Asymptotics of the price oscillations of a vanilla option in a tree model[1]

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Abstract. It is well known that the price of a european vanilla option computed in a binomial tree model converges towards the Black-Scholes price when the time step tends to zero. Moreover, is has been observed empiricaly that this convergence is oscillatory and is of order $1/n$.

In this paper, we compute this oscillatory behaviour using asymtotics of Laplace integrales, giving explicitely the first terms of the asymtotics. This allows to show that there is no asymptotic expansion in the usual sens, but that the rate of convergence is indeed of order $1/n$, as the second term (in $1/\sqrt{n}$) vanishes, the next term being of type $C(n)/n$, with $C(n)$ being some (not $\mathcal{C}^1$) bounded function of n that has no limit when n tends to infinity.

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1 Introduction

There are mainly three kinds of methods to compute the price of financial derivatives : tree methods, numerical methods for solving partial differential equations, and Monte Carlo methods. Even if the tree methods are the most rudimentary,

[1]This research began during a stay of the authors at INRIA's Omega projet in Sophia-Antipolis so as at Oxford University's OCIAM during the fall and winter 1998. The idea of applying the asymptotic methods to problems of mathematical finance goes back to a visit to the Newton Institut in Cambridge during sspring 1995, where two programs on exponential asymptotics and mathematical finance took place simultaneously. These visits provided the oportunity of fruitfull discussions, in particular with Imme van den Berg, Ellis Cumberbatch, Damien Lamberton, Claude Martini, Adri Olde Daalhuis, Bruno Salvy, and Denis Talay.

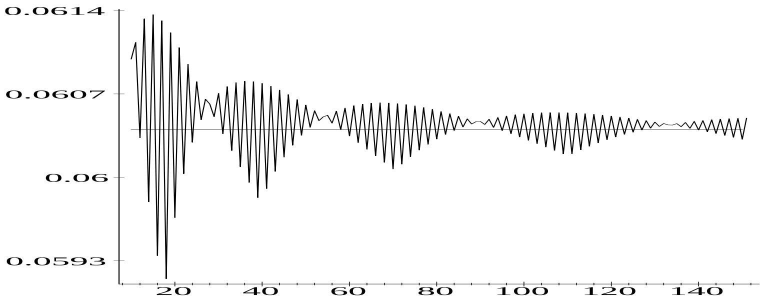


Figure 1: Price $C(n)$ of a european call option as a function of the number n of time steps (the horizontal line is the Black-Scholes price, towards which $C(n)$ tends as n tends to infinity). One can observe scalloped lines with cusp points on the two enveloppe lines and intervals where the distance to the limit decraeses as n increases followed by intervals where this distance increases again.

they stay frequently used, mainly because they are really easy to understand and can be used safely.

Moreover, they converge towards a limit (that we shall denote by BS for "Black-Scholes") as the time steps tend to zero and, consequently, the number n of steps tends to infinity, so it suffice to take a large enough number of time steps to obtain a good enough value.

And yet, only few results exist on the convergence such as its speed (how many steps are needed to obtain some given precision ?), its nature, monotonic or oscillatory (does the obtained result underestimate or overestimate the limit), or concerning the effect of the position of the nodes of the tree with respect to the barrier (in case it is a barrier option) or to the exercice price (is it advisable to choose the tree in such a way that some nodes coincide with these values or not).

One may understand why the answers to these questions are not easy by looking on figure 1, on which the price $C(n)$ of a european call option has been represented as a fonction of the number n of time steps. One observes rather irregular oscillations, and yet the values of $C(n)$ for even values of n and those for odd values of n seem to line up along two curves that envelop the oscillations, curves that exhibit amazing cusp points (scallops).

One way to answer these questions would be to give the equations of these two curves, or at least to compute an approximation of them. This is precisely the aim of this paper. To that purpose, we introduce a new method for computing the asymptotics of the price of an option as a function of the number n of time steps. It consists in replacing the binomial sums that this price exhibits by Euler integrals, and to give estimates of these integrals using an extended Laplace method. We will show, among others, that the distance to the limit, for a european option, has the form $\frac{C_2(n)}{n} + o(\frac{1}{n})$, where $C_2(n)$ is a function that stays bounded as n tends to infinity. An explicit computation of $C_2(n)$ provides an excellent approximation of the equation of the two enveloping curves that one perceives on figure 1. The asymptotics of the price obtained shows that the rate of convergence of $C(n)$ is indeed of order $\frac{1}{n}$ (and not of order $\frac{1}{\sqrt{n}}$ as it has been suggested in [12]) but we will see that the function $C(n)$ (for which we will give an explicite expression in two particular cases (corollary 6 and 7) has no limit as n tends to infiny. This explains why the attempts to show the existence of an asymptotic expansion of type $BS + \frac{C_2}{n} + o(\frac{1}{n})$ stayed fruitless, as such an expansion can not exist if $C_2(n)$ has no limit, as it is the case here.

The existence of oscillations in the price $C(n)$ is known by users of tree methods and has been pointed out by various authors ([6],[13],[15]) who usually tryed to take benefit of the oscillations to improve their computations of the price, yet not giving real explanations. In ([15]), Leisen and Reimer give a nice explanation of the scallops and attempt to provide an upper bound to the distance to the limit. Concerning the rate of convergence of order $\frac{1}{n}$ of quantities like $C(n)$, sharp results have been given by Talay et Tubaro [17] who show that, for an expectation $\mathbb{E}(f(S_T))$, where S_t is a solution of stochastic differential equations such as $dS_t = \sigma S_t dW_t$ and with a smooth function f, the error resulting from using an Euler scheem is of type $\frac{C}{n} + o(\frac{1}{n})$, with an explicit constant C. A convergence of order $\frac{1}{n}$ when f is smooth is also underlined in [12]. But these results can not be applied for a call or a put options, as in that case, f is a continuous but not smooth function. The method used in [17] takes advantage of the fact that, if one considers the random walk as a discretization (according to an Euler scheme) of a continuous-time stochastic process, an expectation like $\mathbb{E}(f(S_T))$ can be considered as the value at one point of the solution of some partial differential equation, and this allows to estimate the difference with the limiting value of the scheme as the sum of approximation errors along one solution of this equation, using in a Taylor expansion the control that one has on successive derivatives of the solutions. Generalized in [2] and [3], this method allows, actually, to get an asymptotic expansion in powers of $\frac{1}{n}$, even in the case when f is measurable, but only in the case when the Euler scheme uses the increments of a brownian motion, which is not the case for a binomial walk for which the increments are Bernouilli random variables. It is also with this idea that Gobet, in [11], for the case of barrier options, and under these same general hypothesis for f, obtains an asymptotic expansion, in powers of $\frac{1}{\sqrt{n}}$ then, which is the best result one can get because, for barrier options, the convergence is not of order $\frac{1}{n}$ but in fact of order $\frac{1}{\sqrt{n}}$. Lamberton [14] has also obtained an estimate of order $\frac{1}{n}$ in case f is only lipschitz, but under the assumption that the discrete underlying asset is of type $\sum X_s/\sqrt{n}$, for a family of i.i.d. random variables X_s such that $\mathbb{E}(X) = \mathbb{E}(X^3) = 0$, $\mathbb{E}(X^2) = 1$ and $\mathbb{E}(X^4) < \infty$, conditions that are also not satisfied here, as the choice of the martingale probability $p(n)$ has to be different of $\frac{1}{2}$ (it is a function of n). In fact, we shall see that the price $C(n)$ is equal to the difference of two terms, the rate of convergence of each, taken separately, is of order $\frac{1}{\sqrt{n}}$, and the rate of convergence of $C(n)$ is of order $\frac{1}{n}$ only because of the cancelation, in the difference, of the two terms of order $\frac{1}{\sqrt{n}}$, that precisely balance each other. Besides, this cancelation phenomenon does not occur for higher order terms, and the expansion, beyond the term $\frac{C_2(n)}{n}$, exhibits a term $\frac{C_3(n)}{n\sqrt{n}}$, that usually does not vanish, and so on.

2 About the model

In this section we first recall how to price a european vanilla option in a Cox-Ross-Rubinstein binomial model, and we introduce some notations that will be used in the sequel. Then we state the main result of this paper : this price admits, with respect to the number n of time steps, an asymptotic expansion of a somewhat unusual type, that we shall call an *asymptotic expansion with bounded coefficients.*

2.1 Model for the underlying asset

Using the approach of Cox, Ross and Rubinstein [7], that was inspired by a suggestion of the economist W. Sharpe, one adopts the following *finite random walk* as the dynamic for the price (S_t) of the underlying asset :

- a finite set of time instants $t \in \{0, \delta t, \ldots, n\delta t\} =: [0..T]_{\delta t}$, with $T = n\delta t$ (and thus, $\delta t = T/n$),
- an initial value S_0 (for $t = 0$),
- a dynamic caracterised by the existence, for each time step, of exactly two possibilities for the next step, the present price S_t of the asset being multiplied by a factor $U_{t+\delta t} := \frac{S_{t+\delta t}}{S_t}$ either equal to u (for *up*), either to d (for *down*), with the condition
$$d < e^{r\delta t} < u, \tag{1}$$
where $r \geq 0$ stands for the riskless interest rate.
- the factors $U_t = \frac{S_t}{S_{t-\delta t}}$ are i.i.d. Bernouilli random variables. One puts $p := P(U_t = u)$ and, consequently, $1 - p = P(U_t = d)$.

The natural choice for p will be defined below ; with this choice the price process (S_t) is thus, for each $t = \nu\delta t$, a binomial random variable, assuming $\nu + 1$ values :

$$S_\nu^j := S_0 u^j d^{\nu-j}, \;\; j = 0, \ldots, \nu$$

with probability

$$P(S_{\nu\delta t} = S_\nu^j) = \binom{\nu}{j} p^j (1-p)^{\nu-j}.$$

2.2 The "exact formula" for the price of a european option

Let (C_t) be the price, at time t, of a european option with exercice date $T = n\delta t$ and pay-off function $\varphi(S_T)$. In the (discret) Cox-Ross-Rubinstein model, this price is equal , as for the (continuous) Black-Scholes model,to value of a self-financing portefolio of final value $C_T = \varphi(S_T)$. Under the hypothesis of absence of arbitrage, a simple reasoning allows to compute the value of such a *hedging portefolio*, by backward induction from its final value. The price is then just the (conditional) expectation of the present value of the pay-off $e^{-r(T-t)}\varphi(S_T)$, provided the probability p is chosen such that

$$pu + (1-p)d = e^{rT/n}. \tag{2}$$

With this value of p, the process $\tilde{S}_t := e^{-rt}S_t$ becomes a martingale. This probability p is usually called the martingale probability or risk-neutral probability.

Let us emphasize here that, as one can observe, this probability p depends on n, as does the term $e^{rT/n}$:

$$p = p(n) = \frac{e^{rT/n} - d}{u - d}. \tag{3}$$

Actually, as will be shown below, u and d will also be chosen depending on n, as they will be expressed as a function of $\delta t := T/n = \delta t(n)$. Consequently, $S_\nu^j := S_0 u^j d^{\nu-j} = S_\nu^j(n)$ will also depend on n. But in view of legibility, we shall no longer write this dependance on n, and shall adopt the notations

$$\delta t, \;\; u, \;\; d, \;\; S_\nu^j, \;\; p, \;\; \text{and later on },\;\; k, \;\; \text{and } q$$

for $\delta t(n)$, $u(n)$, $d(n)$, $S_\nu^j(n)$, $p(n)$, $k(n)$, and $q(n)$.

Now, denote by $C(n)$ the price, at time 0, of a european option with pay-off $\varphi(S_T)$, when $T = n\delta t$. One has :

$$C(n) = e^{-rT}\sum_{j=0}^{n}\binom{n}{j}p(n)^j(1-p(n))^{n-j}\varphi(S_0u^jd^{n-j}). \tag{4}$$

In particular, for a call option, with $\varphi(S_T) = (S_T - K)^+$, one can write this value as the difference of two terms

$$C(n) = S_0\sum_{j=k}^{n}\binom{n}{j}\left(pue^{-r\frac{T}{n}}\right)^j\left((1-p)de^{-r\frac{T}{n}}\right)^{n-j} - Ke^{-rT}\sum_{j=k}^{n}\binom{n}{j}p^j(1-p)^{n-j}$$

the sums beginning at k, where $k = k(n)$ is the smallest integer j such that $S_0u^jd^{n-j} > K$. Let

$$q(n) = q = pue^{-r\frac{T}{n}} \;; \tag{5}$$

from the martingale relation (2), one deduces that $(1-p)de^{-r\frac{T}{n}} = 1 - q$, and thus the price $C(n)$ can finally be written :

$$C(n) = S_0\Phi(n, k(n), q(n)) - Ke^{-rT}\Phi(n, k(n), p(n)), \tag{6}$$

where Φ denotes the *incomplete binomial sum :*

$$\Phi(n,k,p) \ := \sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j}. \tag{7}$$

This formula (6) for the option price $C(n)$, called *exact pricing formula* by Cox and Rubinstein [7], is very similar to the famous Black-Scholes formula ; as for this formula, one recognizes in (6) the two parts of the hedging portefolio, one in the underlying asset and one in cash. There is however an important difference between the two formulas. Here $C(n)$ depends on the integer parametre n, whereas the second is undependant of n. It is well-known that, when n tends to infinity, the "limit" of the Cox-Ross-Rubinstein model is the Black-Scholes model (see for instance [16] or [4]). Thus, one has

$$\lim_{n\to\infty} C(n) = BS$$

where BS denotes the price of a call option for the countinuous model, that is

$$BS := S_0\mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2)$$

with $d_1 = (\ln\frac{S_0}{K} + (r+\frac{\sigma^2}{2})/\sigma\sqrt{T}$, $d_2 = (\ln\frac{S_0}{K} + (r-\frac{\sigma^2}{2})/\sigma\sqrt{T}$, and $\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}dy$. Of course we will find this result again in our asymptotic computations. But to understand the oscillations that can be observed (see figure 1), one has to study more carefully the distance $C(n) - BS$ when n goes to infinity.

2.3 Asymptotics of the option price

Cox and Rubinstein have introduced the discret model where $u = e^{+\frac{\sigma}{\sqrt{n}}}$ and $u = 1/d = e^{-\frac{\sigma}{\sqrt{n}}}$, which is natural in view of the continuous lognormal model, and the condition $d < e^{r\delta t} = e^{r\frac{T}{n}} < u$ is also satisfied for large n. For such a model, the first idea for studying the price $C(n)$ when n tends to infinity, is to look for an asymptotic *expansion in powers* of $1/\sqrt{n}$ (or $1/n$?) of this function of n. As a matter-of-fact, we shall see that there exist no expansion in the usual meaning, even in such a simple case ; nevertheless it is possible to compute explicitely an asymptotic approximation of this price, in a somewhat more general meaning, provided u and d have nice asymptotic properties.

So we make the following assumptions on the random walk (S_t)

$$u \text{ and } d \text{ have a converging expansion in powers of } \frac{1}{\sqrt{n}} \tag{8}$$

of type

$$u(n) = u := 1 + \sigma/\sqrt{n} + O(\tfrac{1}{n}) \ , \ d(n) = d := 1 - \sigma/\sqrt{n} + O(\tfrac{1}{n}) \ , \sigma \neq 0. \tag{9}$$

These assumptions include both the above Cox-Rubinstein model (corollary 6) and the following model with trend : u and d are the quantities $1 \pm \frac{\sigma}{\sqrt{n}} + \frac{\mu}{n}$, with

σ and μ constant (corollary 7). This last example allows to see the effect of a trend μ, which is known to disappear in the limit, but affects the oscillations of the price (see figure 3). In the case of the Cox-Rubinstein model, one has $ud = 1$ so the nodes of the tree line up on horizontal lines $S =$ Cnst ; this geometric property allows to generalize theorem 1 to the case of barrier options.

If (8) and (9) hold for u and d, the quantities p and q in the "exact formula" (6) for $C(n)$ also have an asymptotic expansion in powers of $1/\sqrt{n}$, so it seems reasonable to have an expansion in powers of $1/\sqrt{n}$ for $C(n)$. Actually the difficulty will come from k : by definition $k = k(n)$ is the smallest integer such that $S_0 u^j d^{n-j} > K$. Let $a(n)$ be the quantity

$$a(n) := \frac{\ln(K/S_0) - n\ln d}{\ln u - \ln d} \; ; \tag{10}$$

one has

$$k(n) = [a(n)] + 1 = a(n) + 1 - \{a(n)\} \tag{11}$$

where [.] denotes the integer part and {.} the fractionnal part. So, if under our assumptions on u and d, $a(n)$ has indeed an asymptotic expansion, this is no longer the case for $\{a(n)\}$ that has no limit when n tends to infinity, and nor has the function $k(n)$. Nevertheless the fractionnal part $\{a(n)\}$ stays of course bounded between 0 and 1. So, according to (11), the presence in formula (6) of $\{a(n)\}$, that has no expansion but stays bounded, leads to introduce an extended asymptotic computation :

Definition: We shall call *asymptotic expansion with bounded coefficients* in power series of $\varepsilon > 0$ to the order m any expression of type

$$\sum_{i=0}^{m} f_i(\varepsilon)\varepsilon^i \quad + \quad \varepsilon^m \delta_m(\varepsilon),$$

where the f_i are *bounded* functions of ε, and $\lim_{\varepsilon\to 0^+} \delta_m(\varepsilon) = 0$.

Let $(f_i)_{i\geq 0}$ be a sequence of bounded functions of $\varepsilon > 0$; we shall say that a funcion $f(\varepsilon)$ has an asymptotic expansion in powers of ε with coefficients $(f_i)_{i\geq 0}$ if, for any $m \geq 0$,

$$\lim_{\varepsilon\to 0^+} \varepsilon^{-m}\left(f(\varepsilon) - \sum_{i=0}^{m} f_i(\varepsilon)\varepsilon^i\right) = 0$$

There is of course no uniqueness for the expansion with bounded coefficients of a given function.

Observe that the values $f_i(\varepsilon)$ may be numbers, but also, more generally, elements of a normed vector space, such as the space $\mathcal{L}^1$ of integrable fonctions on $\mathbb{R}$, as it will be sometimes the case here.

Theorem 1 *Assume u and d have an asymptotic expansion in powers of* $1/\sqrt{n}$ *of type*

$$u = 1 + \sigma/\sqrt{n} + O(\tfrac{1}{n}) \quad \textit{and} \quad d = 1 - \sigma/\sqrt{n} + O(\tfrac{1}{n}) \;,\; \sigma \neq 0.$$

Then the price $C(n)$ of a european option for the Cox-Ross-Rubinstein model has an asymptotic expansion with bounded coefficients *of type*

$$C(n) = C_0 + \frac{C_1(n)}{\sqrt{n}} + \frac{C_2(n)}{n} + \frac{C_3(n)}{n\sqrt{n}} + \dots$$

with $C_0 = BS$, $C_1(n) = 0$, and, for $i = 2, 3, \dots$, $C_i(n)$ are bounded functions that can be computed explicitely by the extended Laplace method described below.

Comment: There is no surprise in the fact that the limit C_0 of $C(n)$ is equal to the Black-Scholes price BS. On the other hand, it is notice worth that the coefficient C_1 vanishes, and thus *the convergence is of order $\frac{1}{n}$*. However the coefficient $C_3(n)$ is generally not zero, this expansion is indeed an expansion in powers of $\frac{1}{\sqrt{n}}$. The proof of this theorem has been devided in several parts :

- the introduction of an extended Laplace method (section 3),
- the technical details of the proof have been collected in a Technical Theorem (theorem 8), that has been postponed to a separate appendix (appendix B).
- the method of expansions' effective computation, and the results illustrated by graphics (section 4) ; one gets in this way C_0, $C_1(n)$, that turns out to be zero, and $C_2(n)$; the next values $C_i(n)$ for $i > 2$ could be obtained in a similar way, but this leads to a "complexity explosion" that should not be underestimated. The `Maple` worksheets are given in appendix A.

3 An extended Laplace method

In this section we explain how to compute the asymptotics of the price $C(n)$ given by the *exact formula* (6) :

$$C(n) = S_0\Phi(n, k(n), q(n)) - Ke^{-rT}\Phi(n, k(n), p(n))$$

with Φ the incomplete binomial sum $\Phi(n,k,p) := \sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j}$.

3.1 Asymptotics of the model's parameters

We assume that u and d have an expansion in powers of $\frac{1}{\sqrt{n}}$ of type (9). An elementary asymptotic computation leads to the following result :

Proposition 2 *Assume u and d satisfy (8) and (9). Let*

$$u = u(n) =: \tfrac{1}{2} + \sigma\tfrac{1}{\sqrt{n}} + u_2\tfrac{1}{n} + \dots \quad \textit{and} \quad d = d(n) =: \tfrac{1}{2} - \sigma\tfrac{1}{\sqrt{n}} + d_2\tfrac{1}{n} + \dots$$

Then the quantities p, q, and a defined by (3), (5), and (10) have an asymptotic expansion in powers of $\frac{1}{\sqrt{n}}$ of type

$$\begin{aligned} p(n) &= \tfrac{1}{2} + \tfrac{p_1}{\sqrt{n}} + O(\tfrac{1}{n}), \\ q(n) &= \tfrac{1}{2} + \tfrac{q_1}{\sqrt{n}} + O(\tfrac{1}{n}), \\ a(n) &= \tfrac{1}{2}n + a_{-1}\sqrt{n} + a_0 + O(\tfrac{1}{\sqrt{n}}), \end{aligned}$$

with $p_1 = \frac{1}{4\sigma}(2r-(u_2+d_2))$, $q_1 = \frac{1}{4\sigma}(2r-(u_2+d_2)+2\sigma^2)$, $a_{-1} = \frac{1}{4\sigma}(2\ln(K\backslash S_0)-(u_2+d_2)+\sigma^2)$ *and* $a_0 = \frac{1}{8\sigma^2}(2\ln(K\backslash S_0)-(u_2+d_2)-\sigma^2)(u_2-d_2)$.

Corollary 3 *Let $\overline{\kappa}(n) := \{a(n)\}$. The integer $k(n)$ defined by (10-11) has the following asymptotic expansion with bounded coefficients :*

$$k(n) = \tfrac{1}{2}n + a_{-1}\sqrt{n} + a_0 + 1 - \overline{\kappa}(n) + O(\tfrac{1}{\sqrt{n}}) \tag{12}$$

3.2 Frozen parameter and incomplete Beta function

One important[2] point in the asymptotic computations that we tackle here – and this will be of the utmost importance when leaving the task to `Maple` – is to deal with κ as a *frozen* parameter, this means not to try to expand $\overline{\kappa}(n)$ (besides, such an expansion does not exist), but on the contrary to wait until the end of all the process of asymptotic computations to "remember" that $\kappa = \overline{\kappa}(n)$ depends on n. Recall that the function $\overline{\kappa}$ is bounded, by definition. The asymptotic computations with frozen κ will lead to expressions that are polynomials with respect to κ, typically $d_i(Y,\kappa)$; consequently, the $n \mapsto d_i(\cdot,\overline{\kappa}(n))$ will be bounded functions (with values in the space $\mathcal{L}^1$ of integrable functions of one variable, that here will be denoted by Y).

The intrusion of $\mathcal{L}^1$ – that is, integrals – into this context, that up to now dealt only with finite sums, is connected with the use of the following lemma, that turns out to be a *magic formula*[3]. This formula will allow to rewrite the price $C(n)$, given by formula (6), in an *integral form* that will allow its asymptotic evaluation.

Lemma 4 *For all $n \in \mathbb{N}$, and all k, $0 < k \le n$, one has the following identity :*

$$\sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j} = k\binom{n}{k} \int_0^p y^{k-1}(1-y)^{n-k}\,dy.$$

[2]It is Laurent Prignaux, student of the Ecole des Mines de Paris, trainee at the projet Oméga of Inria Sophia-Antipolis during the spring of 1999, who first introduced a notation into his computations that "froze" what we denote here by $\overline{\kappa}(n)$.

[3]A big thank to Adri Olde Daalhuis who draw our attention on the virtues of the incomplete Beta function ([1], p. 263 and 944).

Proof: This identity follows immediately from an elementary $(n-k)$ fold integration by parts of its right member. $\square$

Using this formula, the price (6) can be written in the following integral form, to which we will be able to apply an adapted version of the *Laplace method.* This new version of the Cox-Ross-Rubinstein exact formula is the key point of the proof of theorem 1 (see appendix B).

Proposition 5 (integral version of the Cox-Ross-Rubinstein formula)

$$
\begin{aligned}
C(n) \quad &= \quad k(n)\binom{n}{k(n)} \times \qquad\qquad (13)\\
\times &\left(S_0 \int_0^{q(n)} y^{k(n)-1}(1-y)^{n-k(n)}dy - Ke^{-rT}\int_0^{p(n)} y^{k(n)-1}(1-y)^{n-k(n)}dy\right).
\end{aligned}
$$

3.3 Applying the extended Laplace method

In the expression (13) just above, there are two very similar integrals that can be dealt with in the same way. We shall restrict ourselves to present the method on one of them, say $I^p(n)$:

$$I^p(n) := \int_0^{p(n)} y^{k(n)-1}(1-y)^{n-k(n)}dy.$$

As the first term of the asymptotic of the function $k(n)$ is $\frac{n}{2}$, it is natural to rewrite the integrant in the following way :

$$y^{k(n)-1}(1-y)^{n-k(n)} = (y(1-y))^{\frac{n}{2}}\left(\frac{y}{1-y}\right)^{k(n)-\frac{n}{2}-1}\frac{1}{1-y},$$

Let $h(y) := \frac{1}{2}\ln y(1-y)$ and $g(y) := (\frac{y}{1-y})^{k(n)-\frac{n}{2}-1}\frac{1}{1-y}$; the integral is thus a *Laplace integral :*

$$\int_0^p e^{nh(y)}g(y)dy. \qquad (14)$$

Let us first recall the principle of the usual Laplace method ; one assumes that the integration interval, here $[0,p]$, contains a unique maximum y_0 of the function $h(y)$. One considers the change of variable (blow-up) $Y = (y-y_0)\sqrt{n}$. With the new variable Y, the integral becomes, with $y_0 = \frac{1}{2} = p_0$,

$$\int_0^p e^{nh(y)}g(y)dy = \int_{-\frac{\sqrt{n}}{2}}^{(p-\frac{1}{2})\sqrt{n}} e^{nh(\frac{1}{2}+\frac{Y}{\sqrt{n}})}g\left(\frac{1}{2}+\frac{Y}{\sqrt{n}}\right)\frac{dY}{\sqrt{n}}$$

and thus, as $h'(\frac{1}{2}) = 0$,

$$\int_0^p e^{nh(y)}g(y)dy = \frac{e^{nh(\frac{1}{2})}}{\sqrt{n}}\int_{-\frac{\sqrt{n}}{2}}^{(p-\frac{1}{2})\sqrt{n}} e^{-\frac{Y^2}{2}(-h''(\frac{1}{2})+\dots)}g\left(\frac{1}{2}+\frac{Y}{\sqrt{n}}\right)dY.$$

A Taylor expansion of h and g leads to an asymptotic expansion in powers of $\frac{1}{\sqrt{n}}$ of the integrant of this new integral ; after integrating term by term this new integral one gets an expansion with gaussian integrals as coefficients. It suffice then to check that integration term by term leads indeed to an asymptotic expansion for the integral.

For usual Laplace integrals, these properties can be shown, giving the desired expansion of the integral, provided one takes also the limit of the (new) integration interval, that is just $Y \in \mathbb{R}$.

In the case under consideration here, it is easy to check that the change of variable $Y = (y - \frac{1}{2})\sqrt{n}$, turns the expression (13) of $C(n)$ into

$$C(n,\kappa) := c(n,\kappa)\left(S_0 \int_{-\frac{\sqrt{n}}{2}}^{(q-\frac{1}{2})\sqrt{n}} \Theta(Y,n,\kappa)dY - Ke^{-rT}\int_{-\frac{\sqrt{n}}{2}}^{(p-\frac{1}{2})\sqrt{n}} \Theta(Y,n,\kappa)dY\right) \tag{15}$$

where the factor $c(n,\kappa)$ and the integrant $\Theta(Y,n,\kappa)$ are given by

$$c(n,\kappa) := k(n,\kappa)\binom{n}{k(n,\kappa)}\frac{2^{1-n}}{\sqrt{n}}, \text{ and} \tag{16}$$

$$\Theta(n,Y,\kappa) := \left(1 - \frac{(2Y)^2}{n}\right)^{\frac{n}{2}} \left(\frac{1-\frac{2Y}{\sqrt{n}}}{1+\frac{2Y}{\sqrt{n}}}\right)^{k(n,\kappa)-\frac{n}{2}-1} \frac{1}{1-\frac{2Y}{\sqrt{n}}}. \tag{17}$$

Thus $C(n) = C(n, \overline{\kappa}(n))$. In the sequel we shall consider the integral

$$I^q(n,\kappa) = \int_{-\frac{\sqrt{n}}{2}}^{(q-\frac{1}{2})\sqrt{n}} \Theta(Y,n,\kappa)dY \tag{18}$$

obtained from $I^q(n)$ by freezin κ, and similar for $I^p(n,\kappa)$.

But two new difficulties show up here that are not present in the usual Laplace method :

- The integration interval $[0, p(n)]$ depends on n and its end $p(n)$ tends, as n goes to infinity, towards the maximum $y_0 = \frac{1}{2}$ around which the change of variable $Y = (y - y_0)\sqrt{n}$ is performed. Under our hypothesis on u and d, this maximum may even not belong to the interval $[0, p(n)]$. This has two consequences : First, as the image under the change of variable of the interval $[0, p(n)]$ is $[-\frac{1}{2\sqrt{n}}, (p(n) - \frac{1}{2})\sqrt{n}]$, the gaussian integrals that form the coefficents of the expansion should not be extended to the whole of $\mathbb{R}$ but only to an interval of type $]-\infty, p_1 + \frac{p_2}{\sqrt{n}} + \ldots + \frac{p_{m+1}}{n^{\frac{m}{2}}}]$. Second, one will have to truncate the expansion of this upper end of the interval to a large enough order, for example keep the term $\frac{p_3}{n}$ if one wishes to compute the term of order $\frac{1}{n}$ of the integral.
- As the factor $g(y)$ depends both on y and n, one has to consider carefully the image of this function under the change of variable $Y = (y - \frac{1}{2})\sqrt{n}$ in

order to check that the resulting function admits an asymptotic expansion in powers of $\frac{1}{\sqrt{n}}$ leading to integrable functions on the considered interval and such that the asymptotic expansion of the integral can be obtained integrating term by term. In the proof of this property (see theorem 8 of appendix B), as the exponent $k(n,\kappa)-\frac{n}{2}-1$ is equal to $a_1\sqrt{n}+a_2-\kappa+o(1)$, one uses identities like

$$\left(\frac{1-\frac{2Y}{\sqrt{n}}}{1+\frac{2Y}{\sqrt{n}}}\right)^{\sqrt{n}} = e^{-4Y}\left(1-\frac{16}{3}\frac{Y^3}{n}+O\left(\frac{1}{n\sqrt{n}}\right)\right),$$

or similar with higher order terms, to get the expansion of g.

4 Results

In this section we give and comment the results obtained when applying theorem 1 to compute the asymtotics of the price of a call option, in two cases : the Cox-Rubinstein model, and a model with trend.

4.1 Case of the Cox-Rubinstein model

Consider first the case where

$$u = e^{\frac{\sigma}{\sqrt{n}}} \text{ and } d = e^{\frac{-\sigma}{\sqrt{n}}},$$

that has been suggested by Cox and Rubinstein, and for which the computations are easier.

Corollary 6 (Cox-Rubinstein model) *If the underlying asset is such that* $u = e^{\frac{\sigma}{\sqrt{n}}}$ *and* $d = e^{\frac{-\sigma}{\sqrt{n}}}$, *then the price at* $t=0$ *of a call option with value* $(u^j d^{n-j}-K)^+$ *at time* $T = n\delta t = 1$ *satisfies*

$$\begin{aligned} C(n,\kappa) = BS \quad &+ \quad \frac{1}{n}\Big\{-\sqrt{2/\pi}Ke^{-r}e^{-\frac{1}{2}(\frac{\sigma}{2}-\frac{r-\ln K}{\sigma})^2}\times \\ &\qquad \times\left(\sigma\kappa(\kappa-1)+\frac{\sigma^4+12(\sigma^2+r^2)+8r\ln K+4(\ln K)^2}{96\sigma}\right)\Big\} \\ &+ \quad O(\frac{1}{n\sqrt{n}}) \end{aligned}$$

where BS *is the Black-Scholes price, and* κ *is the fractionnal part of* $a(n) := \frac{\ln K - n\ln d}{\ln u - \ln d}$.

This corollary shows that the distance between the price $C(n)$ and its limit BS is indeed of order $\frac{1}{n}$ (one has $C(n) = C(n,\overline{\kappa}(n))$ with $\kappa\in[0,1[$) but also shows that, as the quantity κ, even if between 0 and 1, has no limit when n tends to infinity, an asymptotic expansion, in the usual sens, of $C(n)$ can not exist.

The explicit formula given by this corollary also allows to test the quality of the approximation of the price $C(n)$ given by the above asymptotic expansion with bounded coefficient. It allows to compare, for any value n, the quantity $C(n)$, its limit when n tends to infinity (or zero order approximation) and its second order approximation given by this formula. Figure 2 shows, at (a), the second order approximation of $C(n)$ as a function of n, ploted separately for odd and even values of n, and, at (b), the plottings of the price $C(n)$, together with its second order approximation. One observes that it is difficult to see the difference between $C(n)$ and its second order approximation, unless n has very small values. For the same values of the parameter as for figure 2, one has the following datas :

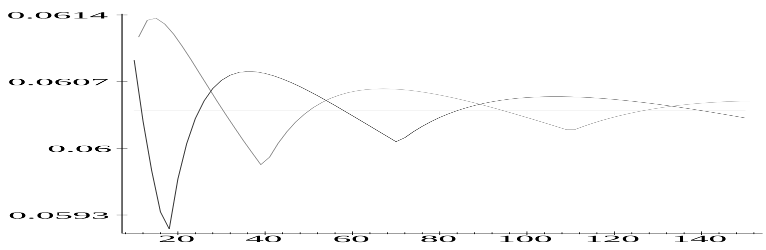
n	$C(n)$	zero order approx. (BS)	second order approx.
50	0.06060695478	0.06040088125	0.06060099755
150	0.06031978406	0.06040088125	0.06031902046

The formula of corollary 6 allows to understand the oscillations in the price of a call and the cusps that can be observed for specific values of n. Indeed, recall that the function $\overline{\kappa}(n)$ it the fractionnal part of $a(n) = \frac{\ln K - n \ln d}{\ln u - \ln d}$ which has an asymptotic expansion of type $a(n) = \frac{1}{2}n + a_{-1}\sqrt{n} + O(\frac{1}{\sqrt{n}})$ in the model under consideration here. The first term gives no contribution to κ for even n (as its fractionnal part is zero), and, to the contrary, brings a contribution $\frac{1}{2}$ to κ for odd n. This explains the oscillations of order $\frac{1}{n}$ between the even and the odd values of n in the price $C(n)$. Moreover, for values of n with same parity, when $a(n)$ is far from an integer value, its fractionnal part κ changes continuously, but κ will have a discontinuity each time $a(n)$ crosses an integer. It is easy to check on the definition of $a(n)$ that this happens at the cusp points on the picture. Observe that, in the case of the call at the money, this is when $K = S_0$, one has $a(n) = \frac{n}{2}$ (as $u = 1/d$ in the model under consideration here) and thus the price $C(n)$ oscillates, as in the general case, but the asymptotic expansion with bounded coefficent is much easier to compute (see [10]) as κ is simply equal to 0 for even n and equal to $\frac{1}{2}$ for odd n. Thus, there is no scallop in this case.

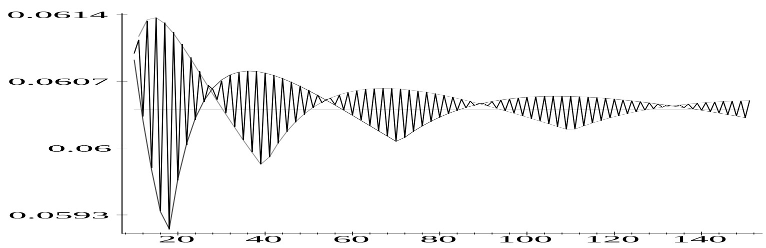
4.2 Case of a model with an explicit drift term

One of the properties often stressed of the Black-Scholes price is that is does not depend of the trend in the model for the underlying asset. The study of a discrete model with drift μ will emphasize that, to the contrary, the Cox-Rubinstein price $C(n)$ indeed depends on the trend (the following formula will show precisely how) and it is only in the limit that the trend disappears.

Corollary 7 *If the binomial model for the underlying asset is such that* $u = 1 + \frac{\sigma}{\sqrt{n}} + \frac{\mu}{n}$ *and* $d = 1 - \frac{\sigma}{\sqrt{n}} + \frac{\mu}{n}$, *than the price at* $t = 0$ *of a european call with*



(a)



(b)

Figure 2: (a) Graph of the function of n given by the right member of the formula of corollary 6, plotted separately for the even and for the odd values of n, after replacing κ by its value as a function of n (κ is the fractionnal part of $a(n)$), thus neglecting the tail-term $O(\frac{1}{n\sqrt{n}})$. The chosen model is the Cox-Rubinstein one (that is $u = exp(\frac{\sigma}{\sqrt{n}})$ and $d = 1/u$) ; the values of the parameter are : $K = 1,1$, $r = 0,05$ and $\sigma = 0,2$. (b) Simultaneous plotting of the value of $C(n)$ given by the Cox and Rubinstein exact formula (formulas 6 or 7), and its second order approximation already plotted at (a). Unless n is very small, the difference between exact and second order approximation is difficult to see. The horizontal line is for the Black-Scholes price BS, limit of $C(n)$ when n tends to infinity (or zero order approximation)

value $(u^j d^{n-j} - K)^+$ *at* $T = n\delta t = 1$ *satisfies the following formula :*

$$C(n,\kappa) = BS \quad + \quad \frac{1}{n}\Big\{-\sqrt{2/\pi}\; K e^{-r} e^{-\frac{1}{2}(\frac{\sigma}{2}-\frac{r-\ln K}{\sigma})^2}\,(\sigma\kappa(\kappa-1) + \\ \frac{-9\sigma^4 + 12(\sigma^2+r^2) + (2r+\ln K)(4\ln K - 16\mu + 8\sigma^2) + 24\mu(\sigma^2+\mu)}{96\sigma}\Big)\Big\} \\ + \quad O(\frac{1}{n\sqrt{n}})$$

where BS *stands for the Black-Scholes price, and et* κ *denotes the fractionnal part of the quantity* $a(n) := \frac{\ln K - n\ln d}{\ln u - \ln d}$.

Figure 3 shows the oscillation of the Cox-Rubinstein price $C(n)$ for three different values of the parameter μ. In this model, as for the Cox-Rubinstein model above, one can check that the approximation obtained with the asymptotic formula (corollary 7), neglecting the tail $O(\frac{1}{n\sqrt{n}})$, gives excellent approximations of the price $C(n)$. It gives a better understanding of the influence of the drift μ on the prices.

Appendicies

A Effective computation of the expansion using `Maple`

Theorem 1 states the existence of an asymptotic expansion with bounded coefficients of $C(n)$ for any Cox-Ross-Rubinstein model satifying conditions (8)-(9). This expansion is effective but difficult and boring ; for these reasons it appeared essential, in order to perform these computations, to use a computer algebra system ; we chose `Maple`.

In this section we first explain some of the trics we had to introduce in the program below to help `Maple` to compute the three first terms of the expansion with bounded coefficients of $C(n)$. Finally we give the program which plots the two curves of figure 2 using these three coefficients $C0$, $C1$ and $C2$ which were obtained in this way.

The computation of $C(n)$ follows four steps :

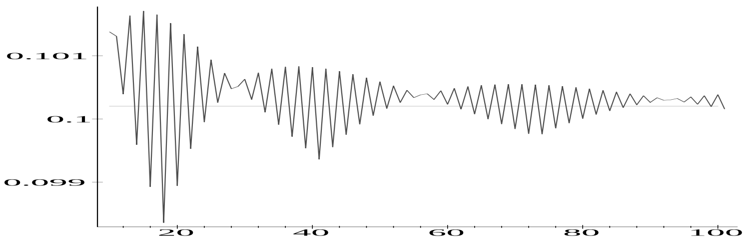
1. First one introduces the model ; here any $u(n)$ and $d(n)$ satisfying (8)-(9) may be chosen. We used

   $$u(n) = 1 + \frac{\mu}{n} + \frac{\sigma}{\sqrt{n}}\ ,\ d(n) = 1 + \frac{\mu}{n} - \frac{\sigma}{\sqrt{n}},$$

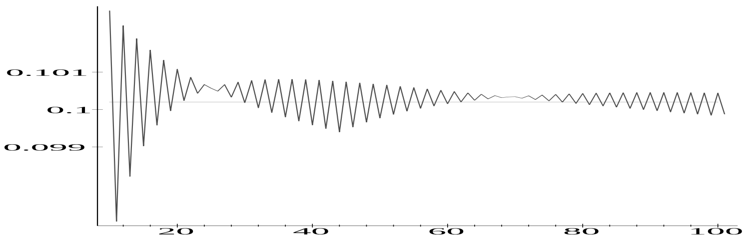
   and thus one has :

   $$p(n) = \frac{e^{\frac{r}{n}} - d(n)}{u(n) - d(n)}\ ,\ a(n) = \frac{L - n\ln d(n)}{\ln u(n) - \ln d(n)}\ ,\ \text{and } k(n) = a(n) + 1 - \kappa.$$

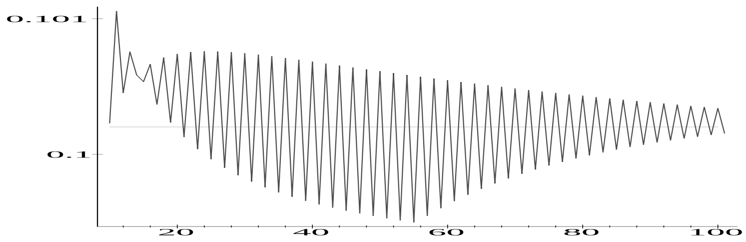
   Here L stands for the constant $\ln K$ and κ is frozen and dealt with as if it was independant of n.



($\mu = 0$)



($\mu = 0.05$)



($\mu = 0.10$)

Figure 3: Oscillations of the option price (formulas 6 and 7) in a Cox-Ross-Rubinstein model where $u = 1 + \frac{\sigma}{\sqrt{n}} + \frac{\mu}{n}$ and $d = 1 - \frac{\sigma}{\sqrt{n}} + \frac{\mu}{n}$ for three different values of the drift μ. The values of the other parameters are $T = 1$, $K = 1.1$, $r = 0.05$ and $\sigma = 0.3$.

2. Then one computes an asymptotic expansion with bounded coefficients of $k(n)$ and of $\binom{n}{k(n)}$. For this last quantity `Maple` meets a problem : it tries to use the Stirling formula to compute an asymptotic expansions of $n!$, $k(n)!$ and $(n-k(n))!$ and fails as it has only an expansion of $k(n)$ that depends on the "constant" κ. We had to help `Maple` by defining the function $Stir(n,m)$ that gives the expansion of $n!$ truncated at order m for any explicit $m = 1, 2, 3, \ldots$, and the function Binom (n, k, m) defined by

$$\text{Binom } (n,k,m) := \frac{\text{Stir } (n,m)}{\text{Stir } (k,m)\text{Stir } (n-k,m)}.$$

This allows `Maple` to compute finally the expansion of the factor (16):

$$c(n,\kappa) = k(n,\kappa)\binom{n}{k(n,\kappa)}\frac{2^{1-n}}{\sqrt{n}}.$$

3. To compute the expansion of the integral (18)

$$I^p(n,\kappa) = \int_{-\frac{\sqrt{n}}{2}}^{(p-\frac{1}{2})\sqrt{n}} \left(1-\frac{(2Y)^2}{n}\right)^n \left(\frac{1-\frac{2Y}{\sqrt{n}}}{1+\frac{2Y}{\sqrt{n}}}\right)^{k(n,\kappa)-\frac{n}{2}-1} \frac{dY}{1+\frac{2Y}{\sqrt{n}}}$$

and the similar integral $I^q(n,\kappa)$, one first computes the asymptotic expansion of the integrant truncated at a suitable order. Then one integrates this finite sum (involving powers of $\frac{1}{\sqrt{n}}$) for Y ranging between $-\infty$ and a formal upper bound b. Only then one substitutes to b the expansion of $(p-\frac{1}{2})\sqrt{n}$ (respectively $(q-\frac{1}{2})\sqrt{n}$ for the integral I^q) and then performs a new asymptotic expansion of the resulting expression.

4. Finally, the last step just consists to collect together the pieces of the jigsaw-puzzle as given by the formula

$$C(n,\kappa) = c(n,\kappa)\left(S_0 I^q(n,\kappa) - Ke^{-rT}I^p(n,\kappa)\right)$$

to obtain an expansion which, at the chosen order, is equal to the desired expansion.

A.1 The `Maple` worksheet, with results

In the worksheet bellow, we did not display the results, except for the three last ones which compute the (bounded) coefficients $C0$, $C1$, and $C2$ of the asymptotic expansion of the price $C(n,\kappa)$. The chosen model is a model with drift.

1. Introduction of the model:

```
> u:=proc(n) option remember; 1+mu/n+sigma/sqrt(n) end:
> d:=proc(n) option remember; 1+mu/n-sigma/sqrt(n) end:
> p:=proc(n) option remember; (exp(r/n)-d(n))/(u(n)-d(n)) end:
```

```
> a:=proc(n)(L-ln(d(n))*n)/(ln(u(n))-ln(d(n)))end:### L=ln(K)
```

```
> k:=a(n)+1-kappa:  ### introduction of the frozen parameter
kappa
```

2. Computation of the expansion of the factor:

```
> asymptk:=map(simplify,eval(subs(O=0,asympt(k,n,3)))):
```

```
> stir:=proc(n,m) local nn:
```

```
> eval(subs(nn=n,eval(subs(O=0,asympt(factorial(nn),nn,m)))))
end:
```

```
> binom:=proc(n,k,m) stir(n,m)/stir(k,m)/stir(n-k,m) end:
```

```
> asymptfacteur:=map(simplify,asympt(expand(exp(asympt(expand(
ln(asymptk*binom(n,asymptk,4)*2^(1-n)/sqrt(n))),n,3))),n,3 )):
```

3. Computation of the expansion of both integrals:

```
> integrand:=(1-(2*Y)^2/n)^(n/2)/(1-2*Y/sqrt(n))*
((1+2*Y/sqrt(n))/(1-2*Y/sqrt(n)))^(asymptk-1-n/2):
```

```
> asymptintegrand:=map(expand,asympt(integrand,n,3)):
```

```
> integralb:=
int(eval(subs(O=0,asymptintegrand)),Y=-infinity..b):
```

```
> bornep:=
map(simplify,eval(subs(O=0,asympt((p(n)1/2)*sqrt(n),n,3)))):
```

```
> q:=p(n)*u(n)*exp(-r/n):
```

```
> borneq:=
map(simplify,eval(subs(O=0,asympt((q-1/2)*sqrt(n),n,3)))):
```

```
> integralp:=
map(simplify,asympt(subs(b=bornep,integralb),n,3)):
```

```
> integralq:=
map(simplify,asympt(subs(b=borneq,integralb),n,3)):
```

4. Computation of the expansion of the price $C(n, \kappa)$:

```
> L:=ln(K):
```

```
> AsymptCall:=map(simplify,series(eval(subs(n=1/epsilon^2,
asymptfacteur*(integralq-K*exp(-r)*integralp))),epsilon,3)):
```

```
> C0:=op(1,AsymptCall);
```

$$
\begin{aligned}
\mathit{C0} \quad := \quad & \frac{1}{2}\,\mathrm{erf}\Big(\frac{1}{4}\,\frac{\sqrt{2}\,(\sigma^2 + 2\,r - 2\ln(K))}{\sigma}\Big) + \frac{1}{2} \\
& -\frac{1}{2}\,K\,e^{(-r)}\,\mathrm{erf}\Big(\frac{1}{4}\,\frac{\sqrt{2}\,(2\,r - 2\ln(K) - \sigma^2)}{\sigma}\Big) - \frac{1}{2}\,K\,e^{(-r)}
\end{aligned}
$$

$C0$ is the Black-Scholes price which can be easily recognized, remembering that $\frac{1}{2}+\frac{1}{2}erf(\frac{x}{\sqrt{2}})=\mathcal{N}(x)$, the (cumulative) distribution function of a gaussian law.

```
> C1:=simplify(expand(eval(
subs(exp(1/2*(sigma^2+r-mu)*(-r+2*ln(K)-mu)/sigma^2)=
exp(ln(K))*exp(-1/2*(r*sigma^2-2*r*ln(K)+r^2-mu^2+mu*sigma^2+
2*mu*ln(K))/sigma^2),op(3,AsymptCall)))));
### one helps Maple to simplify the coefficient of order
1/sqrt(n)
```

$$C1 := 0$$

```
> C2:=simplify(expand(eval(
subs(exp(1/2*(sigma^2+r-mu)*(-r+2*ln(K)-mu)/sigma^2)=
exp(ln(K))*exp(-1/2*(r*sigma^2-2*r*ln(K)+
r^2-mu^2+mu*sigma^2+2*mu*ln(K))/sigma^2),op(5,AsymptCall)))));
### one helps Maple to simplify the coefficient of order 1/n
```

$$\begin{aligned} C2 \;:=\; & -\frac{1}{96}\sqrt{2}\,K^{(1/2\,\frac{\sigma^2+2\,r}{\sigma^2})}\,\big(16\,r\,\sigma^2+12\,r^2+24\,\mu\,\sigma^2-16\ln(K)\,\mu+24\,\mu^2 \\ & +96\,\sigma^2\,\kappa^2+8\ln(K)\,\sigma^2+12\,\sigma^2-32\,\mu\,r-96\,\kappa\,\sigma^2+4\ln(K)^2 \\ & +8\ln(K)\,r-9\,\sigma^4\big)\,e^{(-1/8\,\frac{4\ln(K)^2+\sigma^4+4\,r\,\sigma^2+4\,r^2}{\sigma^2})}\,/(\sigma\sqrt{\pi}) \end{aligned}$$

A.2 Le Maple program plotting the asymptotic enveloping curves

The following program has to be run immediately after the previous one, as it uses the coefficients of $C(n,\kappa)$ computed there.

Plotting of the price, using the Cox-Rubinstein formula, as a function of n:

```
> plus:=proc(x) if evalf(x)>0 then x else 0 fi end:

> call:=proc(n);

> sum(binomial(n,j)*p(n)^j*
(1-p(n))^(n-j)*'plus(u(n)^j*d(n)^(n-j)-K)',
j=0..n)*exp(-r) end:

> n0:=10:n1:=150:K:=1.1:r:=0.05:sigma:=0.2:

> with(plots):

> C:=plot(evalf(['[n,call(n)]'
$n=n0..n1+1]),style=line,color=black):

> display({C}):
```

Plotting of the asymptotic expansion of the price truncated at order zero, and then at order two, as a function of n, separating the values for even n from those for odd n. One has to compute the fractional part of $a(n)$ to substitue it afterwards to κ.

```
> fraca:=proc(n) frac(a(n)) end:

> BS:=evalf(C0);prixBS:=plot([[n,BS]$n=n0..n1],
style=line,colour=black):

> courbepaire:=plot(['[2*m,BS+evalf((1/(2*m))*eval(subs(
kappa=fraca(2*m),C2)))]' $m=n0/2..n1/2],style=line,colour=red):

> courbeimpaire:=plot(['[2*m+1,BS+evalf((1/(2*m+1))*eval(subs(
kappa=fraca(2*m+1),C2)))]'
$m=n0/2..n1/2],style=line,colour=green):

> display({prixBS,courbepaire,C,courbeimpaire});
```

B Proof of the theorem

B.1 A technical theorem

Formula (6) shows that the proof of the fondamental theorem (theorem 1) reduces to the study of the asymptotics of $\Phi(n, k(n), p(n))$ (and of $\Phi(n, k(n)), q(n)))$ for any Cox-Ross-Rubinstein model satisfying (8)-(9). In turn, this reduces to the proof of a technical theorem (theorem 8), in following way : Recall $a(n)$ is defined by (10), that is :

$$a(n) := \frac{\ln(K/S_0) - n \ln d(n)}{\ln u(n) - \ln d(n)} \ ;$$

as by assumptions (8) the expansions (9) of $u(n)$ and $d(n)$ in powers of $1/\sqrt{n}$ converge, an elementary computation shows that a has a Laurent expansion in powers of $1/\sqrt{n}$

$$a(n) = \frac{n}{2} + a_{-1}\sqrt{n} + \overline{\alpha}(1/\sqrt{n}), \tag{19}$$

where $\overline{\alpha}$ is an analytic function. By definition (11), as $\overline{\kappa}(n) := \{a(n)\}$, one has

$$\begin{aligned} k(n) &= a(n) + 1 - \{a(n)\} = \frac{n}{2} + a_{-1}\sqrt{n} + \overline{\alpha}(1/\sqrt{n}) + 1 - \overline{\kappa}(n) &(20)\\ &= \frac{n}{2} + a_{-1}\sqrt{n} + A(n, \overline{\kappa}(n)) &(21)\end{aligned}$$

with

$$A(n, \kappa) := \overline{\alpha}(1/\sqrt{n}) + 1 - \kappa. \tag{22}$$

Define[4]

$$k(n, \kappa) := \frac{n}{2} + a_{-1}\sqrt{n} + A(n, \kappa), \tag{23}$$

[4]We adopt here, for the sake of simplicity, an abuse which is now perfectly mastered in computer languages, in using the same name for different functions, from the moment that they do not have the same number of variables. So, for example, we saw that the choice of $u(n)$ et $d(n)$ leads to a definition of $\overline{\kappa}(n)$. Now, for any function f of *two* variables n et κ, we define $f(n)$ (with only *one* variable) by $f(n) := f(n, \overline{\kappa}(n))$.

where κ is a parameter ; of course $k(n, \overline{\kappa}(n)) = k(n)$.

In order to obtain the desired expansion with bounded coefficents, we shall substitute to $k(n)$ the function with parameter $k(n, \kappa)$, and deal of this question in terms of asymptotics uniform with respect to the parameter κ, $\kappa \in \mathcal{K}$, $\mathcal{K}$ compact ([18], *uniformly asymptotic series*). We can later choose $\mathcal{K} := [0, 1]$, as this compact contains all the values of $\overline{\kappa}(n)$. Finally, substituting $\overline{\kappa}(n)$ to κ, we obtain the desired expansion with bounded coefficient, provided we can check that these coefficients are continuous with respect to the parameter κ – which is true as we shall be able to show that, actually, it is a polynomial function of κ.

Notations: Let us introduce the parameter κ in the definition of the integrant, in the definition of the integral, and in the definition of the factor of this integral : for any $1 \leq k \leq n$, let

$$\theta_{n,k}(y) := 2^{n-1}\sqrt{n}y^{k-1}(1-y)^{n-k}.$$

Still with $k(n, \kappa)$ defined by (23) and $\overline{\kappa}(n) := \{a(n)\}$, let

$$\begin{array}{rcll} \theta(n, y, \kappa) & := & \theta_{n,k(n,\kappa)}(y), & \\ \theta(n, y) & := & \theta(n, y, \overline{\kappa}(n)). & \\ I^p(n, \kappa) & := & \displaystyle\int_0^{p(n)} \theta(n, y, \kappa) dy & \text{, and} \\ I^p(n) & := & I^p(n, \overline{\kappa}(n)) & \text{, and similar for } I^q(n, \kappa) \text{ and } I^q(n). \\ c(n, \kappa) & := & k(n, \kappa)\displaystyle\binom{n}{k(n, \kappa)}\frac{2^{1-n}}{\sqrt{n}} & \text{, and} \\ c(n) & := & c(n, \overline{\kappa}(n)). & \end{array}$$

By lemma 4, using these notations we have thus

$$\begin{array}{rcl} \Phi(n, k(n), p(n)) & = & c(n)I^p(n) = c(n, \overline{\kappa}(n))I^p(n, \overline{\kappa}(n)) \\ \text{and } \Phi(n, k(n), q(n)) & = & c(n)I^q(n) = c(n, \overline{\kappa}(n))I^q(n, \overline{\kappa}(n)), \end{array}$$

and by proposition 5, one has

$$C(n) = S_0 c(n, \overline{\kappa}(n))I^q(n, \overline{\kappa}(n)) - Ke^{-rT}c(n, \overline{\kappa}(n))I^p(n, \overline{\kappa}(n)), \qquad (24)$$

for which we shall get the bounded coefficients asymptotics, computing the asymptotics uniform with respect to κ, $\kappa \in \mathcal{K}$, of $c(n, \kappa)I^p(n, \kappa)$ (and similar for $c(n, \kappa)I^q(n, \kappa)$). This can be done using the following technical theorem, which completes the proof of the fondamental theorem.

Theorem 8 (technical)
Let $u(n)$ and $d(n)$ satisfy conditions (8)-(9). Then, for any compact $\mathcal{K} \subseteq \mathbb{R}$,

1. *the factor $c(n,\kappa)$ has an expansion in powers of $1/\sqrt{n}$, uniform with respect to κ, for $\kappa \in \mathcal{K}$,*
$$c(n,\kappa) = c_0(\kappa) + \frac{c_1(\kappa)}{\sqrt{n}} + \frac{c_2(\kappa)}{n} + \dots$$
the coefficients $c_i(\kappa)$ being polynomial functions of κ.
2. *the integral $I^p(n,\kappa)$ has an expansion in powers of $1/\sqrt{n}$, uniform with respect to κ, for $\kappa \in \mathcal{K}$, $I^p(n,\kappa) = I^p_{i_0}(n,\kappa) + o(n^{-i_0/2})$, where the expansion $I^p_{i_0}(n,\kappa)$ of $I^p(n,\kappa)$ truncated at order i_0 can be obtained in the following way : let*
$$\Theta(n,Y,\kappa) := \frac{1}{\sqrt{n}}\theta\left(n, \frac{1}{2} + \frac{Y}{\sqrt{n}}, \kappa\right)$$
be the integrant obtained by the change of variable
$$y = \frac{1}{2} + \frac{Y}{\sqrt{n}}$$
in $\int_0^{p(n)} \theta(n,y,\kappa)dy$. Let $\Theta_{i_0}(n,Y,\kappa)$ denote its asymptotic expansion in powers of $1/\sqrt{n}$, truncated at order i_0 ; it is of the type
$$\Theta_{i_0}(n,Y,\kappa) = e^{-2Y^2+4a_{-1}Y} \sum_{i=0}^{i_0} d_i(Y,\kappa)\left(\frac{1}{\sqrt{n}}\right)^i,$$
where the $d_i(Y,\kappa)$ are polynoms in Y and κ. Let
$$Q_{i_0}(n,P,\kappa) := \int_{-\infty}^{P} \Theta_{i_0}(n,Y,\kappa)dY$$
be the antiderivative vanishing at $P=-\infty$ of this expansion, truncated at order i_0, of $\Theta(n,Y,\kappa)$; let
$$P^n_{i_0} := p_1 + \frac{p_2}{n^{\frac{1}{2}}} + \dots + \frac{p_{i_0+1}}{n^{i_0/2}}$$
be the expansion of $P := \sqrt{n}(p - \frac{1}{2})$ truncated at order i_0. Let $\hat{Q}_{i_0}(n,\kappa)$ be the expansion in powers of $1/\sqrt{n}$ of $Q_{i_0}(n, P_{i_0}(n), \kappa)$, truncated at order i_0. The expansion $I^p_{i_0}(n,\kappa)$ of $I^p(n,\kappa)$, in powers of $1/\sqrt{n}$, uniform with respect to κ, for $\kappa \in \mathcal{K}$, truncated at order i_0, is given by
$$I^p_{i_0}(n,\kappa) = \hat{Q}_{i_0}(n,\kappa). \tag{25}$$

B.2 Proof of the technical theorem : some nonstandard asymptotics

We want to compute the asymtotics of
$$\Phi(n,k(n),p(n)) = c(n,\overline{\kappa}(n))I^p(n,\overline{\kappa}(n)),$$

B.2.1 Asymptotics of the factor $c(n,\kappa)$

The Stirling formula immediately gives the indicated asymptotics of the factor $c(n,\kappa)$, and it is effective, for example with `Maple` ; it is the integral $I^p(n)$ that will keep us busy.

B.2.2 Asymptotics of the integral $I^p(n,\kappa)$

The idea of the computation of the asymptotics of $I^p(n,\kappa)$ is similar to that of the Laplace method, which can not be applied as such here. Indeed, we see that here the upper bound p of the integration domain is not constant, but depends on n, and above all, tends to the maximum $y_0 = \frac{1}{2}$ of the function h, if we write the integrant θ as in (14) :

$$\theta(n,y,\kappa) = e^{nh(y)}g,$$

Moreover, here the function g depends not only on y, but also on n and κ. This obliges to go all over the proof of the Laplace method to show that it can be adapted to the present case.

The first step consists to show that one can, without changing the asymptotic expansion, reduce the domain of integration $[0,p(n)]$ to a carefully chosen interval $[y^-(n),p(n)]$, that among other, tends to $\{1/2\}$. This step is necessary for the proof, but does not need to be effective : we will show in a later step that, as soon as we have this result, we can change once again the domain of integration to a third one (typically $(-\infty,p]$) on which the computations may be performed in a "naïve" way.

For these steps we shall make use of the language of orders of magnitude in the sens and with the methods of *nonstandard analysis.* We refer to [9] for what we need here. This method will be especially usefull to reduce the problem to the two lemmas 9 and 10, and their proof.

Without loss of generality, we can assume, by transfer, that the parameters of the problem, that is the real numbers S_0 and K are standard, so as the functions u, and d. By transfer, the functions p, q, k, et $\overline{\kappa}$ are thus standard, so as the numbers a_{-1}, as these objets are uniquely defined from S_0, K, u, and d, and so is $\mathcal{K} = [0,1]$. Now, *let n^* be any fixed i-large integer* ; this fixes one value for the functions u, d, p, q, k, and $\overline{\alpha}$. We denote by p^* the value $p(n^*)$, and similar for q^*, k^*, and κ^*. As these are values at some nonstandard point of a standard functions, the *numbers* p^*, q^*, k^*, and κ^* are thus (generally) nonstandard. Nevertheless, any $\kappa \in \mathcal{K} = [0,1]$ is limited. Let $\varepsilon := 1/\sqrt{n^*}$, which thus is a positive infinitesimal. We shall make use of the notation $\varepsilon^{\not\infty}$ to denote a quantity with identically zero ε-shadow expansion[5].

[5] We use the symbols of Van den Berg [5] : $\oslash$ denotes an i-small real number, £ a limited real number (that is, not i-large), @ a positive appreciable real number (that is, not i-small nore i-large), two different occurences of one of these symbols standing for numbers usually not equal ; we refer to [8] for a definition of the ε-shadow expansion of a number f, the nonstandard (and generalised) version of an asymptotic expansion of a standard function $f(\varepsilon)$ in powers of ε.

Proposition 2 implies that, for any standard i_0, we have $p^* = \frac{1}{2} + p_1\varepsilon + \ldots + (p_{i_0+1} + \phi)\varepsilon^{i_0+1}$ and $q^* = \frac{1}{2} + q_1\varepsilon + \ldots + (q_{i_0+1} + \phi)\varepsilon^{i_0+1}$, with all p_i and q_i standard. Let

$$\theta^*(y,\kappa) := \theta(n^*, y, \kappa).$$

Lemma 9 *It exists a real number $y_- \in [0, p^*] \cap [0, \frac{1}{2}[$, with $y_- \simeq \frac{1}{2}$ and $e^{-Y_-^2} = \varepsilon^{\not\infty}$ where $Y_- := \frac{1}{\varepsilon}(y_- - \frac{1}{2})$, such that for all $\kappa \in \mathcal{K}$*

$$I^{*p}(\kappa) := I^{p^*}(n^*, \kappa) := \int_0^{p^*} \theta^*(y,\kappa)dy = \int_{y_-}^{p^*} \theta^*(y,\kappa)dy + \varepsilon^{\not\infty} =: \tilde{I}^{*p}(\kappa) + \varepsilon^{\not\infty}, \tag{26}$$

and thus, in particular, $Y_- < 0$ is i-large.

Proof: To begin with, we observe that we have

$$\forall y^- \in [0, p^*] \quad y^- \not\simeq \frac{1}{2} \;\; \Rightarrow \;\; \forall y \in [0, y^-] \;\; \forall \kappa \in \mathcal{K} \;\; 0 \leq \theta^*(y,\kappa) \leq \varepsilon^{\frac{1}{\varepsilon}} \tag{27}$$

Indeed, let z be such that $y = \frac{1}{2} - z$; thus $0 \lneqq z \leq \frac{1}{2}$, and $y \simeq \frac{1}{2}$ if and only if $z \simeq 0$. We have

$$\begin{aligned}
\theta^*(y,\kappa) &= \frac{2^{n^*-1}}{\varepsilon} y^{k(n^*,\kappa)-1}(1-y)^{n^*-k(n^*,\kappa)} \\
&= \frac{2^{n^*-1}}{\varepsilon}\left(\frac{1}{2} - z\right)^{\frac{n^*}{2}+l\sqrt{n^*}} \left(\frac{1}{2} + z\right)^{\frac{n^*}{2}-l\sqrt{n^*}} \frac{2}{1+2z} = \frac{1}{\varepsilon} e^{-\frac{1}{\varepsilon^2}\varphi(z)},
\end{aligned}$$

where $l := a_{-1} + \varepsilon(\overline{\alpha}(\varepsilon) - \kappa)$ is limited, and it is elementary to check that

$$\varphi(z) := \ln \frac{1}{\sqrt{1-4z^2}} - \varepsilon l \ln \frac{1-2z}{1+2z} - \varepsilon^2 \ln \frac{1}{1+2z}$$

is a function such that, for any considered z such that $z \not\simeq 0$ one has $\varphi(z) > 0$ and $\varphi(z) \not\simeq 0$. Thus, $\frac{1}{\varepsilon} e^{-\frac{1}{\varepsilon^2}\varphi(z)} < \varepsilon^{\frac{1}{\varepsilon}}$, so (27) holds.

Now, applying the Cauchy permanence principle ([9]) , to the external property $y^- \not\simeq \frac{1}{2}$ in (27), one obtains

$$\exists y^- \in [0, p^*] \quad y^- \simeq \frac{1}{2} \;\; \text{and} \;\; \forall y \in [0, y^-] \;\; \forall \kappa \in \mathcal{K} \;\; 0 \leq \theta(y) \leq \varepsilon^{\frac{1}{\varepsilon}}. \tag{28}$$

We have to show moreover that for $Y_- := \frac{1}{\varepsilon}(y_- - \frac{1}{2})$ one has $e^{-Y_-^2} = \varepsilon^{\not\infty}$, which is true after having possibly replaced y_- by Min $\left\{y_-, \frac{1}{2} - \varepsilon^{\frac{2}{3}}\right\}$. The lemma follows now from the fact that $\int_0^{y_-} \varepsilon^{\frac{1}{\varepsilon}} dy = y_- \varepsilon^{\frac{1}{\varepsilon}} = \varepsilon^{\omega}$, for some $\omega \simeq \frac{1}{\varepsilon}$, which is i-large and positive. □

B.3 End of the proof of the technical theorem

Let us choose y_- and Y_- as in lemma 9. In (26) we have defined

$$\tilde{I}^{*p}(\kappa) := \int_{y_-}^{p^*} \theta^*(y,\kappa)dy.$$

Lemma 9 shows precisely that $\tilde{I}^{*p}$ has same ε-shadow expansion as I^{*p}. On the interval $[y_-, p^*]$ on which the integral $\tilde{I}^{*p}$ is taken, the Laplace method may be applied after proper adapting. Essentially, it consists in changing the singular perturbation (the factor $\frac{1}{\varepsilon^2}$ contained in θ^*) into a gaussian integration kernel using the microscope (change of variable leading to a change of scale)

$$y = \frac{1}{2} + \varepsilon Y,$$

which, in particular, changes the infinitesimal interval $[y_-, p]$ into the infinitely large one $[Y_-, P]$, with

$$P := \frac{1}{\varepsilon}\left(p - \frac{1}{2}\right) = p_1 + p_2\varepsilon + \ldots + p_{i_0+1}\varepsilon^{i_0} + \phi\varepsilon^{i_0} = P_{i_0} + \phi\varepsilon^{i_0} \quad \text{, where}$$

$$P_{i_0} := p_1 + p_2\varepsilon + \ldots + p_{i_0+1}\varepsilon^{i_0}.$$

In this way, we obtain that

$$\tilde{I}^{*p}(\kappa) = \int_{Y_-}^{P} \Theta^*(Y,\kappa)dY \quad \text{, with}$$

$$\begin{aligned}
\Theta^*(Y,\kappa) &:= e^{\frac{1}{\varepsilon^2}\ln(1-4\varepsilon^2Y^2)} e^{\frac{a-1}{\varepsilon}\ln\frac{1+2\varepsilon Y}{1-2\varepsilon Y}} e^{(\overline{\alpha}(\varepsilon)-\kappa)\ln\frac{1+2\varepsilon Y}{1-2\varepsilon Y}} \frac{1}{1+2\varepsilon Y} \\
&=: H(Y)D(Y,\kappa) \text{ , with}
\end{aligned}$$

$$\begin{aligned}
H(Y) &:= e^{-2Y^2+4a_{-1}Y} \text{ , and} \\
D(Y,\kappa) &:= \Theta^*(Y,\kappa)/H(Y) \\
&= e^{\frac{1}{\varepsilon^2}(4\varepsilon^2Y^2+\ln(1-4\varepsilon^2Y^2))} e^{\frac{a-1}{\varepsilon}(-4\varepsilon Y+\ln\frac{1+2\varepsilon Y}{1-2\varepsilon Y})} e^{(\overline{\alpha}(\varepsilon)-\kappa)\ln\frac{1+2\varepsilon Y}{1-2\varepsilon Y}} \frac{1}{1+2\varepsilon Y}.
\end{aligned}$$

For $y \in [y_-, p^*]$, one has $\varepsilon Y \simeq 0$; as, moreover κ is limited, we can use an elementary Taylor expansion in $\xi := \varepsilon Y$ of the explicit function that occure in the above expression. From our choice of the function H the singular perturbation terms (that is, those involving $1/\varepsilon^2$ and $1/\varepsilon$) cancel out and so we get, for any standard i_0,

$$D(Y,\kappa) = \sum_{i=0}^{i_0} d_i(Y,\kappa)\varepsilon^i + \varepsilon^{i_0}\delta_{i_0}(Y,\kappa), \tag{29}$$

with $\delta_{i_0}(Y,\kappa) \simeq 0$ for all $Y \in [Y_-, P]$ and all $\kappa \in \mathcal{K}$, where the functions d_i are standard polynoms. The degree of d_i in Y is $2i$ if i is even, and $2i-1$ else, and its degree in κ is i. For any $E \in \mathbb{R}$, and not only for $E = \varepsilon = 1/\sqrt{n^*}$ as we had before, let

$$D_{i_0}(Y,\kappa,E) := \sum_{i=0}^{i_0} d_i(Y,\kappa)E^i \ ;$$

it is a polynom in Y, κ, and E, standard as soon as i_0 is standard. Observe that the identity (29) implies that

$$\begin{aligned} \Theta^*(Y,\kappa) &= H(Y)D_{i_0}(Y,\kappa,\varepsilon) + \varepsilon^{i_0} H(Y)\delta_{i_0}(Y,\kappa) && (30)\\ &= \sum_{i=0}^{i_0} H(Y)d_i(Y,\kappa)\varepsilon^i + \varepsilon^{i_0}\gamma_{i_0}(Y,\kappa), && (31) \end{aligned}$$

with $\gamma_{i_0}(Y,\kappa) := H(Y)\delta_{i_0}(Y,\kappa)$; one checks easily that, as κ is limited, and as $\delta_{i_0}(.,\kappa)$ is i-small in $\mathcal{L}^\infty_{[Y_-,P]}$, we have $\gamma_{i_0}(.,\kappa) \simeq 0$ *both in* $\mathcal{L}^\infty_{[Y_-,P]}$ *and in* $\mathcal{L}^1_{[Y_-,P]}$.

Observe that, for any $Y \in [Y_-, P]$ and any $\kappa \in \mathcal{K}$, one has $\Theta^*(Y,\kappa) = \sum_{i=0}^{i_0} H(Y)d_i(Y,\kappa)\varepsilon^i + \varepsilon^{i_0}\phi$, thus $\sum_{i=0}^{i_0} H(Y)d_i(Y,\kappa)\varepsilon^i$ is an ε-shadow expansion of $\Theta^*(Y,\kappa)$, uniform in $\kappa \in \mathcal{K}$, and thus

$$\sum_{i=0}^{i_0} H(Y)d_i(Y,\kappa)\varepsilon^i = \Theta_{i_0}(n^*,Y,\kappa), \tag{32}$$

where $\Theta_{i_0}(n,Y,\kappa)$ stands, as in the statement of the theorem, for the asymptotic expansion of $\Theta(n,Y,\kappa)$ in powers of $1/\sqrt{n}$, truncated at order i_0.

Lemma 10 *With Y_- such that $e^{-Y_-^2} = \varepsilon^{\not\infty}$, one has, for any standard i_0, and limited κ ,*

$$\int_{-\infty}^{Y_-} \Theta_{i_0}(n^*,Y,\kappa) = \varepsilon^{\not\infty}. \tag{33}$$

Proof: We have

$$\begin{aligned} \left| \int_{-\infty}^{Y_-} \Theta_{i_0}(n^*,Y,\kappa) \right| &\le \sum_{i=0}^{i_0} \varepsilon^i \left| \int_{-\infty}^{Y_-} H(Y)d_i(Y,\kappa)dY \right| \\ &\le \sum_{i=0}^{i_0} \varepsilon^i e^{-Y_-^2} \int_{-\infty}^{Y_-} e^{-Y_-^2+4a_{-1}Y} \left| d_i(Y,\kappa)\right| dY. \end{aligned}$$

But, as κ is limited, it admits a standard part κ_0 ; one has, for any standard $i \le i_0$,

$$d_i(Y,\kappa) = d_i(Y,\kappa_0) + \Delta_i(Y),$$

where Δ_i in a polynom of standard degree and i-small coefficients ; as for any standard s the function $Y^s e^{-Y_-^2+4k_1 Y}$ is standard and integrable, we see that

$\int_{-\infty}^{Y_-} e^{-Y_-^2+4k_1Y} |d_i(Y,\kappa)|\, dY$ is i-small, as $Y_- \le 0$ is i-large. As we have that $e^{-Y_-^2} = \varepsilon \not\infty$, (33) holds. $\square$

Now, as $\Theta(.,\kappa) = \sum_{i=0}^{i_0} H(.) d_i(.,\kappa)\varepsilon^i + \varepsilon^{i_0}\gamma_{i_0}(.,\kappa)$, with $\gamma(.,\kappa) \simeq 0$ in $\mathcal{L}^1_{[Y_-,P]}$, we have, in $\mathcal{L}^1_{[Y_-,P]}$ and for any limited κ,

$$
\begin{aligned}
\tilde{I}^{*p}(\kappa) &:= \int_{[Y_-,P]} \Theta^*(.,\kappa) \\
&= \int_{[Y_-,P]} \Theta_{i_0}(n^*,.,\kappa) + \varepsilon^{i_0} \int_{[Y_-,P]} \gamma(.,\kappa) \quad \text{, by (31) and (32)} \\
&= \int_{]-\infty,P]} \Theta_{i_0}(n^*,.,\kappa) - \int_{]-\infty,Y_-]} \Theta_{i_0}(n^*,.,\kappa) + \varepsilon^{i_0}\phi \\
&= \int_{]-\infty,P]} \Theta_{i_0}(n^*,.,\kappa) - \varepsilon \not\infty + \varepsilon^{i_0}\phi \quad \text{by (33)} \\
&= \int_{]-\infty,P]} \Theta_{i_0}(n^*,.,\kappa) + \varepsilon^{i_0}\phi \quad \text{as } \varepsilon^{i_0}\phi + \varepsilon \not\infty = \varepsilon^{i_0}\phi \\
&= Q(n^*,P,\kappa) + \varepsilon^{i_0}\phi \qquad \text{by definition of } Q(n,P,\kappa) \\
&= \hat{Q}_{i_0}(n^*,\kappa) + \varepsilon^{i_0}\phi + \varepsilon^{i_0}\phi \qquad \text{by definition of } \hat{Q}_{i_0}(n,\kappa) \\
&= \hat{Q}_{i_0}(n^*,\kappa) + \varepsilon^{i_0}\phi \qquad \text{as } \varepsilon^{i_0}\phi + \varepsilon^{i_0}\phi = \varepsilon^{i_0}\phi.
\end{aligned}
$$

and thus, by lemma 9, for any $\kappa \in \mathcal{K}$ thus limited,

$$
I^{*p}(\kappa) := \tilde{I}^{*p}(\kappa) + \varepsilon \not\infty = \hat{Q}_{i_0}(n^*,\kappa) + \varepsilon^{i_0}\phi + \varepsilon \not\infty = \hat{Q}_{i_0}(n^*,\kappa) + \varepsilon^{i_0}\phi,
$$

which, as n^* is any i-large integer, is precisely the nonstandard version of (25), which ends the proof of the technical theorem.

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