A closed-form extension to the Black-Cox model

Aurélien Alfonsi

Joint work with Jérôme Lelong

Work available at: http://cermics.enpc.fr/~alfonsi

Nice, September 30th, 2009

Structure of the talk

Single-name default models

Usually, single default models are divided into two main categories.

- Structural models : they aim at explaining the default time with other economic variables of the firm. (Merton (1974), Black and Cox (1976), Leland (1994), Briys and de Varenne 1997, ...)
- Intensity (or reduced form) models : these models describe the instantaneous probability of default with an exogeneous process $(\lambda_t, t \geq 0)$:

$$\mathbb{P}(\tau \in [t, t+dt) | \mathcal{G}_t) \approx \lambda_t dt.$$

(Jarrow and Turnbull (1995), Lando (1998), Duffie and Singleton (1999), Brigo and A. (2005) ...)

Merton's model (1974)

Merton's pioneering model assume that the value of a firm is the sum of its equity and its debt : $V_t = E_t + D_t$.

- The debt brings on a notional value L with a single maturity T.
- Default occurs at time T if the debt cannot be payed back (i.e. $V_T \leq L$). In that case, debtholders are first reimbursed $D_T = V_T$ and equityholders get 0. Otherwise, $D_T = L$ and $E_T = V_T - L$ and ge have :

$$\tau = T\mathbf{1}_{\{V_T \leq L\}} + \infty\mathbf{1}_{\{V_T > L\}}, D_T = \min(L, V_T), E_T = (V_T - L)^+$$

- The firm value is supposed to follow a geometric Brownian motion under a martingale measure $\mathbb{P}$:

$$dV_t = rV_t dt + \sigma V_t dW_t.$$

$\implies$ closed formula for $\mathbb{P}(\tau = T)$ and the equity appears as a Call option on the firm value.

Black and Cox model (1976)

One of the major drawback of Merton's model is that a firm can default only at one (deterministic) time : the maturity of its debt T. To correct this, Black and Cox extend Merton's model by assuming in addition that the debtholders can force the firm to bankruptcy when $V_t \leq C\,\mathrm{e}^{\alpha t}$, with $C\,\mathrm{e}^{\alpha T} < L$ and $C < V_0$:

$$\tau = \min(\inf\{t \geq 0, V_t \leq C\,\mathrm{e}^{\alpha t}\}, T\mathbf{1}_{\{V_T \leq L\}} + \infty\mathbf{1}_{\{V_T > L\}}).$$

- Closed formulas for $\mathbb{P}(\tau \leq t)$, the equity and the debt values at time $t \in [0, T]$.
- Many extensions of this model called First-passage-time models. (Leland (1994), Briys and de Varenne 1997, Brigo and Tarenghi (2004)...)

Structural models : Pro's and con's

- (+) The default event has a clear and meaningful origin.
- (+) Default and equity are explicitly related.
- ($\approx$) Not meant originally for being calibrated to market data. Calibration to Credit Default Swaps (CDS) data has been recently investigated for some first passage time models (e.g. Brigo and Morini (2006) and Dorfleitner, Schneider and Veza (2008)).
- (-) Structural model are often inappropriate to manage hedging portfolios since the default is predictable (unless considering jumps, e.g Zhou (2001)). Said differently, default probabilities and credit spreads are underestimated for short maturities.

Intensity models : Pro's and con's

- (+) Intensity models are in general easy to calibrate to CDS data and reflect the information known by the market on the single default (Jarrow and Protter (2004)).
- (+) They are more tractable to manage hedging portfolios.
- (-) The default event remains disconnected from other rationales of the firm like its debt or its equity values. This is unfortunate for pricing in a coherent manner equity and credit products, but also for understanding the dependence between defaults since much more information is available on equity market.

Hybrid models

To try to get the advantages of both structural and reduced-form models, models in between called "hybrid models" have been suggested, where the default intensity is a function of the stock or of the firm value (e.g. Madan and Unal (1998,2000), Atlan and Leblanc (2005), Carr and Linetsky (2006)...)

The model setup

We consider a very simple hybrid extension of the Black-Cox model :

- $(\mathcal{F}_t)$ denotes the default-free filtration and $(W_t, t \geq 0)$ a $(\mathcal{F}_t)$-Brownian motion.
- Firm value : $dV_t = rV_t dt + \sigma V_t dW_t$.
- Default intensity :

$$\lambda_t = \mu_+ \mathbf{1}_{\{V_t \leq C\, e^{\alpha t}\}} + \mu_- \mathbf{1}_{\{V_t > C\, e^{\alpha t}\}}, \tag{1}$$

  where $C > 0$, $\alpha \in \mathbb{R}$, and $\mu_+ > \mu_- \geq 0$.
- More precisely, let ξ be ax exponential r.v. of parameter 1 independent of $\mathcal{F}$, the default is defined by :

$$\tau = \inf\{t \geq 0, \int_0^t \lambda_s ds \geq \xi\}. \tag{2}$$

  As usual, we also introduce $(\mathcal{H}_t, t \geq 0)$ the filtration generated by $(\tau \wedge t, t \geq 0)$ and define $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$.

Comments on the model

- Very simple parametrization with a clear meaning. We expect to have $V_0/C > 1$ for healthy firms, and $V_0/C < 1$ for firms in difficulty.
- The Black-Cox model appears as the limit case $\mu_- = 0$, $\mu_+ \to +\infty$ when $V_0 > C$.
- Contrary to the Black-Cox model, we do not consider the additional possibility of default at time T since it would make the default predictable in some cases.
- In the Black-Cox model, the barrier $C\,\mathrm{e}^{\alpha t}$ is a safety covenant that allow bondholders to force bankruptcy. Here, the barrier has rather to be seen as a border between two credit grades.

The main result

Theorem 1

Let us set $b = \frac{1}{\sigma}\log(C/V_0)$, $m = \frac{1}{\sigma}(r-\alpha-\sigma^2/2)$ *and* $\mu_b = \mu_+ \mathbf{1}_{\{b>0\}} + \mu_- \mathbf{1}_{\{b<0\}}$. *The default cumulative distribution function* $P(\tau \le t)$ *is a function of* t, b, m, μ_- *and* μ_+ *and is fully characterized by its Laplace transform defined for* $\lambda \in \mathbb{C}_+ := \{z \in \mathbb{C}, \mathcal{R}\mathrm{e}\,(z) > 0\}$,

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-\lambda t} P(\tau \le t)dt \quad &= \quad \frac{1}{\lambda} - \frac{1}{\lambda+\mu_b} \\ &+ \mathrm{e}^{mb-|b|\sqrt{2(\lambda+\mu_b)+m^2}} \times \Bigg\{ \frac{1}{\lambda+\mu_b} - \frac{1}{\sqrt{\lambda+\mu_-+m^2/2}}\frac{1}{\sqrt{\lambda+\mu_++m^2/2}} \\ &- \frac{m}{\mu_+-\mu_-}\left(\sqrt{2(\lambda+\mu_+)+m^2} - \sqrt{2(\lambda+\mu_-)+m^2}\right) \times \Bigg[\frac{1}{2(\lambda+\mu_-)} \\ &- \frac{1}{2(\lambda+\mu_+)} + \frac{m}{2(\lambda+\mu_-)\sqrt{2(\lambda+\mu_-)+m^2}} + \frac{m}{2(\lambda+\mu_+)\sqrt{2(\lambda+\mu_+)+m^2}}\Bigg]\Bigg\}.\end{aligned} \tag{3}$$

Other structural models based on closed formula for Parisian options

The calculation of the Laplace Transform mainly relies on formulas obtained by Chesney, Jeanblanc and Yor (1997) to calculate Parisian option prices.
Other extensions of the Black Cox model based on these formulas have been considered by Moraux (2002), Yu (2004) and Chen and Suchanecki (2007), who consider for $D > 0$ the two following cases :

- $\tau = \inf\{t \geq D, \forall u \in [t-D, t], V_u \leq C\, e^{\alpha u}\}$ (Parisian time)
- $\tau = \inf\{t \geq 0, \int_0^t \mathbf{1}_{\{V_u \leq C\, e^{\alpha u}\}} du > D\}$ (cumulated Parisian time or ParAsian time).

Nonetheless, in both cases, the default is predictable.

Sketch of the proof

The calculation of the Laplace transform is split into three steps :

Step 1 Change of the probability measure and reduction to the case $\mu_- = 0$.

Step 2 Calculation of the Laplace transform when $b = 0$.

Step 3 Calculation of the Laplace transform when $b > 0$ and $b < 0$.

Step 1 I

$V_u = V_0 \exp((r - \frac{1}{2}\sigma^2)u + \sigma W_u)$ and
$\mathbf{1}_{\{V_u \le C\, e^{\alpha u}\}} = \mathbf{1}_{\{W_u + \frac{1}{\sigma}(r-\alpha-\sigma^2/2)u \le \frac{1}{\sigma}\log(C/V_0)\}}$.
We set for an arbitrary $T > t$,

$$b = \frac{1}{\sigma}\log(C/V_0),\ m = \frac{1}{\sigma}(r-\alpha-\sigma^2/2),\ \text{and}\ \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\Big|_{\mathcal{G}_T} = \exp(-mW_T - m^2T/2).$$

$(\tilde{W}_u := W_u + mu, u \in [0,T])$ is a std Brownian motion under $\tilde{\mathbb{P}}$, and

$$\tau = \inf\{t \ge 0, \int_0^t \mu_+ \mathbf{1}_{\{\tilde{W}_u \le b\}} + \mu_- \mathbf{1}_{\{\tilde{W}_u > b\}} du \ge \xi\}.$$

$\Longrightarrow\ \mathbb{P}(\tau \ge t) = \tilde{\mathbb{E}}[\exp(m\tilde{W}_t - m^2t/2)\mathbf{1}_{\{\tau \ge t\}}]$ is a function of $t,\ b,\ m,\ \mu_-$ and μ_+.

Step 1 II

Notations :
$t \geq 0,\ P_{b,m,\mu_-,\mu_+}(t) = \mathbb{P}(\tau \leq t)$ (resp. $P^c_{b,m,\mu_-,\mu_+}(t) = 1 - P_{b,m,\mu_-,\mu_+}(t)$),
$\lambda \in \mathbb{C}_+,\ L_{b,m,\mu_-,\mu_+}(\lambda) = \int_0^{+\infty} \mathrm{e}^{-\lambda t} P_{b,m,\mu_-,\mu_+}(t)dt$
(resp. $L^c_{b,m,\mu_-,\mu_+}(\lambda) = 1/\lambda - L_{b,m,\mu_-,\mu_+}(\lambda)$).
$\tau \overset{law}{=} \min(\xi^1/\mu_-, \inf\{t \geq 0, \int_0^t (\mu_+ - \mu_-)\mathbf{1}_{\{V_u \leq C\,\mathrm{e}^{\alpha u}\}} du \geq \xi^2\})$ with $\xi^1, \xi^2 \sim Exp(1)$ independent of $\mathcal{F}$.

$$P^c_{b,m,\mu_-,\mu_+}(t) = \mathrm{e}^{-\mu_- t} P^c_{b,m,0,\mu_+-\mu_-}(t)$$
$$L^c_{b,m,\mu_-,\mu_+}(\lambda) = L^c_{b,m,0,\mu_+-\mu_-}(\lambda + \mu_-).$$

We set $\mu_- = 0$ and $\mu = \mu_+$ in the following of the proof.

Step 2 : $b = 0$ I

For $D > 0$, we introduce

$$\tau^D = \inf\{t \geq 0, \int_0^t \mathbf{1}_{\{\tilde{W}_u \leq 0\}} du \geq D\}.$$

Given $\{\xi/\mu = D\}$, $\tau^D \overset{law}{=} \tau$.
From Chesney and al. (1997), the law of $(\tilde{W}_t, A_t^- := \int_0^t \mathbf{1}_{\{\tilde{W}_u \leq 0\}} du)$ is known : $x \in \mathbb{R}$ and $D > 0$, we have :

$$\tilde{\mathbb{P}}(\tilde{W}_t \in dx, A_t^- \leq D) = \mathbf{1}_{\{t>D\}} \Bigg(\mathbf{1}_{\{x<0\}} \left[\frac{-x}{2\pi} \int_{t-D}^t \frac{D+s-t}{\sqrt{s^3(t-s)^3}} \exp(-\frac{x^2}{2(t-s)}) ds \right]$$
$$+ \mathbf{1}_{\{x>0\}} \left[\int_0^D \frac{x \exp(-\frac{x^2}{2(t-s)})}{2\pi\sqrt{s(t-s)^3}} ds + \int_D^t \frac{xD \exp(-\frac{x^2}{2(t-s)})}{2\pi\sqrt{s^3(t-s)^3}} ds \right] \Bigg) + \mathbf{1}_{\{t \leq D\}} \frac{\exp(-\frac{x^2}{2t})}{\sqrt{2\pi t}}.$$

Step 2 : $b = 0$ II

$\{\tau^D \geq t\} = \{A_t^- \leq D\}$, and we get integrating w.r.t x :

$$
\begin{aligned}
\mathbb{P}(\tau^D \geq t) &= \tilde{\mathbb{E}}[\exp(m\tilde{W}_t - m^2 t/2)\mathbf{1}_{\{A_t^- \leq D\}}] \\
&= \mathbf{1}_{\{t \leq D\}} + \mathbf{1}_{\{t > D\}} \left(\mathrm{e}^{-m^2 t/2} \left(\frac{1}{2} + \frac{1}{\pi} \arctan \left(\frac{D - t/2}{\sqrt{D(t-D)}} \right) \right) \right. \\
&\quad + \frac{m}{\sqrt{2\pi}} \left[\int_0^D \frac{\mathrm{e}^{-m^2 s/2}\, \Phi(m\sqrt{t-s})}{\sqrt{s}} ds + D \int_D^t \frac{\mathrm{e}^{-m^2 s/2}\, \Phi(m\sqrt{t-s})}{\sqrt{s^3}} ds \right. \\
&\quad \left. \left. - \int_{t-D}^t \frac{(D+s-t)\, \mathrm{e}^{-m^2 s/2}\, \Phi(-m\sqrt{t-s})}{\sqrt{s^3}} ds \right] \right).
\end{aligned}
$$

Step 2 : $b = 0$ III

Using $\mathbb{P}(\tau \geq t) = \int_0^{+\infty} \mathbb{P}(\tau^\delta \geq t)\mu \, e^{-\mu\delta} \, d\delta$, we get :

$$\begin{aligned}
P^c_{0,m,0,\mu}(t) \quad = \quad & e^{-\mu t}(1 - e^{-m^2 t/2}) + \frac{1}{\pi}\int_0^t \frac{e^{-(\mu+m^2/2)s}}{\sqrt{s}} \frac{e^{-m^2(t-s)/2}}{\sqrt{t-s}} ds \\
& + \frac{m}{\sqrt{2\pi}}\Bigg[- \int_0^t \frac{e^{-(\mu+m^2/2)s}}{\sqrt{s}} \Phi(m\sqrt{t-s}) \, e^{-\mu(t-s)} \, ds \\
& + \frac{1}{\mu}\int_0^t \frac{e^{-m^2 s/2} - e^{-(\mu+m^2/2)s}}{\sqrt{s^3}} \Phi(m\sqrt{t-s}) \, ds \\
& - \frac{1}{\mu}\int_0^t \frac{e^{-m^2 s/2} - e^{-(\mu+m^2/2)s}}{\sqrt{s^3}} \, e^{-\mu(t-s)} \, \Phi(-m\sqrt{t-s}) ds \\
& + \int_0^t \frac{e^{-(\mu+m^2/2)s}}{\sqrt{s}} \Phi(-m\sqrt{t-s}) \, e^{-\mu(t-s)} \, ds \Bigg].
\end{aligned}$$

Step 2 : $b = 0$ IV

We recognize convolution products of

$$\int_0^\infty t^{-1/2}\, \mathrm{e}^{-\lambda t}\, dt = \sqrt{\frac{\pi}{\lambda}}, \int_0^\infty \mathrm{e}^{-\lambda t}\, \Phi(m\sqrt{t}) dt = \frac{1}{2\lambda} + \frac{m}{2\lambda\sqrt{2\lambda + m^2}},$$
$$\int_0^\infty t^{-3/2}\, \mathrm{e}^{-\lambda t}(1 - \mathrm{e}^{-\mu t}) dt = 2\sqrt{\pi}(\sqrt{\lambda + \mu} - \sqrt{\lambda}),$$

and get after some simplifications :

$$\begin{aligned} L^c_{0,m,0,\mu}(\lambda) &= \frac{1}{\sqrt{\lambda + m^2/2}} \frac{1}{\sqrt{\lambda + \mu + m^2/2}} + \frac{m}{\mu}\left(\sqrt{2(\lambda+\mu) + m^2} - \sqrt{2\lambda + m^2}\right) \\ &\times \left[\frac{1}{2\lambda} - \frac{1}{2(\lambda+\mu)} + \frac{m}{2\lambda\sqrt{2\lambda + m^2}} + \frac{m}{2(\lambda+\mu)\sqrt{2(\lambda+\mu) + m^2}}\right]. \end{aligned}$$

Step 3, $b < 0$

We set $\tau_b = \inf\{u \geq 0, \tilde{W}_u = b\}$ (recall $\tilde{\mathbb{E}}[e^{-\lambda\tau_b}] = e^{-\sqrt{2\lambda}|b|}$) and $\tau' = \inf\{t \geq 0, \int_0^t \mathbf{1}_{\{\tilde{W}_{\tau_b+u} - b \leq 0\}} du \geq \xi/\mu\}$, so that

$$\tau = \tau_b + \tau'.$$

$$\begin{aligned}
&P^c_{b,m,0,\mu}(t) = \\
&\tilde{\mathbb{E}}[\exp(m\tilde{W}_t - m^2 t/2)\mathbf{1}_{\{\tau_b \geq t\}}] + \tilde{\mathbb{E}}[\exp(m\tilde{W}_t - m^2 t/2)\mathbf{1}_{\{\tau_b < t\}}\mathbf{1}_{\{\tau' \geq t-\tau_b\}}] \\
&= 1 - \tilde{\mathbb{E}}[\exp(m\tilde{W}_t - m^2 t/2)\mathbf{1}_{\{\tau_b < t\}}] \\
&\qquad + \tilde{\mathbb{E}}[e^{mb - m^2\tau_b/2}\mathbf{1}_{\{\tau_b < t\}} e^{m(\tilde{W}_{\tau_b+(t-\tau_b)} - b) - m^2(t-\tau_b)/2}\mathbf{1}_{\{\tau' \geq t-\tau_b\}}] \\
&= 1 - \tilde{\mathbb{E}}[\exp(mb - m^2\tau_b/2)\mathbf{1}_{\{\tau_b < t\}}] \text{ (Doob)} \\
&\qquad + \tilde{\mathbb{E}}[\exp(mb - m^2\tau_b/2)\mathbf{1}_{\{\tau_b < t\}}P^c_{0,m,0,\mu}(t-\tau_b)]. \text{ (Strong Markov)}
\end{aligned}$$

$$\Longrightarrow L^c_{b,m,0,\mu}(\lambda) = \frac{1}{\lambda} + e^{mb + b\sqrt{2\lambda+m^2}}\left(L^c_{0,m,0,\mu}(\lambda) - \frac{1}{\lambda}\right).$$

Step 3, $b > 0$

We set $\tau'' = \inf\{t \geq 0, \int_0^t \mathbf{1}_{\{\tilde{W}_{\tau_b+u}-b\leq 0\}}du \geq \xi/\mu\}$, so that

$$\mathbf{1}_{\{\tau\geq t\}} = \mathbf{1}_{\{\tau_b\geq t\}}\mathbf{1}_{\{\xi/\mu\geq t\}} + \mathbf{1}_{\{\tau_b< t\}}\mathbf{1}_{\{\xi/\mu\geq \tau_b\}}\mathbf{1}_{\{\tau''\geq t-\tau_b\}}.$$

$P^c_{b,m,0,\mu}(t) = e^{-\mu t}(1 - \tilde{\mathbb{E}}[\exp(m\tilde{W}_t - m^2t/2)\mathbf{1}_{\{\tau_b<t\}}])$
$+\tilde{\mathbb{E}}[e^{mb-m^2\tau_b/2}\mathbf{1}_{\{\tau_b<t\}}\mathbf{1}_{\{\xi/\mu\geq\tau_b\}}\, e^{m(\tilde{W}_{\tau_b+(t-\tau_b)}-b)-m^2(t-\tau_b)/2}\mathbf{1}_{\{\tau''\geq t-\tau_b\}}]$
$= e^{-\mu t}(1 - \tilde{\mathbb{E}}[e^{mb-m^2\tau_b/2}\mathbf{1}_{\{\tau_b<t\}}])+$ Doob
$\tilde{\mathbb{E}}[e^{mb-m^2\tau_b/2}\mathbf{1}_{\{\tau_b<t\}}\mathbf{1}_{\{\xi/\mu\geq\tau_b\}}P^c_{0,m,0,\mu}(t-\tau_b)]$ Markov + lack of memory
$= e^{-\mu t}(1-\tilde{\mathbb{E}}[e^{mb-m^2\tau_b/2}\mathbf{1}_{\{\tau_b<t\}}])+\tilde{\mathbb{E}}[e^{mb-(\mu+m^2/2)\tau_b}\mathbf{1}_{\{\tau_b<t\}}P^c_{0,m,0,\mu}(t-\tau_b)]$.

$$\Longrightarrow L^c_{b,m,0,\mu}(\lambda) = \frac{1}{\lambda+\mu} + e^{mb-b\sqrt{2(\lambda+\mu)+m^2}}\left(L^c_{0,m,0,\mu}(\lambda) - \frac{1}{\lambda+\mu}\right).$$

The geometric Brownian motion barrier case

$$\lambda_t = \mu_+ \mathbf{1}_{\{V_t \le C\,\mathrm{e}^{(\alpha-\eta^2/2)t+\eta Z_t}\}} + \mu_- \mathbf{1}_{\{V_t > C\,\mathrm{e}^{(\alpha-\eta^2/2)t+\eta Z_t}\}}, \text{ with } \langle W, Z\rangle_t = \rho t.$$

We exclude the trivial case $\rho = 1$ with $\eta = \sigma$, and set $\varsigma = \sqrt{\sigma^2 + \eta^2 - 2\rho\sigma\eta} > 0$. $B_t = (\sigma W_t - \eta Z_t)/\varsigma$ is a std B.m.
Since

$$\mathbf{1}_{\{V_t \le C\,\mathrm{e}^{(\alpha-\eta^2/2)t+\eta Z_t}\}} = \mathbf{1}_{\{B_t + \frac{1}{\varsigma}(r-\alpha-(\sigma^2-\eta^2)/2)t \le \frac{1}{\varsigma}\log(C/V_0)\}},$$

we can proceed like in step 1 and we get the Laplace transform of $\mathbb{P}(\tau \le t)$ in that case by simply taking

$$b = \frac{1}{\varsigma}\log(C/V_0) \text{ and } m = \frac{1}{\varsigma}(r - \alpha - (\sigma^2 - \eta^2)/2).$$

Mathematical properties of $\mathbb{P}(\tau \leq t)$ I

Proposition 2

For any $t \geq 0$, the function $P_{b,m,\mu_-,\mu_+}(t)$ is nondecreasing with respect to b, μ_- and μ_+, and is nonincreasing with respect to m.

$$P^c_{b,m,\mu_-,\mu_+}(t) = \mathbb{E}\left[e^{-\int_0^t \mu_+ \mathbf{1}_{\{W_u+mu \leq b\}} + \mu_- \mathbf{1}_{\{W_u+mu > b\}} du}\right] = e^{-\mu_- t}\, \mathbb{E}\left[e^{-\int_0^t (\mu_+ - \mu_-) \mathbf{1}_{\{W_u+mu \leq b\}} du}\right].$$

Mathematical properties of $\mathbb{P}(\tau \leq t)$ II

Proposition 3

When $b \neq 0$, the functions $P_{b,m,\mu_-,\mu_+}(t)$ and $\partial_p P_{b,m,\mu_-,\mu_+}(t)$ for $p \in \{b, m, \mu_-, \mu_+\}$ are $\mathcal{C}^\infty$ on $[0,\infty)$. Moreover, for any $\varepsilon > 0$, we have

$$\forall k \in \mathbb{N}^*, P^{(k)}_{b,m,\mu_-,\mu_+}(t) \underset{t\to\infty}{=} O(\mathrm{e}^{(\varepsilon-\mu_-)t}),\ \forall k \in \mathbb{N}, \partial_p P^{(k)}_{b,m,\mu_-,\mu_+}(t) \underset{t\to\infty}{=} O(\mathrm{e}^{(\varepsilon-\mu_-)t}).$$

When $b = 0$, P_{0,m,μ_-,μ_+} is $\mathcal{C}^1$ but not $\mathcal{C}^2$ on $[0,\infty)$ and $\mathcal{C}^\infty$ on $(0,\infty)$.

Proposition 4

The functions $P_{b,m,\mu_-,\mu_+}(t)$ and $\partial_p P_{b,m,\mu_-,\mu_+}(t)$ are continuous w.r.t. (b, m, μ_-, μ_+) and $t \geq 0$.

Framework and goal

$f : \mathbb{R} \to \mathbb{R}$: Real valued function such that $\forall t < 0, f(t) = 0$ and $f(t)e^{-\gamma t}$ is integrable for some $\gamma > 0$.

$$\hat{f}(z) = \int_0^{\infty} \mathrm{e}^{-zt} f(t)dt, \ \mathcal{R}\mathrm{e}\,(z) \geq \gamma : \text{known function}$$

We want to recover f using the inversion formula :

$$f(t) = \frac{\mathrm{e}^{\gamma t}}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{-ist} \hat{f}(\gamma - is)ds, \quad t > 0.$$

Typically, f will be either $P_{b,m,\mu_-,\mu_+}(t)$ or $\partial_p P_{b,m,\mu_-,\mu_+}(t)$ for $p \in \{b, m, \mu_-, \mu_+\}$.
To compute f, one has to discretize the integral and also to truncate the integration domain : we want to quantify these errors.

The discretization error

We introduce

$$f_h(t) = \frac{h\,\mathrm{e}^{\gamma t}}{2\pi} \sum_{k=-\infty}^{\infty} \mathrm{e}^{-ikht} \widehat{f}(\gamma - ikh) = \frac{h}{2\pi} \sum_{k=-\infty}^{\infty} \mathrm{e}^{\gamma t + ikht} \widehat{f}(\gamma + ikh).$$

Proposition (Abate, Choudhury, Whitt, 1999)

If f is C^0 bounded, we have $\forall t < 2\pi/h$, $|f(t) - f_h(t)| \leq \|f\|_\infty \frac{\mathrm{e}^{-2\pi\gamma/h}}{1-\mathrm{e}^{-2\pi\gamma/h}}$.

Poisson summation Formula :

$$\sum_{k=-\infty}^{\infty} f(t + \frac{2\pi k}{h})\,\mathrm{e}^{-\gamma(t+2\pi k/h)} = \frac{h}{2\pi} \sum_{k=-\infty}^{\infty} \mathrm{e}^{ikht} \widehat{f}(\gamma + ikh).$$

$$\Longrightarrow f_h(t) - f(t) = \sum_{k\in\mathbb{Z}^*} \underbrace{f(t + \frac{2\pi k}{h})}_{=0 \text{ when } k<0,\ t<2\pi/h} \mathrm{e}^{-\gamma 2\pi k/h}.$$

The truncation error

We approximate $f_h(t) = \frac{h\,e^{\gamma t}}{2\pi}\hat{f}(\gamma) + \frac{h\,e^{\gamma t}}{\pi}\mathcal{R}\mathrm{e}\left(\sum_{k=1}^{\infty} e^{-ikht}\widehat{f}(\gamma - ikh)\right)$ by the following finite sum

$$f_h^N(t) = \frac{h\,e^{\gamma t}}{2\pi}\hat{f}(\gamma) + \frac{h\,e^{\gamma t}}{\pi}\mathcal{R}\mathrm{e}\left(\sum_{k=1}^{N} e^{-ikht}\widehat{f}(\gamma - ikh)\right).$$

Proposition 5

$f : \mathcal{C}^3$ on $\mathbb{R}_+$ such that $f(0) = 0$ and $\exists \epsilon > 0,\ \forall k \leq 3,\ f^{(k)}(s) = \mathcal{O}(e^{(\gamma-\epsilon)s})$, when $s \to +\infty$. Let $h \in (0, 2\pi/T)$. Then,

$$\exists K > 0, \forall t \in (0, T],\ |f_h^N(t) - f_h(t)| \leq K(1 + 1/t)\frac{e^{\gamma t}}{N^2}.$$

Proof : Control of the remainder of the sum, using that
$\widehat{f}(\gamma - ikh) = \frac{-f'(0)}{(ikh-\gamma)^2} + \frac{f''(0)}{(ikh-\gamma)^3} - \int_0^{\infty} \frac{f^{(3)}(u)}{(ikh-\gamma)^3}\, e^{(ikh-\gamma)u}\, du$ (Int. by parts).

The Laplace inversion using FFT I

FFT : very efficient algorithm to compute $(\hat{x}_l, l = 0, \dots, N-1)$ from $(x_k, k = 0, \dots, N-1)$, where

$$\hat{x}_l = \sum_{k=0}^{N-1} \mathrm{e}^{-2i\pi kl/N} x_k, \quad \text{for } l = 0, \dots, N-1.$$

We consider the time-grid $t_l = 2\pi l/(Nh)$ for $1 \leq l \leq N$.

$$\begin{aligned} f_h^N(t_l) &= \frac{h\,\mathrm{e}^{\gamma t_l}}{2\pi}\hat{f}(\gamma) + \frac{h\,\mathrm{e}^{\gamma t_l}}{\pi}\,\mathcal{R}\mathrm{e}\left(\sum_{k=1}^{N} \mathrm{e}^{-2i\pi kl/N}\widehat{f}\,(\gamma - ikh)\right) \\ &= \frac{h\,\mathrm{e}^{\gamma t_l}}{2\pi}\hat{f}(\gamma) + \frac{h\,\mathrm{e}^{\gamma t_l}}{\pi}\,\mathcal{R}\mathrm{e}\left(e^{-2i\pi(l-1)/N}\sum_{k=1}^{N} \mathrm{e}^{-2i\pi(k-1)(l-1)/N}\, e^{-2ik\pi/N}\widehat{f}\,(\gamma - ikh)\right) \end{aligned}$$

can be computed with a FFT on $(e^{-2ik\pi/N}\widehat{f}\,(\gamma - ikh)\,, k = 1, \dots, N)$.

The Laplace inversion using FFT II

Corollary 6

Under the assumptions of Prop 5, $\exists K > 0$ s.t.

$$\forall l \geq 1, t_l \leq T, \ |f_h^N(t_l) - f(t_l)| \leq K \max\left(\frac{e^{\gamma T}}{N^2}, \frac{h}{2\pi N}\right) + \|f\|_\infty \frac{e^{-2\pi\gamma/h}}{1 - e^{-2\pi\gamma/h}}.$$

Parameters to achieve a precision of order ε :

$$h < 2\pi/T, \ \frac{2\pi\gamma}{h} = \log(1 + 1/\varepsilon), \ N > \max\left(\frac{h}{2\pi\varepsilon}, \sqrt{\frac{e^{\gamma T}}{\varepsilon}}\right)$$

- Very efficient method that allows to recover f on a whole time-grid.
- The time grid has to be regular, and it is convenient to take it such that it includes the standard payment grids (quarterly).

The Euler summation technique I

- It is an efficient alternative to the FFT.
- Contrary to the FFT, it compute $f(t)$ at a single time and not on a regular time grid, which can be convenient for bespoke products.
- The trick is to take $h = \pi/t$ to get an alternating series and replace $f^N_{\pi/t}(t) = \frac{e^{\gamma t}}{2t}\hat{f}(\gamma) + \frac{e^{\gamma t}}{t}\sum_{k=1}^N (-1)^k \mathcal{R}e\left(\hat{f}\left(\gamma + i\frac{\pi k}{t}\right)\right)$ by a more accurate proxy of $f_{\pi/t}(t) : E(q,N,t)$.

Proposition 7

$q \in \mathbb{N}^*$, f C^{q+4} function s.t. $\exists \epsilon > 0,\ \forall k \leq q+4,\ f^{(k)}(s) = \mathcal{O}(e^{(\gamma-\epsilon)s})$. We set $E(q,N,t) = \sum_{k=0}^q \binom{q}{k} 2^{-q} f^{N+k}_{\pi/t}(t)$. Then,

$$\left|f_{\pi/t}(t) - E(q,N,t)\right| \leq \frac{t\, e^{\gamma t}\left|f'(0) - \gamma f(0)\right|}{\pi^2} \frac{N!\,(q+1)!}{2^q\,(N+q+2)!} + \mathcal{O}\left(\frac{1}{N^{q+3}}\right), N \to \infty.$$

The Euler summation technique II

Practical choice of parameters : We take $q = N = 15$ and $\gamma = 11.5/t$.

- We have $\left|f_{\pi/t}(t) - E(q,N,t)\right| \leq \mathrm{e}^{\gamma t} \frac{N!\,(q+1)!}{2^q\,(N+q+2)!} \approx 3.13 \times 10^{-10}$.
- Discretization error : $|f_{\pi/t}(t) - f(t)| \leq \|f\|_\infty \frac{e^{-2\gamma t}}{1-e^{-2\gamma t}}$, is of order 10^{-10}.
- The overall error is therefore of order 10^{-10}.
- For a fixed t, the computation cost of $E(q,N,t)$ is proportional to $N+q$.

Pricing of CDS I

We consider a CDS with maturity T on a unit notional value, and assume a deterministic recovery rate $1 - LGD \in [0, 1]$ and short interest rate r.

Default Leg : $DL(0, T) = \mathbb{E}[e^{-r\tau} \mathbf{1}_{\{\tau \le T\}} LGD] = LGD \left[e^{-rT} \mathbb{P}(\tau \le T) + \int_0^T r e^{-ru} \mathbb{P}(\tau \le u) du \right]$.

Payment Leg : Payment grid : $T_0 = 0 < T_1 < \cdots < T_n = T$.
$\beta(t) \in \{1, \ldots, n\}$ is the index s.t. $T_{\beta(t)-1} \le t < T_{\beta(t)}$.

$$\begin{aligned} PL(0,T) &= R \times \mathbb{E}\left[\sum_{i=1}^{n} (T_i - T_{i-1}) e^{-rT_i} \mathbf{1}_{\{\tau > T_i\}} + (\tau - T_{\beta(\tau)-1}) e^{-r\tau} \mathbf{1}_{\{\tau \le T\}} \right] \\ &= R \left[\int_0^T e^{-ru} \mathbb{P}(\tau > u) du - \int_0^T r e^{-ru} (u - T_{\beta(u)-1}) \mathbb{P}(\tau > u) du \right]. \end{aligned}$$

Pricing of CDS II

Proposition 8

With a deterministic interest rate $r > 0$ and a deterministic recovery rate $1 - LGD \in [0, 1]$, the CDS fair rate within the model (1) is given by :

$$R^{\mathbf{model}}(0,T) = LGD \frac{\mathrm{e}^{-rT} P_{b,m,\mu_-,\mu_+}(T) + \int_0^T r\,\mathrm{e}^{-ru} P_{b,m,\mu_-,\mu_+}(u)du}{\int_0^T \mathrm{e}^{-ru} P^c_{b,m,\mu_-,\mu_+}(u)du - \int_0^T r\,\mathrm{e}^{-ru}(u - T_{\beta(u)-1}) P^c_{b,m,\mu_-,\mu_+}(u)du},$$

where $b = \frac{1}{\sigma}\log(C/V_0)$ and $m = \frac{1}{\sigma}(r - \alpha - \sigma^2/2)$. Moreover, if we neglect the second integral in the denominator this rate is nondecreasing with respect to C, α, μ_- and μ_+, and we get the following bounds :

$$\mu_- \lesssim \frac{R^{\mathbf{model}}(0,T)}{LGD} \lesssim \mu_+. \quad (4)$$

The calibration procedure I

Our aim is not to present the ultimate calibration procedure, but rather to see qualitatively how this Black-Cox extension can fit different kinds of market data.

- We have $\nu = 8$ CDS market data and want to minimize :

$$\min_{b,m\in\mathbb{R},0<\mu_-<\mu_+} \sum_{i=1}^{\nu} (R^{\mathbf{model}}(0,T^{(i)}) - R^{\mathbf{market}}(0,T^{(i)}))^2.$$

- Minimization is achieved with a gradient algorithm (FFT to get $P_{b,m,\mu_-,\mu_+}(t)$ and $\partial_p P_{b,m,\mu_-,\mu_+}(t)$, and integration using Simpson's rule) with the following starting point :

$$b=0,\ m=0,\ \mu_- = \min_{i=1,\dots,\nu} \frac{R^{\mathbf{market}}(0,T^{(i)})}{LGD},\ \mu_+ = \max_{i=1,\dots,\nu} \frac{R^{\mathbf{market}}(0,T^{(i)})}{LGD}.$$

It takes few seconds.

The calibration procedure II

We have tested this procedure trying to get back parameters from computed prices : it can be worth to have a better prior on (b, m).

- We take a finite set $\mathcal{S} \subset \mathbb{R}^2$, typically $\mathcal{S} = \{-B + 2iB/n, i = 0, \dots, n\} \times \{-M + 2iM/n, i = 0, \dots, n\}$ for some $B, M > 0, n \in \mathbb{N}^*$. For $(b, m) \in \mathcal{S}$, we minimize the criterion with respect to μ_- and $\mu+$, keeping b and m constant. In practice, we have mostly taken $B, M \in \{1, 2\}$ and $n = 8$.
- Then, we select $(b, m) \in \mathcal{S}$ that achieves the smallest score and use it (with the optimized parameters μ_- and $\mu+$) as the initial point of the gradient algorithm for the global minimization.

The calibration procedure III

This improves the algorithm but the problem is anyway ill-posed : two different set of parameters can give very similar default cdf. For example, the constant intensity model $\lambda > 0$ corresponds to the following parametrizations :

1. $\mu_- = \mu_+ = \lambda$, with $b, \lambda \in \mathbb{R}$ arbitrarily chosen,
2. $\mu_- = \lambda, b \to -\infty$, with $m \in \mathbb{R}$ and $\mu_+ > \mu_-$ arbitrarily chosen,
3. $\mu_+ = \lambda, b \to +\infty$, with $m \in \mathbb{R}$ and $\mu_+ > \mu_-$ arbitrarily chosen.

The calibration procedure IV

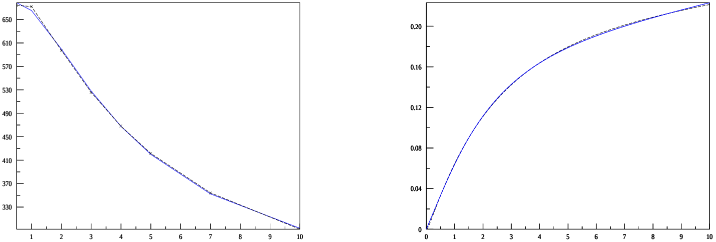


FIG.: Left picture : CDS prices in function of the maturities. Prices are in bps (10^{-4}) with $LGD = 1$ and $r = 5\%$. Right picture : associated cumulative distribution functions. Dashed line : is $b = -0.2$, $m = 0.6$, $\mu_- = 0.005$ and $\mu_+ = 0.3$. Solid line : $b = 2.168849$, $m = 0.912237$, $\mu_- = 0.008414$ and $\mu_+ = 0.067515$.

Calibration on market data I

- Data from 2006 to 2009 on Crédit Agricole, PSA, Ford and Saint-Gobain.
- We have taken $r = 5\%$, $LGD = 0.6$ except for CA ($LGD = 0.8$).
- Market data are in dotted lines and Calibrated data in solid line.
- We have indicated V_0/C and α using the 1Y ATM implied vol as a proxy of the firm value volatility.
- We have plotted each time the default cdf since it is what we really calibrate.

We have split the results into three cases :

- The curve $T \mapsto R^{\textbf{market}}(0,T)$ is mostly increasing.
- The curve $T \mapsto R^{\textbf{market}}(0,T)$ is mostly decreasing.
- The curve is rather flat.

Increasing rates I

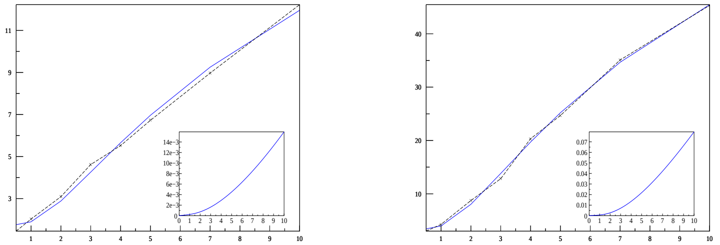
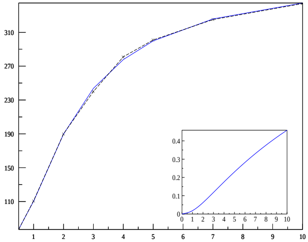


FIG.: *Left, CA 08/31/06 :* $b = -2.3415,\ m = -0.2172,\ \mu_- = 2.164 \times 10^{-4},\ \mu_+ = 5.597 \times 10^{-3},\ V_0/C = 1.753,\ \alpha = -1.78 \times 10^{-2}$. *Right, PSA 05/03/06 :* $b = -2.3878,\ m = -0.3745,\ \mu_- = 5.581 \times 10^{-4},\ \mu_+ = 2.214 \times 10^{-2},\ V_0/C = 1.757,\ \alpha = 2.038 \times 10^{-2}$.

Increasing rates II



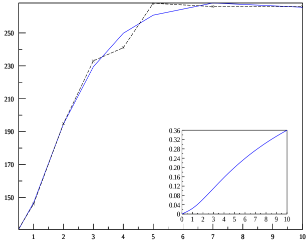


FIG.: *Left, Ford 11/30/06 :* $b = -1.734$, $m = -1.363$, $\mu_- = 1.2 \times 10^{-2}$, $\mu_+ = 7.05 \times 10^{-2}$, $V_0/C = 2.173$, $\alpha = 0.436$. *Right, SG 10/08/08 :* $b = -1.897$, $m = 0.1725$, $\mu_- = 2.135 \times 10^{-2}$, $\mu_+ = 0.652$, $V_0/C = 2.8506$, $\alpha = -0.3213$.

Decreasing rates I

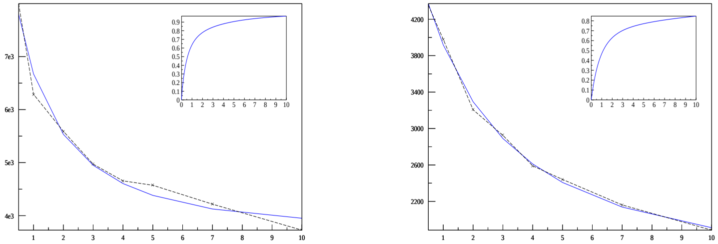


FIG.: *Left, Ford 11/24/08 :*
$b = 0.209,\ m = 0.344,\ \mu_- = 0.2014,\ \mu_+ = 1.986,\ V_0/C = 0.716,\ \alpha = -1.3.$
Right, Ford 02/25/09 : $b = 0.8517,\ m = 0.5277,\ \mu_- = 6.85 \times 10^{-2},\ \mu_+ = 0.7806,\ V_0/C = 0.3355,\ \alpha = -1.2676$

Decreasing rates II

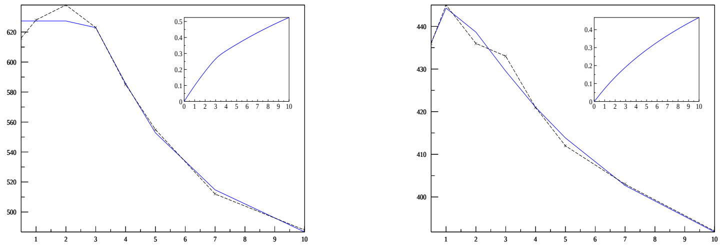


FIG.: *Left, PSA 03/06/09 :* $b = 15.55,\ m = 4.889,\ \mu_- = 6.055 \times 10^{-2},\ \mu_+ = 0.104,\ V_0/C = 6.32 \times 10^{-5},\ \alpha = -3.3$. *Right, SG 12/01/08 :* $b = -0.268,\ m = 0.567,\ \mu_- = 5.46 \times 10^{-2},\ \mu_+ = 0.154,\ V_0/C = 1.1837,\ \alpha = -0.6213$.

Flat rates

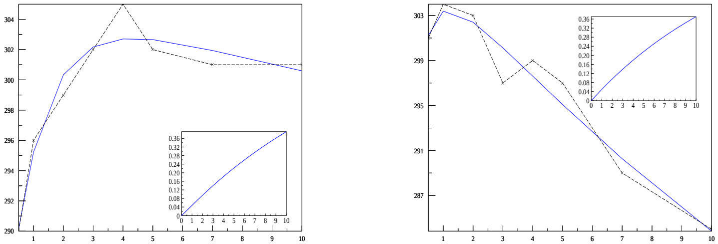


FIG.: *Left, SG 10/21/08 :* $b = -1.032,\ m = 0.493,\ \mu_- = 4.75 \times 10^{-2},\ \mu_+ = 9.23 \times 10^{-2},\ V_0/C = 1.83,\ \alpha = -0.531.$ *Right, SG 10/31/08 :* $b = -3.42 \times 10^{-2},\ m = 4.69 \times 10^{-2},\ \mu_- = 1.45 \times 10^{-2},\ \mu_+ = 9.295 \times 10^{-2},\ V_0/C = 1.021,\ \alpha = -0.282.$

Conclusion

- We have proposed an hybrid extension to the Black-Cox model.
- Its has a very clear and simple parametrization.
- The default probabilities can be computed easily with Laplace inversion methods, which provide a very fast calibration procedure.
- The model seems to fit qualitatively well a wide range of CDS data.
- The calibration to the CDS only ensures a good fit of the default cdf, not of the parameters.

Further work : We would like ton investigate how this model can be used in a multiname setting, using the "bottom-up" approach.