of each of the stressed undamaged cables.
- k undamaged srping constant,
- from which "undamaged" tension $T_0 = k(L - L_0)$,
- L_1 (with $L_0 < L_1 < L$) increased lenth of the damaged cable,
- from which, $L_2 = 2L - L_1$ decreased lenth of the undamged cable,
- from which "damaged"tension $T_{0d} = k(L_2 - L_0)$,
- from which spring constant of the damaged cable $k_1 = \frac{T_{od}}{L_1 - L_0}$

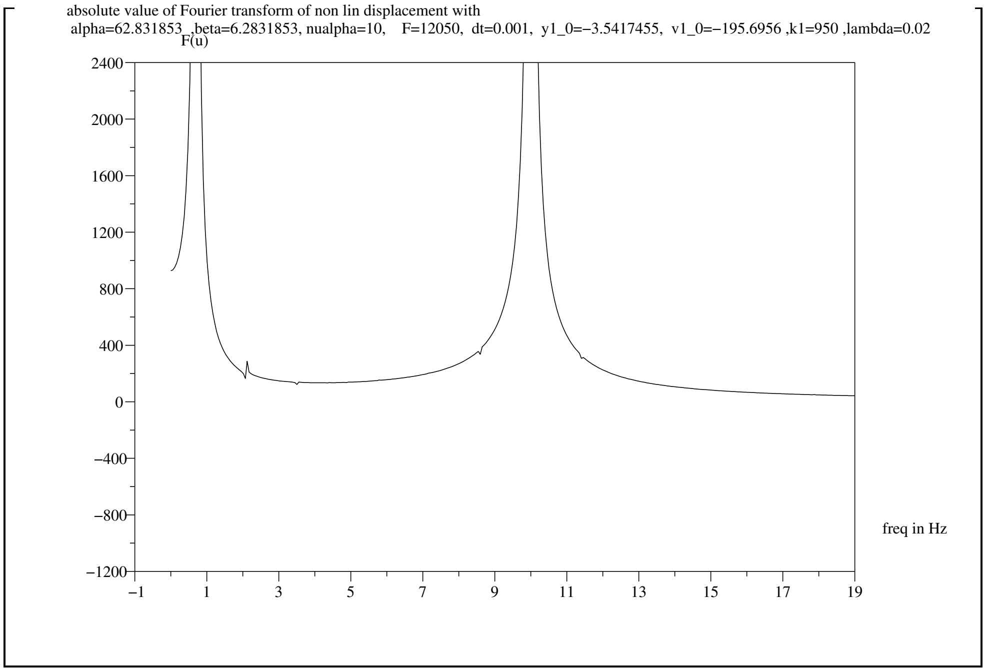


Figure 14: ynl,f10,z,F12050

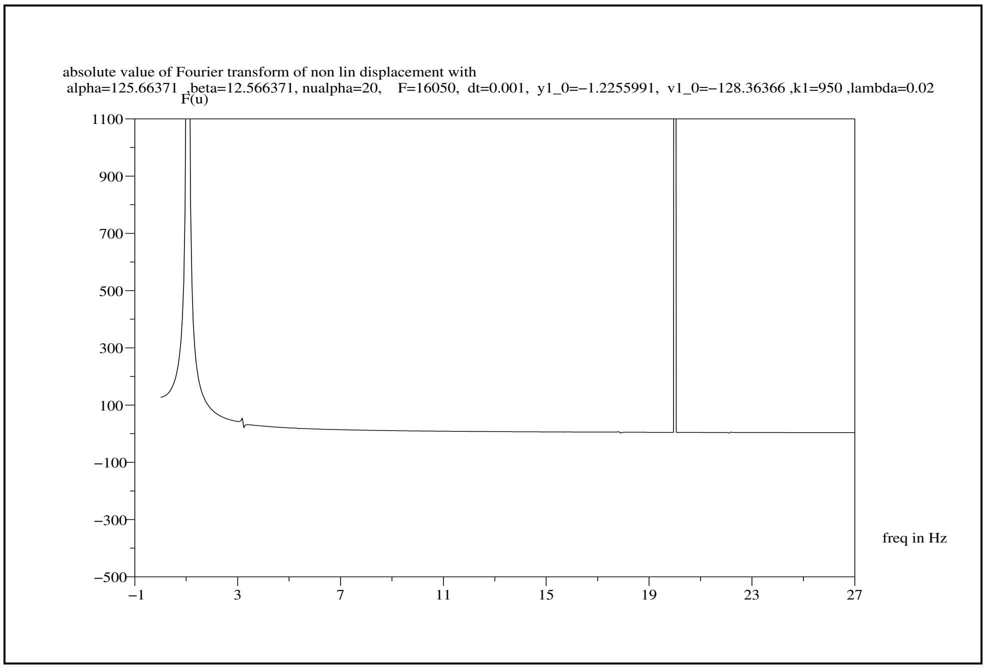


Figure 15: ynl,f20,z,F16050

5 A non linear string model

A model of non linear string has been introduced first by Kirchoff in 1877 and rederived by Carrier in 1945.

$$y_{tt} - T(\int_0^l y_x^2) y_{xx} = f \tag{5.1}$$

For the classical linear string model, T is the tension of the string, **assumed to be constant**; in a next step, a natural asumption is:

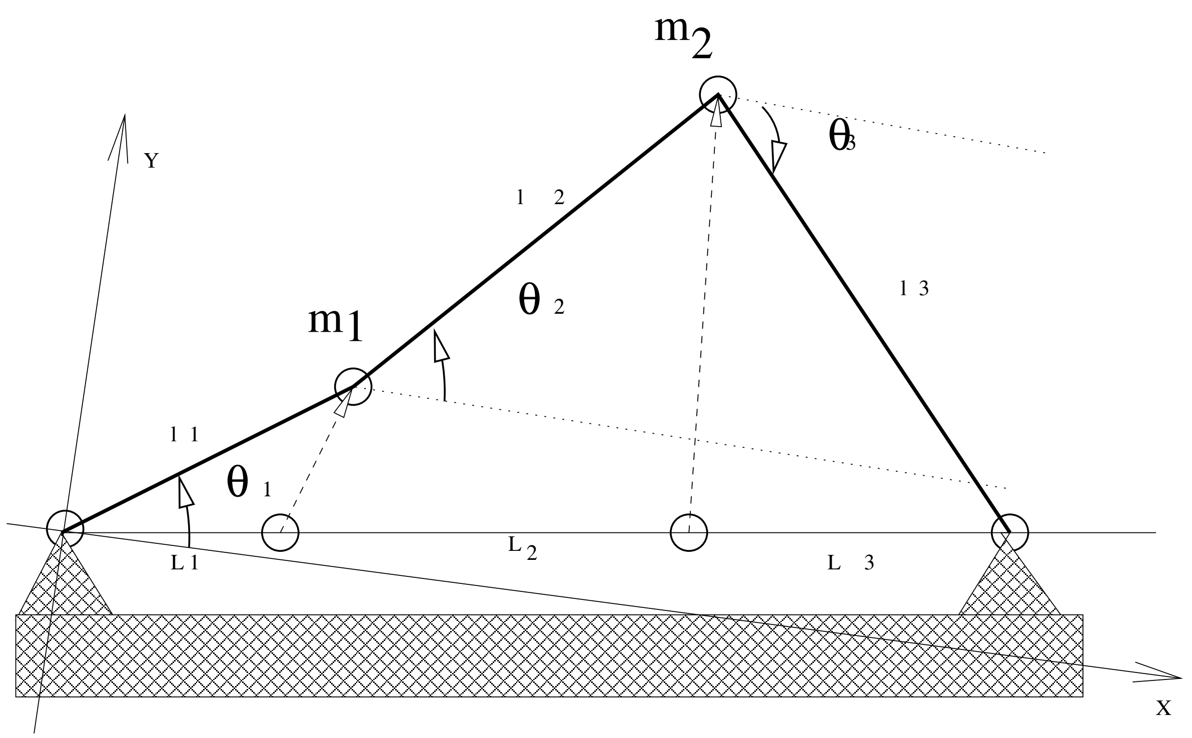
$$T = T_0 + k \int_0^l y_x^2$$

it involves the linearized change of lenth as the length of the deformed string is:

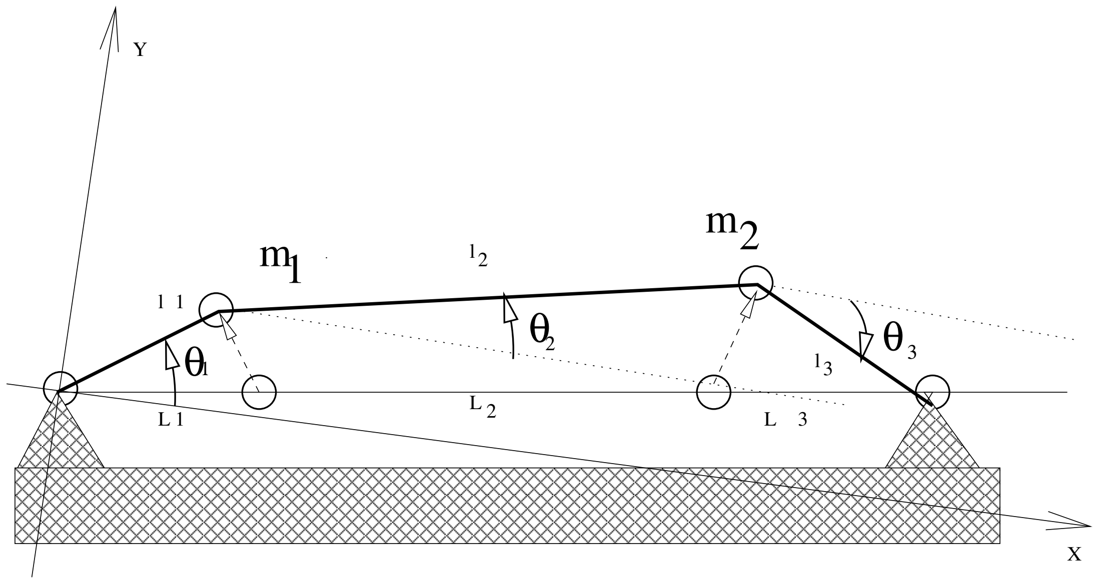
$$l(y) = \int_0^l \sqrt{1 + y_x^2}$$

- Several mathematical studies of this type of equations have been performed recently (Medeiros(1994), Clark- Lima (1997).
- Following the lines of the discrete model, we intend to **investigate** a string made of two materials (*safe and dameged*).
- For a damaged string, k will be small on a small portion of the string:

$$T = T_0 + k\int_0^{d-\epsilon} y_x^2 + k_d\int_{d-\epsilon}^{d+\epsilon} y_x^2 + k\int_{d+\epsilon}^{l} y_x^2$$



Two masses on stretched cables moving freely



Two masses on stretched cables (cable 2 damaged) moving freely

6 Masses free to move in a plane

Here, we assume that the masses can move freely; we denote:

$$\begin{pmatrix} X_i \\ Y_i \end{pmatrix} \text{ the position at rest, } \begin{pmatrix} x_i \\ y_i \end{pmatrix} \text{ the curent position} \tag{6.1}$$

- L_i lenth at rest; l_i lenth at time t; as the masses are moving freely:

$$l_i(x,y)^2 = L_i^2 + ((x_i - x_{i-1}) + (y_i - y_{i-1}))^2$$

- and the change of tension of the linear elastic spring due to the change of of lenth $T_i = T_{0,i} + k_i[l_i(x,y) - L_i] =$. this tension is directed along the axis of the spring:

$$\vec{T_i} = T_i \vec{\tau_i}$$

- Denote by θ_i, the angle of the spring with the horizontal axis, we have,

$$\vec{\tau}_i = \begin{pmatrix} cos\theta_i \\ sin\theta_i \end{pmatrix}$$

- $y_1 = x_1 tan(\theta_1)$, $\quad y_i - y_{i-1} = (x_i - x_{i-1}) tan(\theta_i) \quad y_n - y_{n-1} = x_n tan(\theta_n)$; but here, it is more convenient to use:

$$y_i - y_{i-1} = l_i(x,y) sin\theta_i, \quad x_i - x_{i-1} = l_i(x,y) cos\theta_i$$

,

Equation of the dynamics

$$m_i x_i" = -T_i cos(\theta_i) + T_{i+1} cos(\theta_{i+1}) + f_i \quad i = 1 \dots n \tag{6.2}$$

$$m_i y_i" = -T_i sin(\theta_i) + T_{i+1} sin(\theta_{i+1}) + g_i \quad i = 1 \dots n \tag{6.3}$$

We can express θ_i with respect to x_i, y_i, to obtain:

$$m_i x_i" = -T_i \frac{x_i - x_{i-1}}{l_i(x,y)} + T_{i+1} \frac{x_{i+1} - x_i}{l_{i+1}(x,y)} + f_i \quad i = 1 \dots n \tag{6.4}$$

$$m_i y_i" = -T_i \frac{y_i - y_{i-1}}{l_i(x,y)} + T_{i+1} \frac{y_{i+1} - y_i}{l_{i+1}(x,y)} + g_i \quad i = 1 \dots n \tag{6.5}$$

7 Actively controled system, non destructive testing

- The case of an actively controled system is prospective; real experiments are not yet performed.
- **Idea:** to detect damage in real time taking adavantage of the data processed by the real time actuators used for the optimal control; real time control, research group: “Echtzeit Optimierung grosser Systeme”in Germany.
- Example of the vibrating masses: the forces u_i are now the control we consider the simple case of a quadratic functional:

$$F(u) = \int_0^{t_f} \left(\sum_i u_i^2(t) \right) dt$$

  with final time conditions:

$$y_i(t_f) = 0, \; y_i'(t_f) = 0$$

- The initial conditions may be seen as a perturbation of the system, the active control brings to rest the system;

- this process is supposed to be performed regularly during the lifetime of the system; in practice y_i is measured by sensors and the control u_i is a force performed by actuators; both devices transform electric energy in mechanical energy.
- the communication between both devices goes trough some computer
- If we are able to distinguish the response of a damaged system from an undamaged one, this opens the path of monitoring controled systems in real time as a dayly routine during their life.
- Numerical approach: to solve damaged and undamaged system and compare
- Perturbation approach, introduce a small parameter ϵ and expand the solution with respect to it; theoretical basis: the controled system should satisfy second order sufficient conditions (Malanowski, Maurer ...)

Datas for an example of controlled 2 masses worked out by by K. Theissen (U. Muenster)

$T_{0,1} = T_{0,2} = T_{0,3} =$	1
$k_1 = k_2 = k_3 =$	5
$m_1 = m_2 =$	1
$L_1 = L_2 = L_3 =$	1
t_f	100

Figure 16: Frequences of u_1

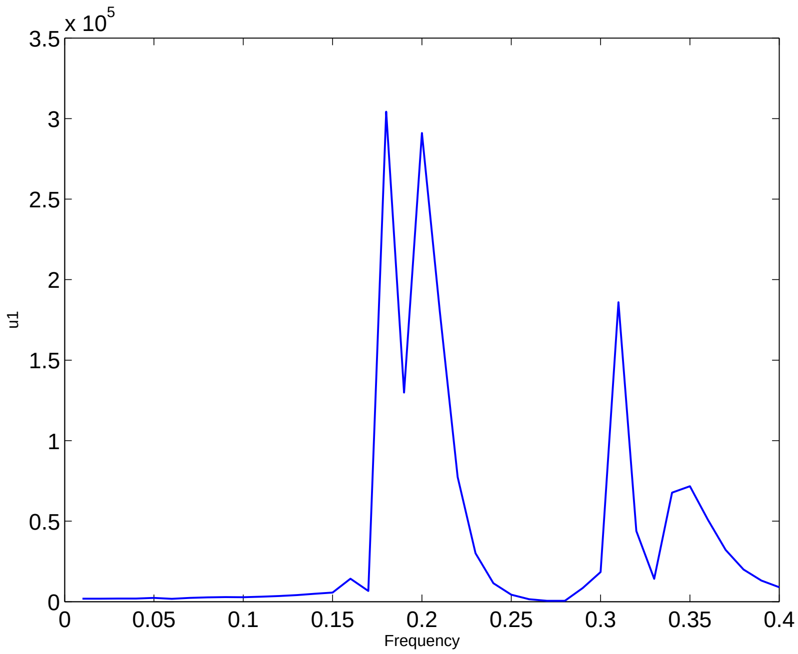
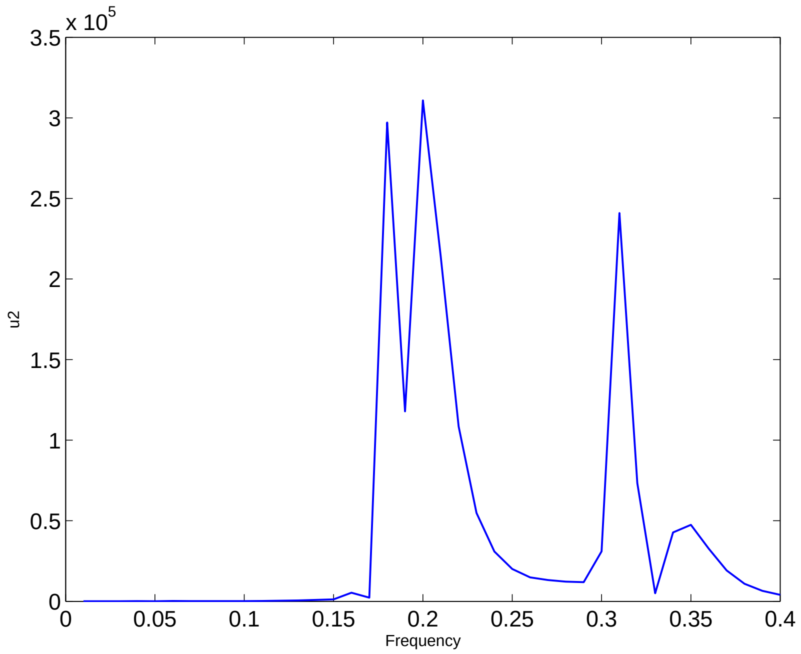
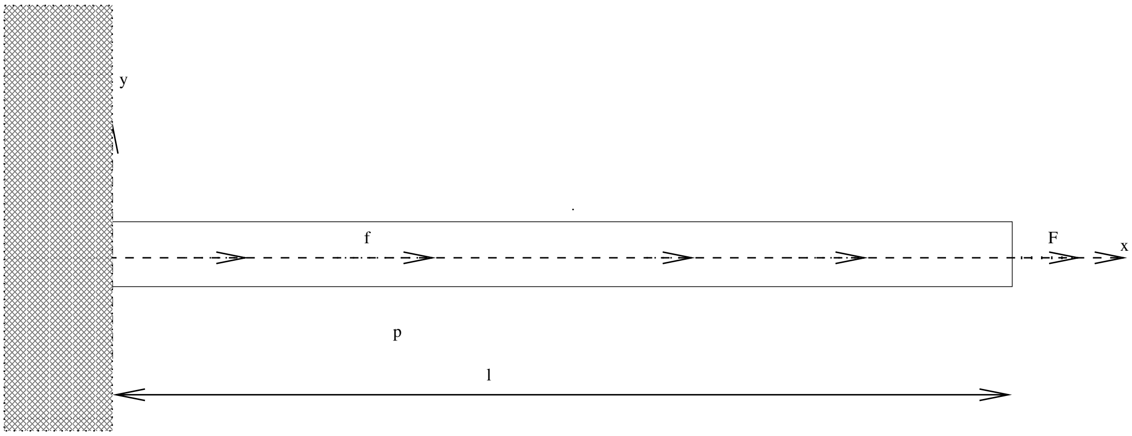


Figure 17: Frequences of u_2





8 Bar models with defects

Bar models with longitudinal waves (dynamical traction and compression) are considered.

$$\rho\frac{\partial^2 u}{\partial t^2} - \frac{\partial n}{\partial x} = f(x,t) \tag{8.1}$$

With a non linear stress-strain law:

$$n = E(A\frac{\partial u}{\partial x} + \epsilon\chi_{[a,b]}(\frac{\partial u}{\partial x})^3) \tag{8.2}$$

Also a linear law is considered with a modified equation::

$$n = EA\frac{\partial u}{\partial x} \tag{8.3}$$

it may correspond to the action of a non linear spring acting on part of the bar :

$$\rho\frac{\partial^2 u}{\partial t^2} - \frac{\partial n}{\partial x} + \epsilon\chi_{[a,b]}u^3 = f(x,t) \tag{8.4}$$

We could as well assume that the applied load is of order epsilon without any assumption on the nonlinearity. Assuming ϵ to be small an approximate solution is searched for with the following "ansatz":

$$u = u_0 + \epsilon u_1 + \dots \quad \text{d'où} \tag{8.5}$$

$$u^3 = u_0^3 + 3\epsilon u_0^2 u_1 + \dots \tag{8.6}$$

$$\frac{\partial u}{\partial x}^3 = \frac{\partial u_0}{\partial x}^3 + 3\epsilon\frac{\partial u_0}{\partial x}^2\frac{\partial u_1}{\partial x} + \dots \tag{8.7}$$

$$\tag{8.8}$$

From which we get for the non linear law:

$$n = E(A\frac{\partial u_0}{\partial x} + \epsilon \left(A\frac{\partial u_1}{\partial x} + \chi_{[a,b]}(\frac{\partial u_0}{\partial x})^3 \right) + \dots \tag{8.9}$$

and for the linear law:

$$n = EA(\frac{\partial u_0}{\partial x} + \epsilon\frac{\partial u_1}{\partial x}) + \dots \tag{8.10}$$

Using these expansions, with the non linear law, the following system is obtained:

$$\rho\frac{\partial^2 u_0}{\partial t^2} - EA\frac{\partial^2 u_0}{\partial x^2} = f(x,t) \tag{8.11}$$

$$\rho\frac{\partial^2 u_1}{\partial t^2} - EA\frac{\partial^2 u_1}{\partial x^2} = -E\frac{\partial}{\partial x}(\frac{\partial u_0}{\partial x})^3\chi_{a,b} \tag{8.12}$$

For the modified equation the same equation for u_0 is found but for u_1:

$$\rho\frac{\partial^2 u_1}{\partial t^2} - EA\frac{\partial^2 u_1}{\partial x^2} = -(u_0)^3\chi_{[a,b]} \tag{8.13}$$

Theoretical justification of the expansions:

Non liner law The situation is complex in full generality: non linear hyperbolic equations exhibit a singularity after a finite time! But: the experiments are performed during a short time interval and the Fourier transforms are computed on these time intervals! Following a suggestion of Guy Metivier we are addressing the problem during a small initial time interval in which the solution is smooth: plan to use an approximation of the equation with a fixed point method proposed in Majda. In any case we should smooth the characteristic function (the material is changing smoothly)!

Modified equation The situation is simpler; we can use a priori inequalities for this type of equation.

8.1 Explicit Solution

Coefficients are assumed to be constant and we consider:

Clamped at both ends: $u(x,0) = 0 = u(x,l)$; Eigenfunctions are introduced :

$$EA\frac{\partial^2\phi}{\partial x^2} = -\lambda\rho\phi \tag{8.14}$$

$$\phi(0) = 0 = \phi(l) \tag{8.15}$$

we find $\lambda_k = \frac{k^2\pi^2}{l^2}\frac{EA}{\rho}$, on pose $\omega_k = \sqrt{\lambda_k}$ and the normalised eigenfunction: $\phi_k = \sqrt{\frac{2}{l}}sin(\frac{k\pi}{l}x)$.

8.1.1 Computation of u_0

let us consider $\boxed{\text{a force of frequency } \frac{\alpha}{2\pi}}$

$$f(x,t) = Fcos(\alpha t)sin(\frac{k\pi}{l}x) \tag{8.16}$$

with initial velocity: $\frac{\partial u}{\partial t}(x,0) = 0$ The solution

$$u_0 = \frac{Fcos(\alpha t)}{\rho(-\alpha^2 + \lambda_k)} sin(\frac{k\pi}{l}x) \tag{8.17}$$

corresponding to an initial condition

$$u_0(x,0) = \frac{F}{\rho(-\alpha^2 + \lambda_k)} sin(\frac{k\pi}{l}x) \frac{\partial u_0}{\partial t}(x,0) \tag{8.18}$$

For the initial condition:

$$u_0(x,0) = a_0 sin(\frac{k\pi}{l}x) \tag{8.19}$$

the solution is:

$$u_0(x,0) = \left[\frac{F}{\rho(-\alpha^2 + \lambda_k)}(cos(\alpha t) - cos(\omega_k t)) + a_0 cos(\omega_k t)\right] sin(\frac{k\pi}{l}x) \tag{8.20}$$

8.1.2 Computation of u_1

Considering the first solution with a global non linearity, we get:

$$u_0^3 = \frac{cos(\alpha t)^3}{\rho^3(-\alpha^2+\lambda_k)^3} sin^3(\frac{k\pi}{l}x) = \tag{8.21}$$

$$\frac{cos(\alpha t)^3}{\rho(-\alpha^2+\lambda_k)^3}\Big[cos(3\alpha t)sin(3\frac{k\pi x}{l}) - 3cos(3\alpha t)sin(\frac{k\pi x}{l}) \tag{8.22}$$

$$+3cos(\alpha t)sin(\frac{3k\pi x}{l}) - 9cos(\alpha t)sin(\frac{k\pi x}{l})\Big] \tag{8.23}$$

$$\frac{\partial u_0}{\partial x}^3 = \frac{k^3\pi^3}{l^3}u_0^3; \qquad \frac{\partial}{\partial x}\frac{\partial u_0}{\partial x}^3 = \frac{k^3\pi^3}{l^3}\frac{\partial u_0^3}{\partial x} = \tag{8.24}$$

$$\tag{8.25}$$

$\boxed{\text{solution } u_1 \text{ with frf́requency } \frac{3\alpha}{2\pi}}$ or $\frac{2\alpha}{2\pi}$ for a quadratic non linearity.

$$\frac{cos(\alpha t)^3}{\rho^3(-\alpha^2+\lambda_k)^3}\frac{k^4\pi^4}{l^4}\Big[3cos(3\alpha t)cos(3\frac{k\pi x}{l}) + 3cos(3\alpha t)cos(\frac{k\pi x}{l}) \tag{8.26}$$

$$+9cos(\alpha t)cos(\frac{3k\pi x}{l}) - 9cos(\alpha t)cos(\frac{k\pi x}{l})\Big] \tag{8.27}$$

Second case

for the second pair of boundary conditions, we set:

$$c = \frac{F}{\rho(-\alpha^2+\lambda_k)} \quad d = \left(-\frac{F}{\rho(-\alpha^2+\lambda_k)} + a_0\right) \tag{8.28}$$

Now we have:

$$u_0 = (c\ cos(\alpha t) + d\ cos(\omega_k t))\ sin(\frac{k\pi x}{l}) \tag{8.29}$$

$$\begin{aligned}(u_0)^3 = \Big[\frac{c^3}{4}cos(3\alpha t) + \frac{3c}{2}(\frac{c^2}{2}+d^2)cos(\alpha t) + \\ \frac{3c^2 d}{4}(cos((\omega_k+2\alpha)t) + cos((\omega_k-2\alpha)t)) + \\ \frac{3cd^2}{4}(cos((2\omega_k+\alpha)t) + cos((2\omega_k-\alpha)t)+) \\ +\frac{3d}{2}(\frac{d^2}{2}+c^2)cos(\omega_k t)\frac{d^3}{4}cos(3\omega_k t)\Big] \\ \frac{1}{4}\left(3sin(\frac{k\pi x}{l}) - sin(\frac{3k\pi x}{l})\right)\end{aligned} \tag{8.30}$$

$$\tag{8.31}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u_0}{\partial x}\right)^3 = \frac{k^3\pi^3}{l^3}\frac{\partial u_0^3}{\partial x} = \tag{8.32}$$

$$\frac{3k^4\pi^4}{4l^4}\Big[\frac{c^3}{4}cos(3\alpha t) + \frac{3c}{2}(\frac{c^2}{2} + d^2)cos(\alpha t) +$$
$$\frac{3c^2d}{4}(cos((\omega_k + 2\alpha)t) + cos((\omega_k - 2\alpha)t)) +$$
$$\frac{3cd^2}{4}(cos((2\omega_k + \alpha)t) + cos((2\omega_k - \alpha)t) +)$$
$$+\frac{3d}{2}(\frac{d^2}{2} + c^2)cos(\omega_k t)\frac{d^3}{4}cos(3\omega_k t)\Big]$$
$$\left(cos(\frac{k\pi x}{l}) - cos(\frac{3k\pi x}{l})\right) \tag{8.33}$$

We notice clearly terms of frequency $\frac{\alpha}{2\pi}$ and $\frac{3\alpha}{2\pi}$ but also cross-modulations: $\frac{\omega_k+2\alpha}{2\pi}$ et $\frac{2\omega_k+\alpha}{2\pi}$ and frequencies $\frac{3\omega_k}{2\pi}$ $\frac{\omega_k}{2\pi}$. This last term provides secular terms for the corrector term u_1; they ought to be eliminated for example by using some renormalization technique:

$$t = s(1 + \epsilon\omega_1 + \dots) \tag{8.34}$$

We notice that the perturbation is larger if α is close to ω_k. this fact is used in practice: the applied load uses two frequencies with the low one at the first resonance in [8]. Here the low frequency is excited by the initial conditions.

9 Conclusion

- Some simple models governed by ODE pr PDE show intermodulations;
- But what is $\boxed{\text{the relative level of secondary peaks}}$ for a given set of datas deserves investigations: indeed $\boxed{\text{it is also the difficulty of the real experiments}}$
- Need to include other behaviors: shocks, friction
- Need of more precise models: non linear beams including tractional, flexural, torsional effects
- Mixture of local models for the defect and global models for the undamaged structure to obtain precise results at low computational cost.

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