Université de Nice Sophia-Antipolis
Master MathMods - Finite Elements - 2008/2009

Exercises - Chapter 3 The Construction of a Finite Element Space

Exercise 1.
Let $\mathcal{P}_k$ denote the set of all polynomials of degree less than or equal to k in one variable.

Let $\widehat{K} = [0\,,1]$, the following triplets $(\widehat{K}\,,\widehat{\mathcal{P}}\,,\widehat{\mathcal{N}})$ are they finite elements? In the favorable case, give the nodal basis of $\widehat{\mathcal{P}}$.

(a) $\widehat{\mathcal{P}} = \mathcal{P}_1$, $\widehat{\mathcal{N}} = \{N_1\,,N_2\}$ where $N_1(v) = v(0)$ and $N_2(v) = v(1)$.

(b) $\widehat{\mathcal{P}} = \mathcal{P}_2$, $\widehat{\mathcal{N}} = \{N_1\,,N_2\,,N_3\}$ where $N_1(v) = v(0)$, $N_2(v) = v(1)$ and $N_3(v) = v(1/2)$.

(c) $\widehat{\mathcal{P}} = \mathcal{P}_3$, $\widehat{\mathcal{N}} = \{N_1\,,N_2\,,N_3\,,N_4\}$ where $N_1(v) = v(0)$, $N_2(v) = v(1)$, $N_3(v) = v(1/3)$ and $N_4(v) = v(2/3)$.

(d) $\widehat{\mathcal{P}} = \mathcal{P}_0$, $\widehat{\mathcal{N}} = \{N_1\}$ where $N_1(v) = \int_0^1 v(x)\,dx$.

(e) $\widehat{\mathcal{P}} = \mathcal{P}_3$, $\widehat{\mathcal{N}} = \{N_1\,,N_2\,,N_3\,,N_4\}$ where $N_1(v) = v(0)$, $N_2(v) = v(1)$, $N_3(v) = v'(0)$ and $N_4(v) = v'(1)$.

Let a, b be reals and $K = [a\,,b]$. In each above example where $(\widehat{K}\,,\widehat{\mathcal{P}}\,,\widehat{\mathcal{N}})$ is a finite element, give a finite element $(K\,,\mathcal{P}\,,\mathcal{N})$ affine equivalent to $(\widehat{K}\,,\widehat{\mathcal{P}}\,,\widehat{\mathcal{N}})$.

Exercise 2.
Let $\widehat{K}$ be the triangle whose vertices are the points $\widehat{a}_1 = (0\,,0)$, $\widehat{a}_2 = (1\,,0)$ and $\widehat{a}_3 = (0\,,1)$ *i.e.* $\widehat{K} = \{(0\,,0)\,,(1\,,0)\,,(0\,,1)\}$, $\widehat{m}_i$ denote the midpoints of his edges, according to $\widehat{m}_1$ is the midpoint of the edge $(\widehat{a}_2\,,\widehat{a}_3)$, $\widehat{m}_2$ is the midpoint of $(\widehat{a}_3\,,\widehat{a}_1)$, $\widehat{m}_3$ is the midpoint of $(\widehat{a}_1\,,\widehat{a}_2)$. The following triplets $(\widehat{K}\,,\widehat{\mathcal{P}}\,,\widehat{\mathcal{N}})$ are they finite elements? In the favorable case, give the nodal basis of $\widehat{\mathcal{P}}$.

(a) $\widehat{\mathcal{P}} = \mathcal{P}_1$, $\widehat{\mathcal{N}} = \{N_1\,,N_2\,,N_3\}$ where $N_i(v) = v(\widehat{a}_i)\,,i = 1\,,2\,,3$.

(b) $\widehat{\mathcal{P}} = \mathcal{P}_1$, $\widehat{\mathcal{N}} = \{N_1\,,N_2\,,N_3\}$ where $N_i(v) = v(\widehat{m}_i)\,,i = 1\,,2\,,3$.

(c) $\widehat{\mathcal{P}} = \mathcal{P}_2$, $\widehat{\mathcal{N}} = \{N_1\,,N_2\,,N_3\,,N_4\,,N_5\,,N_6\}$ where $N_i(v) = v(\widehat{a}_i)\,,i = 1\,,2\,,3$ and $N_i(v) = v(\widehat{m}_{i-3})\,,i = 4\,,5\,,6$.

(d) $\widehat{\mathcal{P}} = \mathcal{P}_0$, $\widehat{\mathcal{N}} = \{N_1\}$ where $N_1(v) = \dfrac{\int_{\widehat{K}} v(\widehat{x}\,,\widehat{y})\,d\widehat{x}\,d\widehat{y}}{|\widehat{K}|}$.

(e) $\widehat{\mathcal{P}} = \mathcal{P}_2$, $\widehat{\mathcal{N}} = \{N_1\,,N_2\,,N_3\,,N_4\,,N_5\,,N_6\}$ where $N_i(v) = v(\widehat{a}_i)\,,i = 1\,,2\,,3$ and $N_i(v) = \boldsymbol{\nabla}\,v(\widehat{m}_{i-3})\cdot \widehat{m}_{i-3}\widehat{a}_{i-3}\,,i = 4\,,5\,,6$, with $\boldsymbol{\nabla}$ the gradient operator, $\widehat{m}_{i-3}\widehat{a}_{i-3}$ the vector whose ends are $\widehat{m}_{i-3}\,,\widehat{a}_{i-3}$ for $i = 4\,,5\,,6$.

Let a_1, a_2, a_3 be points in $\mathbb{R}^2$ and K the triangle whose vertices are the points whose vertices are a_1, a_2, a_3: $K = \{a_1\,, a_2\,, a_3\}$. In each above example where $(\widehat{K}\,, \widehat{\mathcal{P}}\,, \widehat{\mathcal{N}})$ is a finite element, give a finite element $(K\,, \mathcal{P}\,, \mathcal{N})$ affine equivalent to $(\widehat{K}\,, \widehat{\mathcal{P}}\,, \widehat{\mathcal{N}})$.

Exercise 3.

Let $\mathcal{Q}_k = \left\{ \sum_j c_j\, p_j(x)\, q_j(y) \text{ such that } p_j\,, q_j \text{ are polynomials of degrees } \le k \right\}$.

Let $\widehat{K}$ be the squarre whose vertices are the points $\widehat{a}_1 = (0\,,0)$, $\widehat{a}_2 = (1\,,0)$ and $\widehat{a}_3 = (0\,,1)$, *i.e.* $\widehat{K} = \{(0\,,0)\,,(1\,,0)\,,(1\,,1)\,,(0\,,1)\}$. The midpoints of the edges of this squarre are denoted by $\widehat{a}_i$, $i = 5\,,6\,,7\,,8$, according to $\widehat{a}_5$ is the midpoint of the edge $(\widehat{a}_1\,, \widehat{a}_2)$, $\widehat{a}_6$ is the midpoint of $(\widehat{a}_2\,, \widehat{a}_3)$, $\widehat{a}_7$ is the midpoint of the edge $(\widehat{a}_3\,, \widehat{a}_4)$, $\widehat{a}_8$ is the midpoint of $(\widehat{a}_4\,, \widehat{a}_1)$. The center of the squarre is denoted by $\widehat{a}_9$.

(a) Let $\widehat{\mathcal{P}} = \mathcal{Q}_1$, $\widehat{\mathcal{N}} = \{N_1\,, \ldots, N_4\}$ where $N_i(v) = v(\widehat{a}_i)\,, i = 1\,,2\,,3\,,4$. Show that $(\widehat{K}\,, \widehat{\mathcal{P}}\,, \widehat{\mathcal{N}})$ is a finite element.

(b) Let $\widehat{\mathcal{P}} = \mathcal{Q}_2$, $\widehat{\mathcal{N}} = \{N_1\,, \ldots, N_9\}$ where $N_i(v) = v(\widehat{a}_i)\,, i = 1\,, \ldots., 9$. Show that $(\widehat{K}\,, \widehat{\mathcal{P}}\,, \widehat{\mathcal{N}})$ is a finite element.

Exercise 4.

(a) Given a finite element $(K, \mathcal{P}, \mathcal{N})$, let the set $\{\phi_i : 1 \le i \le k\} \cap \mathcal{P}$ be the basis dual of $\mathcal{N}$. If v is a function for which all $N_i \in \mathcal{N}$, $i = 1$, ..., k, are defined, then we defined the local interpolant by

$$\mathcal{I}_K v = \sum_{i=1}^{k} N_i(v)\phi_i\,.$$

Prove that the local interpolant is linear.

(b) Let T_1 be the triangle whose vertices are $a_1 = (0\,,0)$, $a_2 = (1\,,0)$ and $a_3 = (0\,,1)$, T_2 be the triangle whose vertices are $a_2 = (1\,,0)$, $a_4 = (1\,,1)$ and $a_3 = (0\,,1)$. Following finite elements are considered:
$(T_1\,, \mathcal{P}_1\,, \mathcal{N}_1)$ with $\mathcal{N}_1 = \{N_1\,, N_2\,, N_3\}$ where $N_i(v) = v(a_i)\,, i = 1\,,2\,,3$;
$(T_2\,, \mathcal{P}_1\,, \mathcal{N}_2)$ with $\mathcal{N}_2 = \{N_4\,, N_5\,, N_6\}$ where $N_4(v) = v(a_2)\,, N_5(v) = v(a_4)\,, N_6(v) = v(a_3)$.
Finally let f and g be functions in $\mathbb{R}^2$: $f(x,y) = e^{xy}$ and $g(x,y) = \sin(\pi(x+y)/2)$.
Compute the local interpolations $\mathcal{I}_K f$ and $\mathcal{I}_K g$ where $K = T_1\,, T_2$.

Exercise 5.

Let $\mathcal{P}_k^n$ denote the space of polynomials of degree $\le\ k$ in n variables.

Prove that $\dim \mathcal{P}_k^n = \begin{pmatrix} n+k \\ k \end{pmatrix}$, where the latter is the binomial coefficient.