Non destructive testing using non linear vibroacoustic

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1 Background of Fourier transform

1.1 Basic formulas

1.1.1 Fourier serie

For a detailed presentation, see for example Gasquet-Witomski [**?**] and for an engineering view point Lathy [**?**]; for a function f of period T, its expansion in fourier serie is:

$$f(t) = \sum_{n=-\infty}^{+\infty} c_n e^{\frac{2\pi i n t}{T}} \text{ with } c_n = \frac{1}{T}\int_0^T f(t) e^{\frac{-2\pi i n t}{T}} dt \tag{1.1}$$

Discrete Fourier transform: D.F.T. which may be computed quickly by the algorihm of F.F.T. To a sequence $(y_k), k = 0, \ldots, N-1$, is associated an other sequence $(Y_n), n = 0, \ldots, N-1$ with the formulas:

$$Y_n = \frac{\mathrm{T}}{\mathrm{NT}} \sum_{k=0}^{N-1} y_k e^{\frac{-2\pi i n}{T}\frac{kT}{N}} \quad y_k = \sum_{n=0}^{N-1} Y_n e^{\frac{2\pi i n k}{N}} \tag{1.2}$$

An approximation of Fourier coefficients (caution to indixes) may be obtained with :

$$c_n^N = \frac{\mathrm{T}}{\mathrm{NT}} \sum_{k=0}^{N-1} f(\frac{kT}{N}) e^{\frac{2\pi in}{T}\frac{kT}{N}} \quad \text{or with the DFT using} \tag{1.3}$$

$$y_k = f(\frac{kT}{N}) \quad c_n \simeq c_n^N = \begin{cases} Y_n \text{ for } 0 \leq n < \frac{N}{2} \\ Y_{n+N} \text{ for } -\frac{N}{2} \leq n < 0 \end{cases} \tag{1.4}$$

with the error approximation:

$$c_n^N = \sum_{q \neq 0} c_{n+qN} \tag{1.5}$$

Beware to Scilab FFT which provides: $X_n = \sum_{k=0}^{N-1} y_k e^{\frac{2\pi in}{N}k}$ ie the Fourier coefficients are computed from: $Y_n = \frac{1}{N} X_n$.

1.1.2 Fourier transform

$$(\mathcal{F}f)(\nu) = \hat{f}(\nu) = \int_{\mathbf{R}} f(t) e^{-2\pi i \nu t} dt \tag{1.6}$$

$$(\mathcal{F}^{-1} g)(t) = (\bar{\mathcal{F}} g)(t) = \int_{\mathbf{R}} g(\nu) e^{+2\pi i \nu t} d\nu \tag{1.7}$$

$$\widehat{f^m} = (2\pi i \nu)^m \hat{f} \quad \mathcal{F}(-(2\pi i t)^{(m)} f(t)) = \hat{f}^m(\nu) \tag{1.8}$$

$$\tag{1.9}$$

$$\widehat{\chi_{[-A,A]}} = \frac{sin(2\pi\nu A)}{\pi\nu} = 2A.\mathbf{sinc}(2\pi\nu A) \quad \widehat{\chi_{[0,A]}} = e^{-i\pi A\nu}\frac{sin(\pi A\nu)}{\pi\nu} = Ae^{-i\pi A\nu}\mathbf{sinc}(\pi A\nu) \tag{1.10}$$

$$\text{with } \textit{the sampling function ("sinus cardinal")} \quad \boldsymbol{sinc}(t) = \frac{sin(t)}{t} \tag{1.11}$$

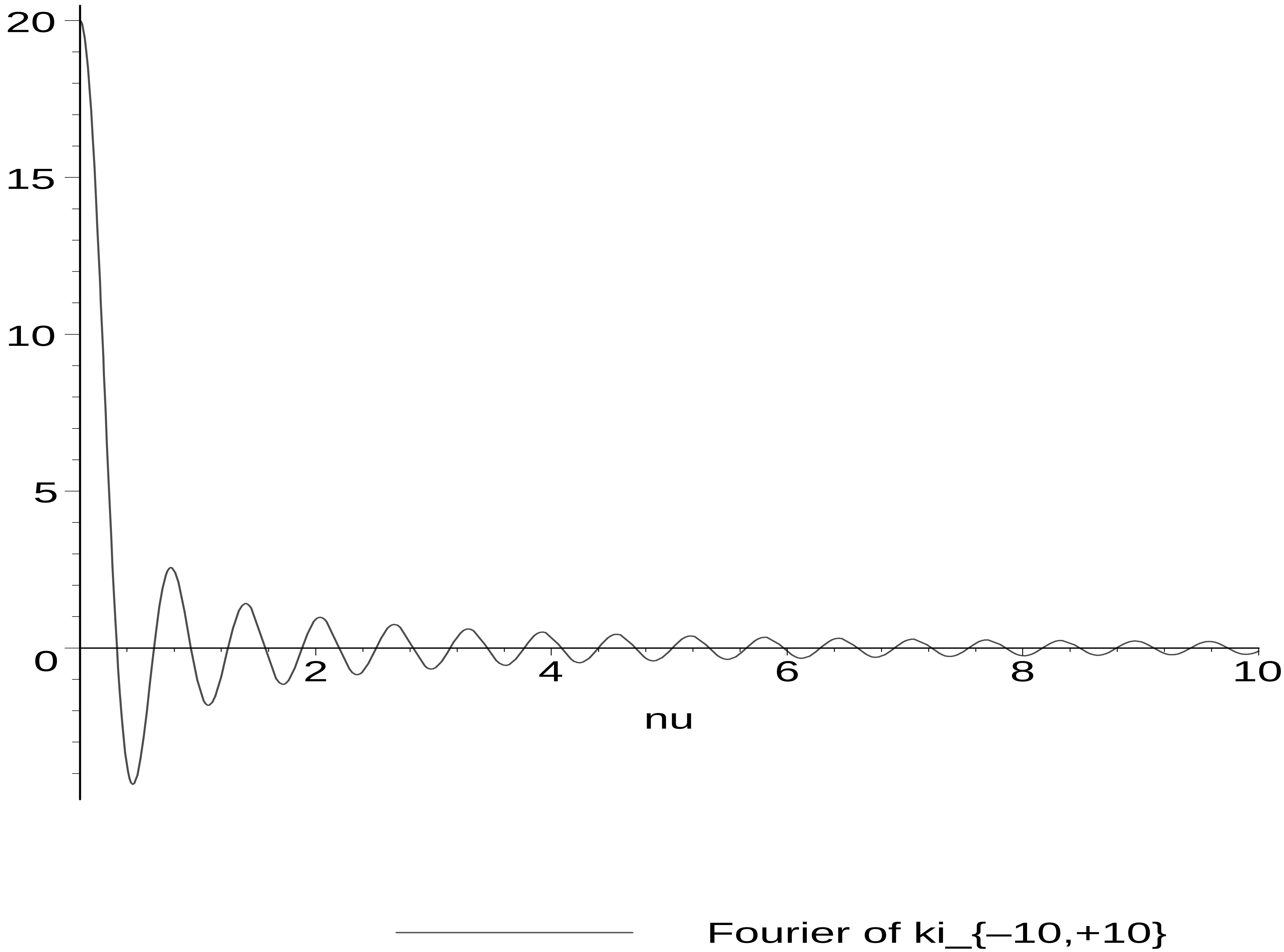
$$\mathcal{F}(e^{2\pi iat}) = \delta_a, \quad \text{and} \quad \mathcal{F}(e^{-2\pi iat}T) = \tau_a\hat{T} = \delta_a * \hat{T} \tag{1.12}$$

$$(\mathcal{F}(cos(2\pi iat)T) = \frac{1}{2}(\tau_a\hat{T} + \tau_{-a}\hat{T}) = \frac{1}{2}(\delta_a * \hat{T} + \delta_{-a} * \hat{T}) \tag{1.13}$$

$$\mathcal{F}(cos(2\pi iat)\chi_{[-A,A]}) = A(\tau_a.\mathbf{sinc}(\mathbf{2}\pi\nu\mathbf{A}) + \tau_{-a}.\mathbf{sinc}(\mathbf{2}\pi\nu\mathbf{A})) \tag{1.14}$$

$$\mathcal{F}(sin(2\pi iat)\chi_{[-A,A]}) = iA(-\tau_a\mathbf{sinc}(\mathbf{2}\pi\nu\mathbf{A}) + \tau_{-a}\mathbf{sinc}(\mathbf{2}\pi\nu\mathbf{A})) \tag{1.15}$$

$$\mathcal{F}(cos(2\pi iat)\chi_{[0,A]}) = \frac{A}{2}(\tau_a e^{-i\pi A\nu}\mathbf{sinc}(\pi\nu\mathbf{A}) + \tau_{-a}e^{-i\pi A\nu}\mathbf{sinc}(\pi\nu\mathbf{A})) \tag{1.16}$$


20
15
10
5
0
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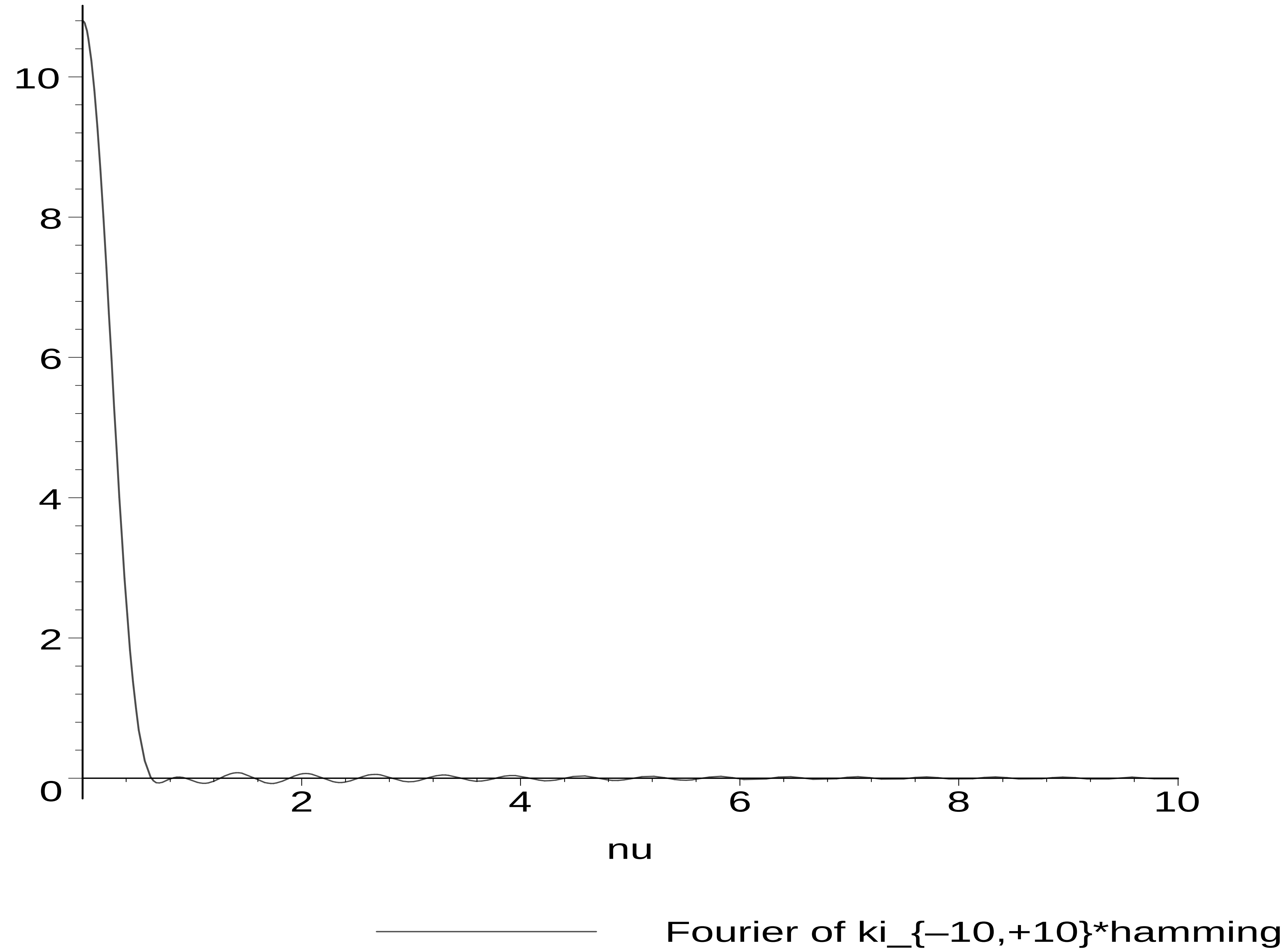
8
10
nu
Fourier of ki_{–10,+10}


10
8
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0
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nu
Fourier of ki_{–10,+10}*hamming

1.2 Sampling

$$\Delta_a = \sum_{n\in\mathbf{Z}} \delta_{na} \quad \widehat{\Delta_a} = \sum_{n\in\mathbf{Z}} e^{-2\pi i\, n\, a\nu} = \frac{1}{a}\Delta_{\frac{1}{a}} \qquad \text{le “peigne”} \tag{1.17}$$

$$\text{also} \qquad \Delta_a = \frac{1}{a}\sum_{n\in\mathbf{Z}} e^{2\pi i n \frac{t}{a}} \tag{1.18}$$

a **sampling** of f is: $af\Delta_a = a\sum_{\mathbf{Z}} f(na)\delta_{na}$ with a the sampling period.

Poisson formula:

$$\sum_{\mathbf{Z}} f(t-na) = \frac{1}{a}\sum_{\mathbf{Z}} \hat{f}(\frac{n}{a})e^{2\pi i n \frac{t}{a}} \text{ (f distribution à support compact)} \tag{1.19}$$

$$\sum_{\mathbf{Z}} \hat{g}(\nu - \frac{n}{a}) = a\sum_{\mathbf{Z}} g(na)e^{-2\pi i n\nu a} \text{ ($\hat{g}$ distribution à support compact)} \tag{1.20}$$

$$a(\widehat{f\Delta_a})(\nu) = \sum_{n\in\mathbf{Z}} \hat{f}(\nu - \frac{n}{a}) = a\sum_{n\in\mathbf{Z}} f(na)e^{-2\pi i\nu na} \tag{1.21}$$

(f tempered and $\hat{f}$ and $\hat{f}$ with support in $[-\nu_c, \nu_c]$)

no overlap of the spectrum for $a \leq \frac{1}{2\nu_c}$.

Shanon For

$$\hat{f} \in L^2(\mathbf{R}) \text{ et } Supp(\hat{f}) \subset [-\nu_c, \nu_c] \tag{1.22}$$

$$\forall a \leq \frac{1}{2\nu_c} \quad f(t) = \sum_{\mathbf{Z}} f(na)\mathbf{sinc}(\frac{\pi}{a}(t - na)) \tag{1.23}$$

$$\textbf{with the sampling function sinc}(t) = \frac{sin(t)}{t} \tag{1.24}$$

Use a low-pass filter before sampling.

1.3 Numerical computation of Fourier transform

$$\hat{f}(\nu) \simeq \int_{-T/2}^{T/2} f(t) e^{-2\pi i \nu t} dt = T c_{\nu T} \quad \text{or with } \nu = \frac{n}{T} \tag{1.25}$$

$$\hat{f}(\frac{n}{T}) \simeq T c_n \quad \text{for } \frac{-N}{2} \leq n \leq \frac{N}{2} \tag{1.26}$$

Fourier coefficients c_n are numerically computed with FFT where $y_k = f(\frac{kT}{N})$:

$$\hat{f}(\nu) \simeq \frac{T}{N} \sum_k f(\frac{kT}{N}) exp(-2\pi i \nu \frac{kT}{N}) \text{ with } \nu = \frac{n}{T} \tag{1.27}$$

$$\simeq \frac{T}{N} \sum_k f(\frac{kT}{N}) exp(-2\pi i n \frac{k}{N}) \text{ with } n = \nu T \tag{1.28}$$

with sampling period $\frac{T}{N} \leq \frac{1}{2\nu_c}$ (no overlap of the spectrum) but ... a function of compact support in time is not of compact support in frequency...

1.4 Exemples

Gate function Commenons par un exemple classique: $f = \chi_{[0,b]}$, sa transformée de Fourier est:

$$\hat{f}(\nu) = \frac{sin(\pi b\nu)}{\pi\nu} exp(-i\pi b\nu) \tag{1.29}$$

Avec $b = 1/2$ et 1000 points utilisés dans l'intervalle $[0, 1]$, on trouve les transformées sur les figures ci jointe. On pourra remarquer que le maximum est correct

Cosinus Pour la fonction $cos(2\pi t)$, il est bien connu que la transformée de Fourier est $\delta_1 + \delta_{-1}$. la transformée discrète est elle mme une approximation numérique de $\int_0^T cos(2\pi t)exp(-2\pi i\nu t)dt$ et l'on trouve un pic de hauteur la moitié de l'intervalle d'intégration. On trouve dans les figures 3 et 4, les transformée de fourier discrète calculée dans $[0, 1]$ puis $[0, 20]$

Exponentielle-cosinus On constate que la transformée de Fourier de $exp(10^{-3}t) * cos(2\pi t)$ est sensiblement égale à celle du cosinus tandis que

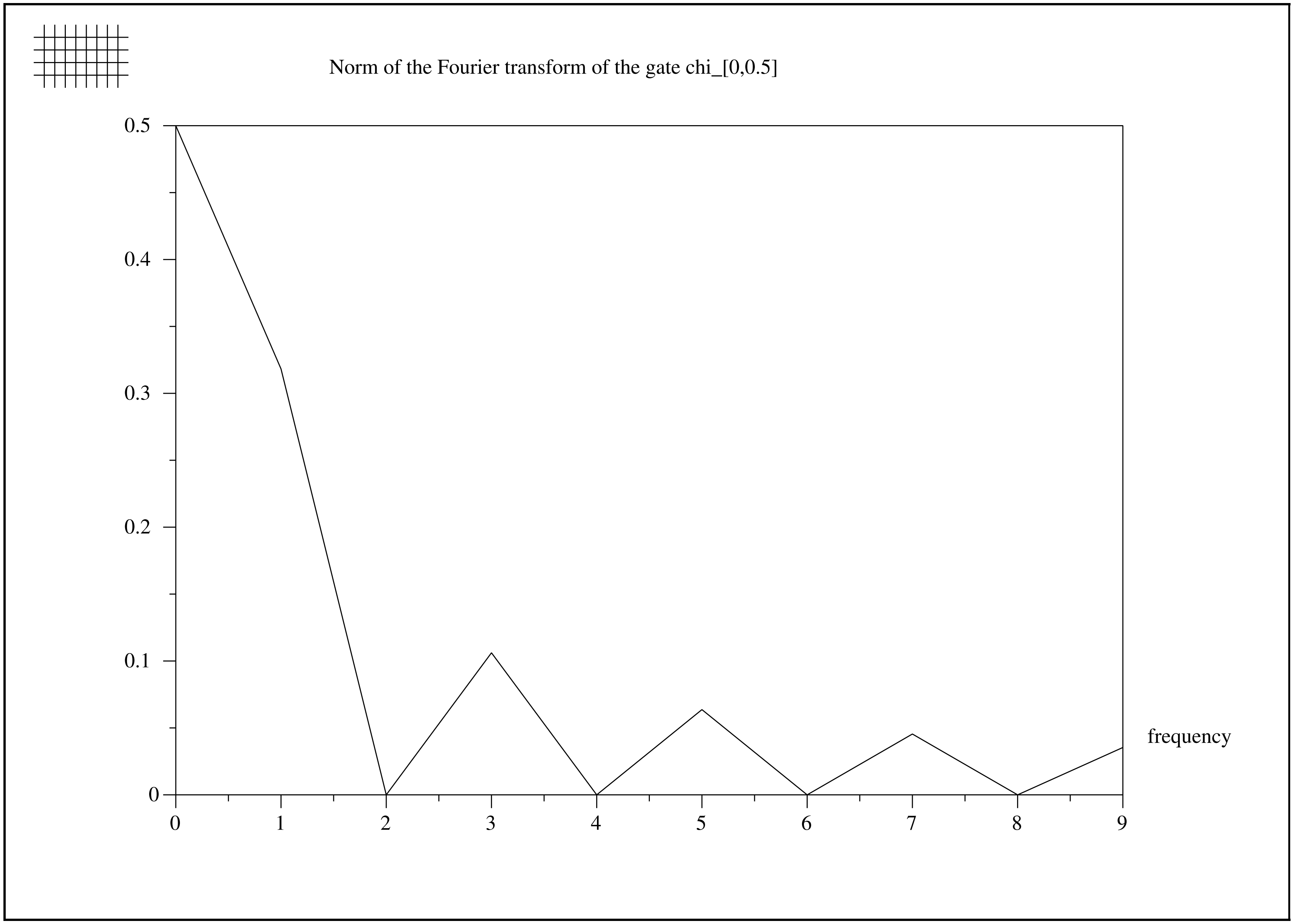

Figure 1: Norm of the Fourier transform of the gate $\chi_{[0,0.5]}$ in $[0, 9]$ herz

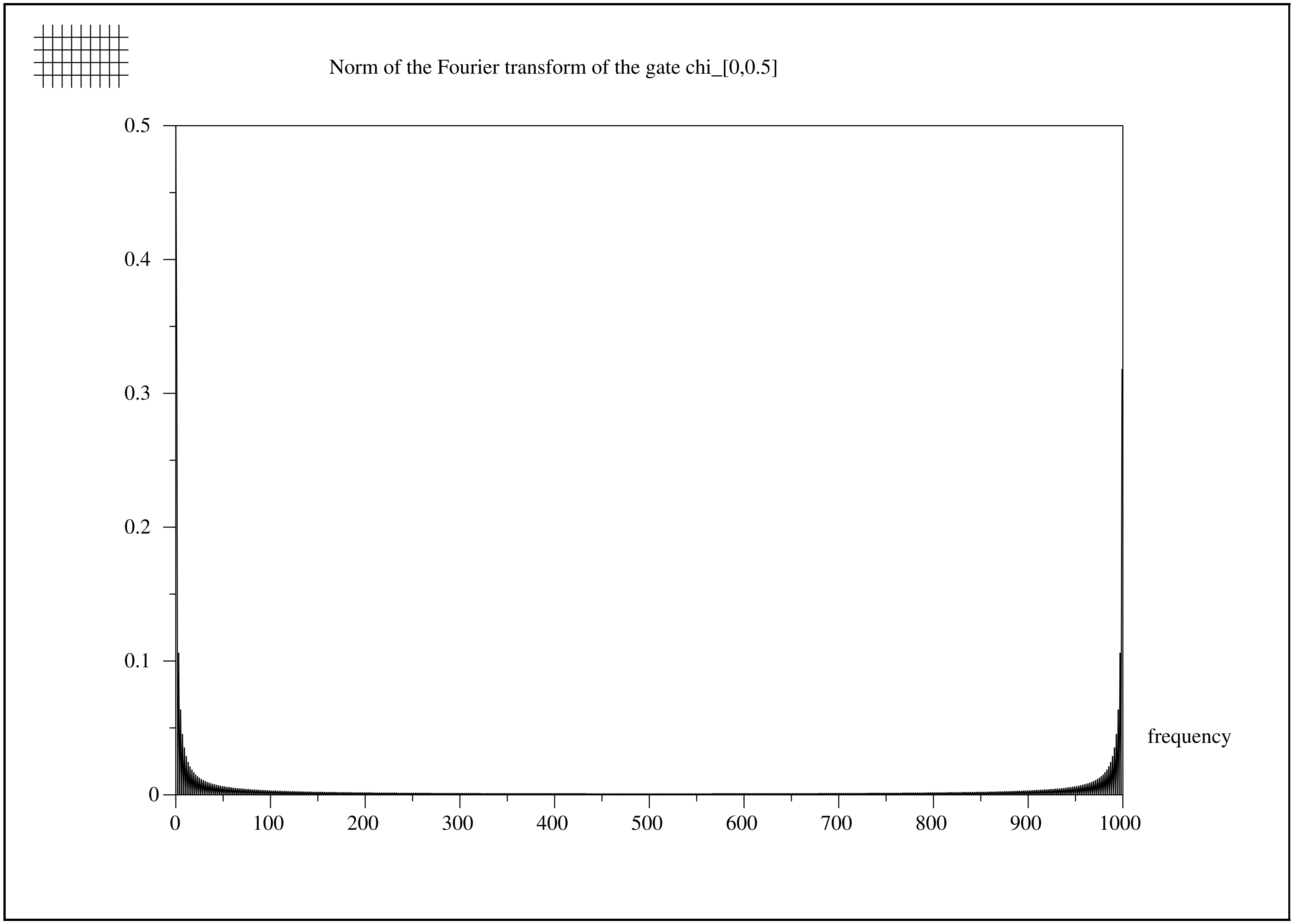

Figure 2: Norm of the Fourier transform of the gate $\chi_{[0,0.5]}$ in $[0, 1000]$ herz

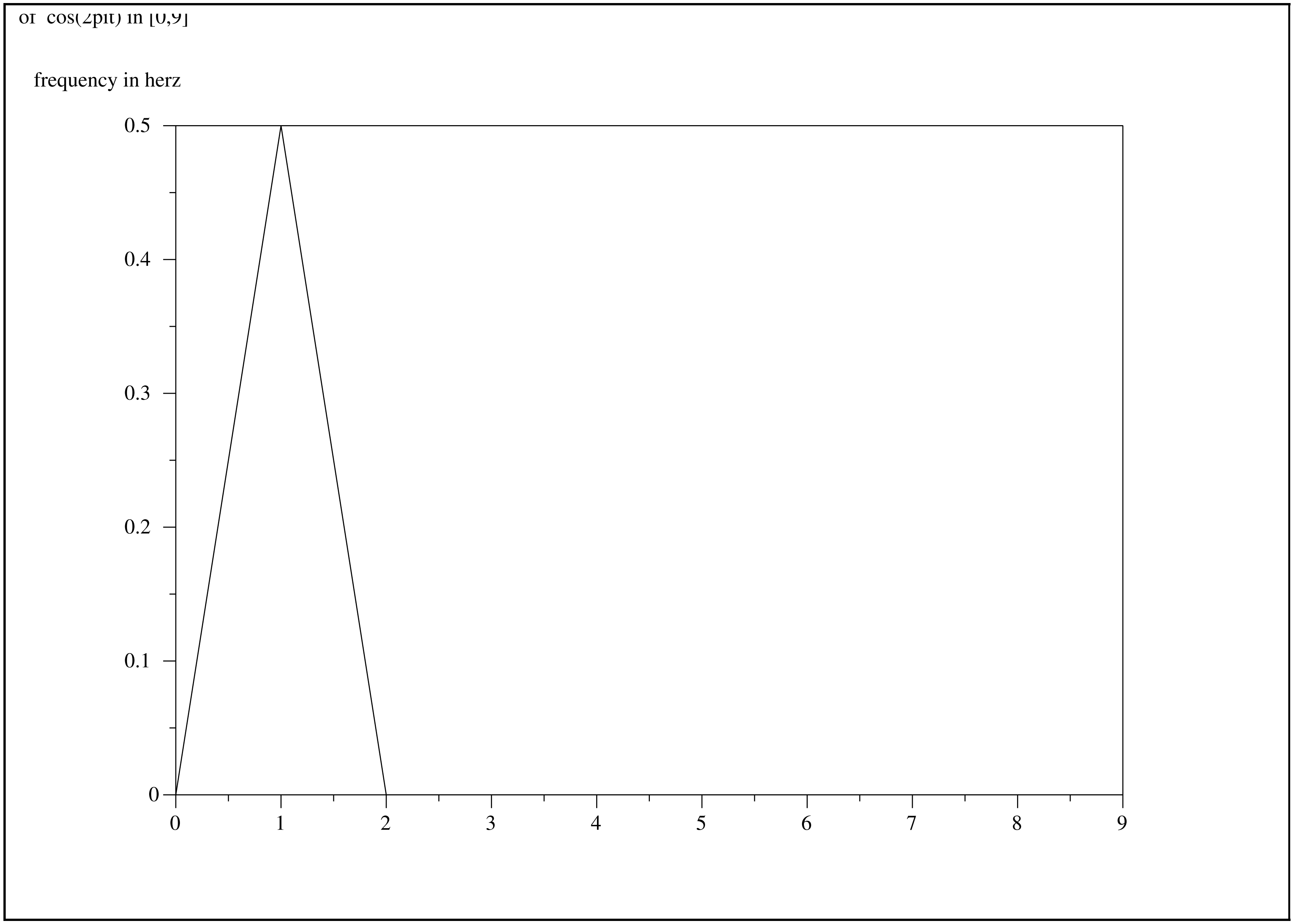

Figure 3: Norm of the Fourier transform of $cos(2\pi t)$ in $[0, 1000]$ herz

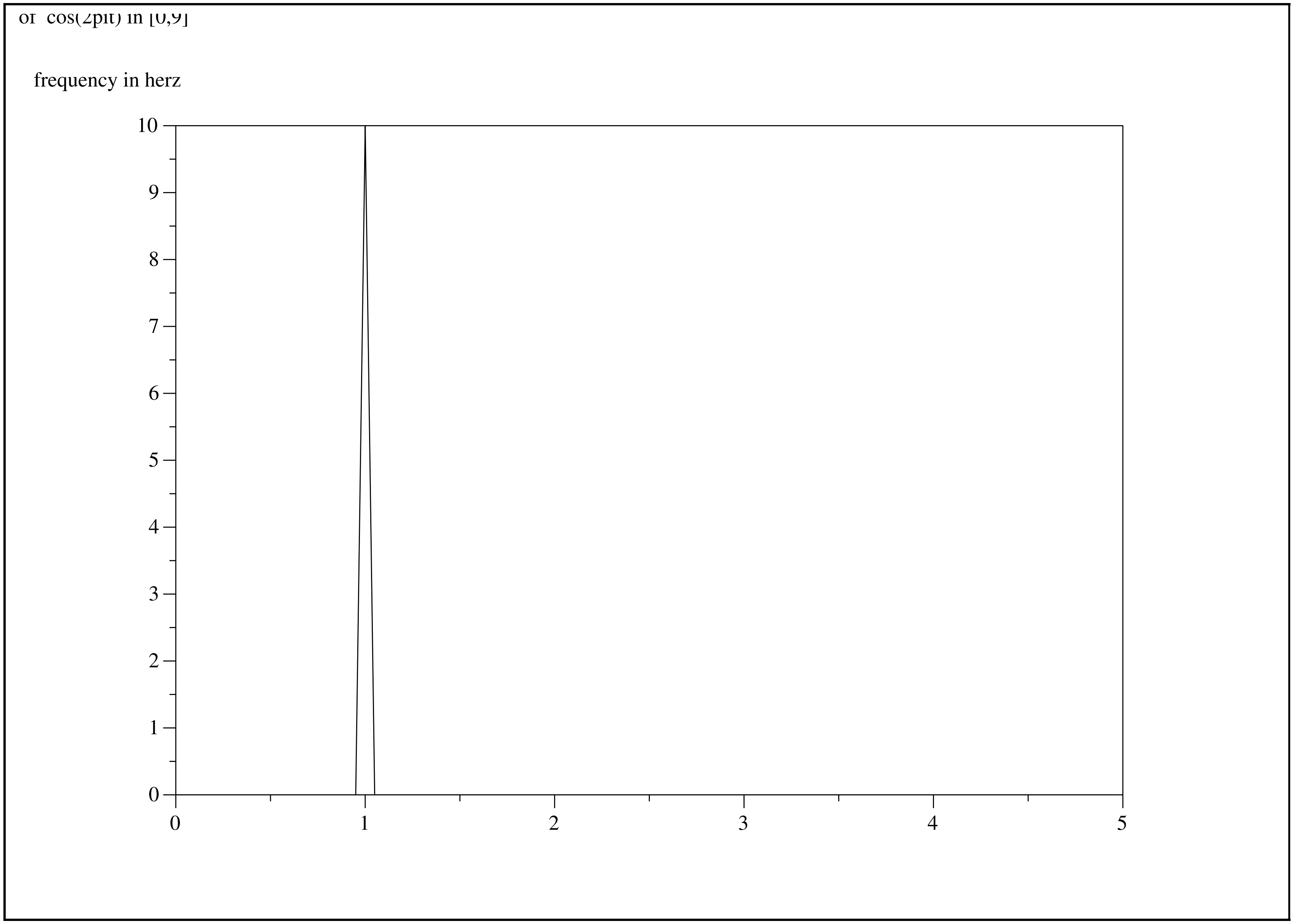

Figure 4: Norm of the Fourier transform of $cos(2\pi t)$ in $[0, 1000]$ herz

celle de $exp(10^{-2}t) * cos(2\pi t)$ est un peu différente, voir les figures 5 et 6

somme de deux sinus

`sin_p_sin.sci`

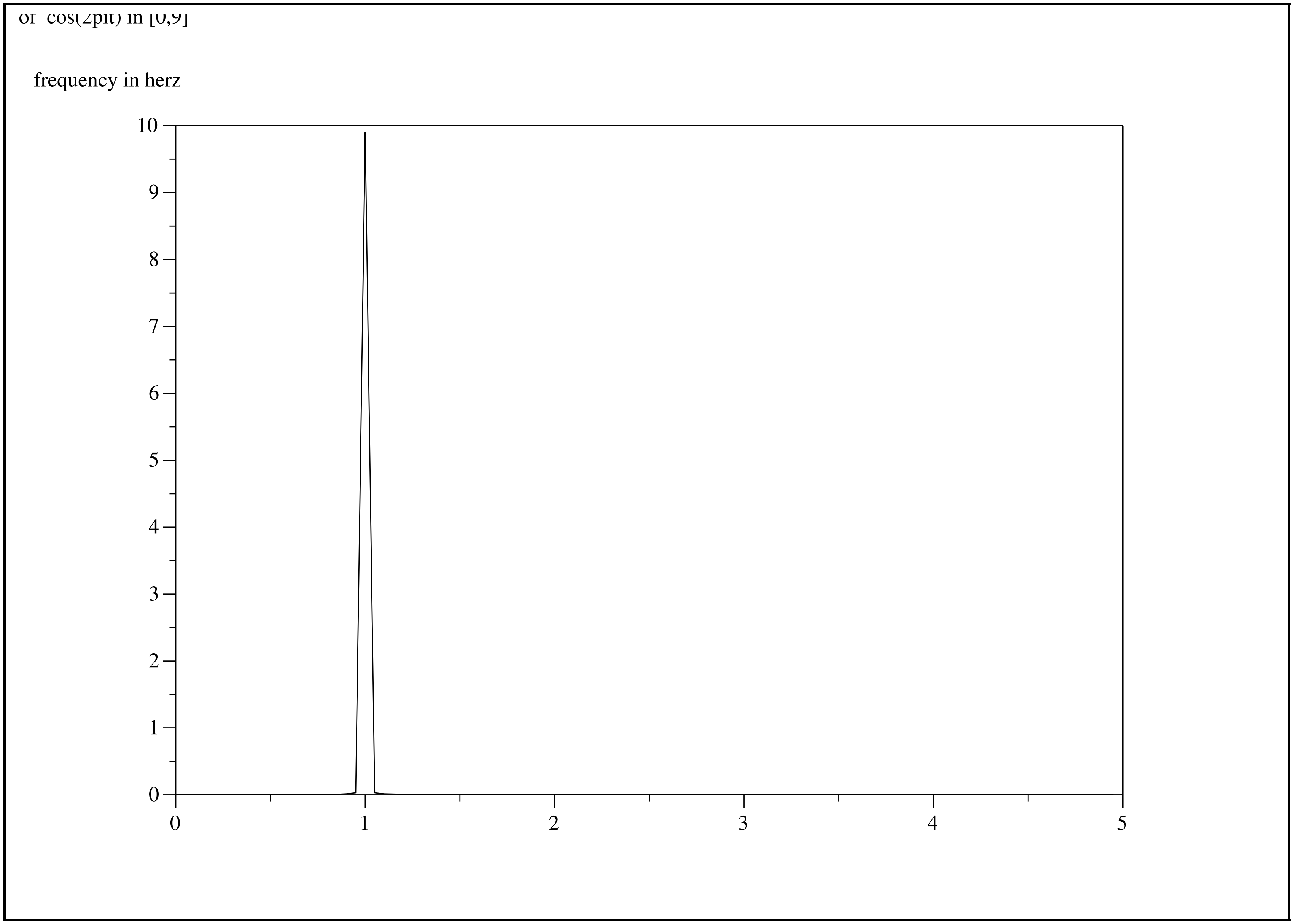

Figure 5: Norm of the Fourier transform of $exp(10^{-3}t) * cos(2\pi t)$ in $[0, 1000]$ herz

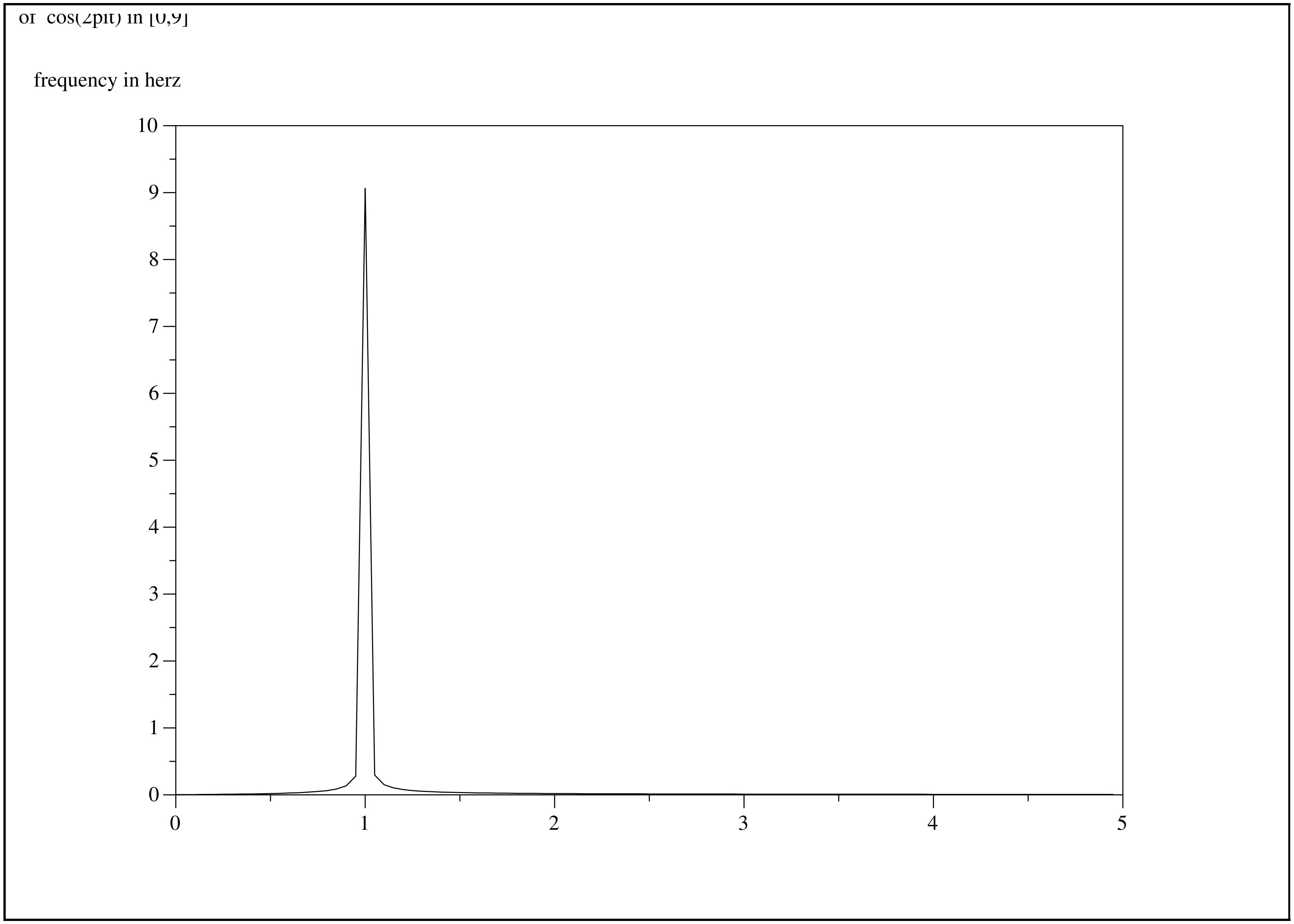

Figure 6: Norm of the Fourier transform of $exp(10^{-2}t) * cos(2\pi t)$ in $[0, 1000]$ herz

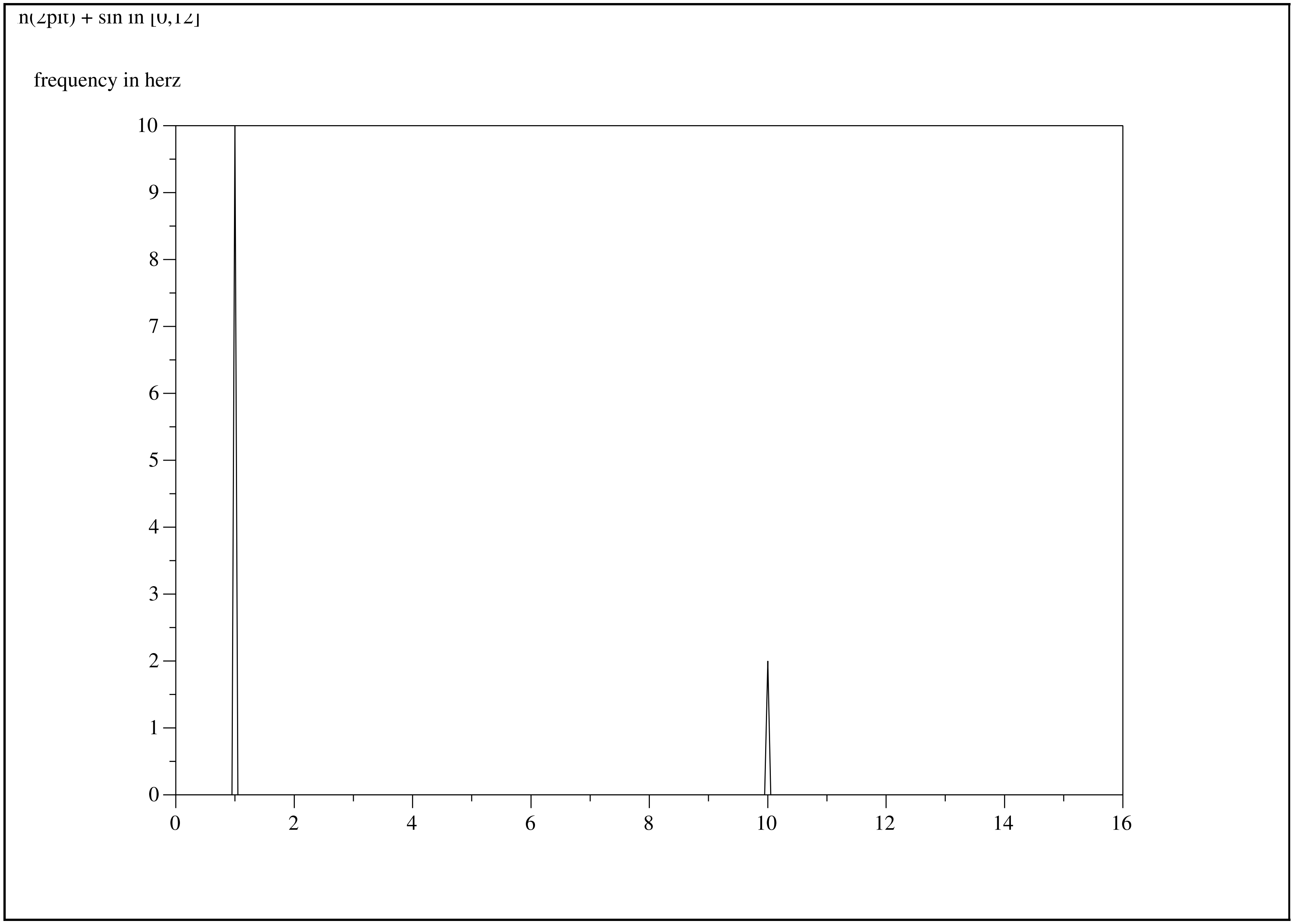

Figure 7: Norm of the Fourier transform of $sin(2\pi t) + .2sin(20\pi t)$ in $[0, 50]$ herz