Density models for credit risk

Monique Jeanblanc, Université d'Évry; Institut Europlace de Finance

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Toy Model

Let us study the case with two random times τ_1, τ_2.

For $i = 1, 2$, we denote by $(H_t^i, t \geq 0)$ the default process associated with τ_i, i.e., $H_t^i = \mathbb{1}_{\{\tau^i \leq t\}}$.

The filtration generated by the process H^i is denoted $\mathbb{H}^i$ and the filtration generated by the two processes H^1, H^2 is $\mathbb{H} = \mathbb{H}^1 \vee \mathbb{H}^2$.

Any $\mathbb{H}$-adapted process Z admits a representation as

$$Z_t = h_0(t)\mathbb{1}_{t<\tau_1\wedge\tau_2} + h_1(t,\tau_1)\mathbb{1}_{\tau_1\le t<\tau_2} + h_2(t,\tau_2)\mathbb{1}_{\tau_2\le t<\tau_1} + h(\tau_1,\tau_2)\mathbb{1}_{\tau_1\vee\tau_2\le t}$$

where h_0, h_1, h_2, h are (deterministic) functions.

We denote by $G(t,s) = \mathbb{Q}(\tau_1 > t, \tau_2 > s)$ the survival probability of the pair (τ_1, τ_2) and we assume that the joint law of (τ_1, τ_2) admits a density $f(u,v)$.

We denote by $\partial_i G$, the partial derivative of G with respect to the i-th variable, $i = 1, 2$.

Simultaneous defaults are precluded in this framework, i.e, $\mathbb{Q}(\tau_1 = \tau_2) = 0$.

The process M^1 defined as

$$M_t^1 \quad := \quad H_t^1 + \int_0^{t\wedge\tau_1\wedge\tau_2} \frac{\partial_1 G(s,s)}{G(s,s)} ds + \int_{t\wedge\tau_1\wedge\tau_2}^{t\wedge\tau_1} \frac{f(s,\tau_2)}{\partial_2 G(s,\tau_2)} ds$$

is a $\mathbb{H}$-martingale.

The processes $H_t^i - \int_0^t \lambda_s^i ds$, $i = 1, 2$, are $\mathbb{H}$-martingales, where

$$\begin{aligned}
\lambda_t^1 &= \mathbb{P}(\tau_1 \in dt | \mathcal{H}_t, \tau_1 > t) \\
&= (1 - H_t^1)\left((1 - H_t^2)\frac{-\partial_1 G(t,t)}{G(t,t)} - H_t^2 \frac{f(t,\tau_2)}{\partial_2 G(t,\tau_2)}\right) \\
&=: (1 - H_t^1)(1 - H_t^2)\widetilde{\lambda}_t^1 + (1 - H_t^1) H_t^2 \lambda_t^{1|2}(\tau_2) \\
\lambda_t^2 &= (1 - H_t^2)\left((1 - H_t^1)\frac{-\partial_2 G(t,t)}{G(t,t)} - H_t^1 \frac{f(\tau_1,t)}{\partial_1 G(\tau_1,t)}\right) \\
&= (1 - H_t^1)(1 - H_t^2)\widetilde{\lambda}_t^2 + H_t^1(1 - H_t^2)\lambda_t^{2|1}(\tau_1)
\end{aligned}$$

where

$$\lambda_t^{1|2}(s) = -\frac{f(t,s)}{\partial_2 G(t,s)}, \quad \lambda_t^{2|1}(s) = -\frac{f(s,t)}{\partial_1 G(s,t)}$$

The goal is to find the dynamics of $Z_t := \mathbb{E}(h(\tau_1, \tau_2)|\mathcal{H}_t)$ and to give an hedging strategy based on CDSs

The price of the contingent claim $h(\tau_1, \tau_2)$ is

$$\begin{aligned} Z_t &= h(\tau_1, \tau_2) H_t^1 H_t^2 + \psi_{1,0}(\tau_1, t) H_t^1 (1 - H_t^2) + \psi_{0,1}(t, \tau_2) H_t^2 (1 - H_t^1) \\ &\quad + (1 - H_t^1)(1 - H_t^2)\psi_{0,0}(t) \end{aligned}$$

with

$$\begin{aligned} \psi_{1,0}(u, t) &= \frac{-1}{\partial_1 G(u, t)} \int_t^\infty h(u, v) f(u, v) dv \\ \psi_{0,1}(t, v) &= \frac{-1}{\partial_2 G(t, v)} \int_t^\infty h(u, v) f(u, v) du \\ \psi_{0,0}(t) &= \frac{1}{G(t, t)} \int_t^\infty du \int_t^\infty dv \, h(u, v) f(u, v) \end{aligned}$$

It can be proved that

$$\begin{aligned} dZ_t &= \Big(\big(h(t,\tau_2) - \psi_{0,1}(t,\tau_2)\big)H_t^2 + \big(\psi_{1,0}(t,t) - \psi_{0,0}(t)\big)(1 - H_t^2)\Big)dM_t^1 \\ &\quad + \Big(\big(h(\tau_1,t) - \psi_{1,0}(\tau_1,t)\big)H_t^1 + \big(\psi_{0,1}(t,t) - \psi_{0,0}(t)\big)(1 - H_t^1)\Big)dM_t^2 \\ &= \pi_t^1 dM_t^1 + \pi_t^2 dM_t^2 \end{aligned}$$

where

$$\begin{aligned} \psi_{1,0}(u,t) &= \frac{-1}{\partial_1 G(u,t)} \int_t^\infty h(u,v) f(u,v) dv \\ \psi_{0,1}(t,v) &= \frac{-1}{\partial_2 G(t,v)} \int_t^\infty h(u,v) f(u,v) du \\ \psi_{0,0}(t) &= \frac{1}{G(t,t)} \int_t^\infty du \int_t^\infty dv\, h(u,v) f(u,v) \end{aligned}$$

We consider a CDS

- with a constant spread κ
- which delivers $\delta(\tau_1)$ at time τ_1 if $\tau_1 < T$, where δ is a deterministic function.

The value of the CDS is, for $t < \tau_1$

$$V_t = 1\!\!1_{t<\tau_1}\mathbb{E}(\delta(\tau_1)1\!\!1_{\tau_1 \le T} - \kappa(\,(T \wedge \tau_1) - t)|\mathcal{H}_t) = \widetilde{V}_t 1\!\!1_{\{t<\tau_2\}} + V_t^{1|2}(\tau_2)1\!\!1_{\{\tau_2 \le t\}}$$

where

$$\widetilde{V}_t = \frac{1}{G(t,t)}\left(-\int_t^T \delta(u)\partial_1 G(u,t)\,du - \kappa\int_t^T G(u,t)\,du\right)$$

$$V_t^{1|2}(s) = \frac{-1}{\partial_2 G(t,s)}\left(\int_t^T \delta(u) f(u,s)\,du + \kappa\int_t^T \partial_2 G(u,s)\,du\right).$$

The dynamics of the price of the CDS are

$$\begin{aligned} dV_t &= (1 - H_t^1)\Big(\kappa - \delta(t)\Big((1 - H_t^2)\widetilde{\lambda}_t^1 + H_t^2 {\lambda_t}^{1|2}(\tau_2)\Big)\Big)\,dt \\ &\quad - V_{t-} dM_t^1 + (1 - H_t^1)({V_t}^{1|2}(t) - V_{t-})dM_t^2 \end{aligned}$$

The dynamics of the **cumulative price** of the CDS are

$$dV_t^{\mathrm{cum}} = (\delta(t) - V_{t-}^{\mathrm{cum}})dM_t^1 + (1 - H_t^1)({V_t}^{1|2}(t) - V_{t-})dM_t^2$$

Assume now that a CDS written on τ_2 is also traded in the market, and that the interest rate is null. We denote by $V^i, i = 1, 2$ the prices of the two CDSs.

A self financing strategy consisting in ϑ^i shares of CDS's and ϑ^0 shares of savings account has value $X_t = \vartheta^0_t + \vartheta^1_t V^1_t + \vartheta^2_t V^2_t$ and dynamics

$$\begin{aligned} dX_t &= \left(-\vartheta^1_t V^1_{t-} + \vartheta^2_t (1 - H^2_t)(V^{2|1}_t(t) - \widetilde{V}^2_t)\right) dM^1_t \\ &\quad + \left(\vartheta^1_t (1 - H^1_t)(V^{1|2}_t(t) - \widetilde{V}^1_t) - \vartheta^2_t V^2_{t-}\right) dM^2_t \\ &= (X^1_t - X_{t-}) dM^1_t + (X^2_t - X_{t-}) dM^2_t \end{aligned}$$

where we have taken into account that CDSs are paying dividends and $X^1_t = \vartheta^0_t + (1 - H^2_t)\vartheta^2_t V^{2|1}_t(t)$.

In order to hedge $Z = \mathbb{E}(Z) + \int_0^T \pi_t^1 dM_t^1 + \int_0^T \pi_t^2 dM_t^2$, it remains to solve the linear system (with unknown ϑ^i)

$$\begin{aligned}
-\vartheta_t^1 V_{t-}^1 + \vartheta_t^2 (1 - H_t^2)(V_t^{2|1}(t) - \widetilde{V}_t^2) &= \pi_t^1 \\
\vartheta_t^1 (1 - H_t^1)(V_t^{1|2}(t) - \widetilde{V}_t^1) - \vartheta_t^2 V_{t-}^2 &= \pi_t^2
\end{aligned}$$

Ordered Defaults

Let us now assume that $\tau_1 < \tau_2$, a.s. In that case, $G(t,s) = G(t,t)$ for $s \le t$,

$$M_t^1 = H_t^1 + \int_0^{t\wedge\tau_1} \frac{\partial_1 G(s,s)}{G(s,s)} ds = H_t^1 - \int_0^{t\wedge\tau_1} \frac{f_1(s)}{G_1(s)} ds$$

where

$$G_1(s) = \mathbb{Q}(\tau_1 > s) = G(s,s) = \int_s^\infty f_1(u)du\,.$$

The process M^1 is $\mathbb{H}^1$-adapted, hence is an $\mathbb{H}^1$-martingale and it follows that any $\mathbb{H}^1$-martingale is a $\mathbb{H}$ martingale. Furthermore, the intensity of τ_2 vanishes on the set $t < \tau_1$ and

$$M_t^2 = H_t^2 + \int_{t\vee\tau_1}^{t\wedge\tau_2} \frac{f(\tau_1,s)}{\partial_1 G(\tau_1,s)} ds\,.$$

Let V^i be the price of a CDS on τ_i, with spread κ_i and recovery δ_i.

The $\mathbb{H}$-dynamics of V^1 is

$$dV_t^1 = -V_{t-}^1 dM_t^1 + (1 - H_t^1)(\kappa_1 - \delta_1(t)\widetilde{\lambda}_1(t))dt$$

with $\widetilde{\lambda}_1(t) = \frac{f_1(t)}{G_1(t)}$.

The $\mathbb{H}$-dynamics of V^2 is

$$dV_t^2 = -V_{t-}^2 dM_t^2 + (1-H_t^2)\kappa_2 dt - (1-H_t^2)H_t^1 \delta_2(t)\lambda_t^{2|1}(\tau_1)dt + (V_t^{2|1}(t) - V_{t-}^2)dM_t^1 .$$

More than two defaults

In the filtration generated by three default processes,

$$\begin{aligned} V_t^1 &= \widetilde{V}_t^1 \mathbb{1}_{t<\tau_1\wedge\tau_2\wedge\tau_3} + V_t^{1|2}(\tau_2)\mathbb{1}_{\tau_2\le t<\tau_1\wedge\tau_3} + V_t^{1|3}(\tau_3)\mathbb{1}_{\tau_3\le t<\tau_1\wedge\tau_2} \\ &\quad + V_t^{1|23}(\tau_2,\tau_3)\mathbb{1}_{\tau_2\vee\tau_3\le t<\tau_1} \end{aligned}$$

where

$$\begin{aligned} V_t^1 &= \frac{1}{G(t,t,t)}\left(-\int_t^T \delta(u)\partial_1 G(u,t,t)dt - \kappa\int_t^T G(u,t,t)du\right), \\ V_t^{1|2}(x) &= \frac{-1}{\partial_2 G(t,x,t)}\left(\int_t^T \delta(u)\partial_1\partial_2 G(u,x,t)du + \kappa\int_t^T \partial_2 G(u,x,t)du\right) \\ V_t^{1|3}(y) &= \frac{-1}{\partial_3 G(t,t,y)}\left(\int_t^T \delta(u)\partial_1\partial_3 G(u,t,y)du + \kappa\int_t^T \partial_3 G(u,t,y)du\right) \\ V_t^{1|23}(x,y) &= \frac{1}{\partial_2\partial_3 G(t,x,y)}\left(\int_t^T \delta(u)f(u,x,y)du - \kappa\int_t^T \partial_2\partial_3 G(u,x,y)du\right) \end{aligned}$$

and the price of the CDS follows

$$
\begin{aligned}
dV_t &= (1-H_t^1)\kappa dt - (1-H_t^1)\delta(t)(1-H_t^2)(1-H_t^3)\widetilde{\lambda}_t^1 dt \\
&\quad -(1-H_t^1)\delta(t)\left[(1-H_t^2)H_t^3\lambda_t^{1|3}(\tau_3) + (1-H_t^3)H_t^2\lambda_t^{1|2}(\tau_2)\right]dt \\
&\quad -(1-H_t^1)H_t^2H_t^3\delta(t)\lambda_t^{1|23}(\tau_2,\tau_3)dt \\
&\quad -V_{t-}dM_t^1 + (1-H_t^1)\left((1-H_t^3)V_t^{1|2}(t) + H_t^3V_t^{1|32}(\tau_3) - V_{t-}\right)dM_t^2 \\
&\quad +(1-H_t^1)\left((1-H_t^2)V_t^{1|3}(t) + H_t^2V_t^{1|23}(\tau_2) - V_{t-}\right)dM_t^3
\end{aligned}
$$

where the intensities are given by

$$
\begin{aligned}
\widetilde{\lambda}_t^1 &= \frac{-1}{G(t,t,t)}\partial_1 G(t,t,t) \\
\lambda_t^{1|2}(s) &= \frac{-1}{\partial_2 G(t,s,t)}\partial_1\partial_2 G(t,s,t), \quad \lambda_t^{1|3}(s) = \frac{-1}{\partial_3 G(t,t,s)}\partial_1\partial_3 G(t,t,s) \\
\lambda_t^{1|23}(s_2,s_3) &= \frac{f(t,s_2,s_3)}{\partial_2\partial_3 G(t,s_2,s_3)}
\end{aligned}
$$

More generally, the value of the contingent claim $h(\tau_1, \tau_2, \tau_3)$ is

$$\begin{aligned} &\psi_{000}(t)\mathbb{1}_{t<\tau_1\wedge\tau_2\wedge\tau_3} \\ &\quad + \quad \psi_{001}(t,\tau_3)\mathbb{1}_{\tau_3\le t<\tau_1\wedge\tau_2} + \psi_{010}(t,\tau_2)\mathbb{1}_{\tau_2\le t<\tau_1\wedge\tau_3} + \psi_{100}(t,\tau_1)\mathbb{1}_{\tau_1\le t<\tau_2\wedge\tau_2} \\ &\quad + \quad \psi_{011}(t,\tau_2,\tau_3)\mathbb{1}_{\tau_2\vee\tau_3\le t<\tau_1} + \psi_{110}(t,\tau_1,\tau_2)\mathbb{1}_{\tau_1\vee\tau_2\le t<\tau_3} \\ &\quad + \quad \psi_{101}(t,\tau_1,\tau_3)\mathbb{1}_{\tau_1\vee\tau_3\le t<\tau_2} + h(\tau_1,\tau_2,\tau_3)\mathbb{1}_{\tau_1\vee\tau_2\vee\tau_3\le t} \end{aligned}$$

where

$$\begin{aligned} \psi_{000}(t) &= \frac{1}{G(t,t,t)}\int_t^T du \int_t^T dv \int_t^T dw h(u,v,w) f(u,v,w) \\ \psi_{001}(t,s) &= \frac{-1}{\partial_3 G(t,t,s)}\int_t^T du \int_t^T dv h(u,v,s) f(u,v,s) \\ \psi_{011}(t,s_2,s_3) &= \frac{1}{\partial_2\partial_3 G(t,s_2,s_3)}\int_t^T du h(u,s_2,s_3) f(u,s_2,s_3) \end{aligned}$$

and similar expressions for the remaining terms.

Density Hypothesis, Single default

Let $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ be a filtered probability space.

A strictly positive and finite random variable τ (the default time) is given. We assume the following **density hypothesis:**

$$G_t(\theta) := \mathbb{P}(\tau > \theta | \mathcal{F}_t) = \int_\theta^\infty f_t(u)du$$

Let

$$G_t := G_t(t) = \mathbb{P}(\tau > t | \mathcal{F}_t) = \int_t^\infty f_t(u)du$$

In what follows, we assume $G_t > 0$.

The family $f_t(.)$ is called the **conditional density** of τ given $\mathcal{F}_t$.

Note that

- $G_t(\theta) = \mathbb{E}(G_\theta|\mathcal{F}_t)$ for any $\theta \geq t$
- the law of τ is $\mathbb{P}(\tau > \theta) = \int_\theta^\infty f_0(u)du$
- for any t, $\int_0^\infty f_t(u)du = 1$
- For an integrable $\mathcal{F}_T \otimes \sigma(\tau)$ r.v. $Y_T(\tau)$, one has, for $t \leq T$:

$$\mathbb{E}(Y_T(\tau)|\mathcal{F}_t) = \mathbb{E}(\int_0^\infty Y_T(u) f_T(u) du|\mathcal{F}_t)$$

The process

$$\mathbb{1}_{\{\tau \leq t\}} - \int_0^{t \wedge \tau} \lambda_s^{\mathbb{F}} ds$$

is a $\mathbb{G}$-martingale, where

$$\lambda_s^{\mathbb{F}} = \frac{f_s(s)}{G_s}.$$

G admits the multiplicative decomposition

$$G_t = L_t^{\mathbb{F}} e^{-\int_0^t \lambda_s^{\mathbb{F}} ds}$$

where $L^{\mathbb{F}}$ is an $\mathbb{F}$-martingale. Conversely, if $G_t = n_t e^{-\Gamma_t}$ where n is an $\mathbb{F}$-martingale and Γ a continuous increasing process, then $\mathbb{1}_{\{\tau \leq t\}} - \Gamma_{t \wedge \tau}$ is a $\mathbb{G}$-martingale.

Pricing formulae

Terminal payoff $X \in \mathcal{F}_T$:

$$\mathbb{E}(X 1\!\!1_{\{T<\tau} | \mathcal{G}_t) = 1\!\!1_{t<\tau} \frac{1}{G_t} \mathbb{E}(G_T X | \mathcal{F}_t)$$

Recovery

$$\mathbb{E}(Z_\tau 1\!\!1_{\{t<\tau\leq T} | \mathcal{G}_t) = 1\!\!1_{t<\tau} \frac{1}{G_t} \mathbb{E}(-\int_t^T Z_u dG_u | \mathcal{F}_t) = 1\!\!1_{t<\tau} \frac{1}{G_t} \mathbb{E}(\int_t^T Z_u f_u(u) du | \mathcal{F}_t)$$

$G_t(\theta) := \mathbb{P}(\tau > \theta | \mathcal{F}_t) = \int_\theta^\infty f_t(u) du$

Problem: given a martingale n and an increasing process Γ (such that $0 < n_t e^{-\Gamma_t} < 1$), construct τ such that $G_t = n_t e^{-\Gamma_t}$.

If $n = 1$, this is the Cox model.

In a general case, the problem admits various solutions.

Immersion property

Immersion property holds if any $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale. It is equivalent to

$$f_t(s) = f_s(s), \forall t > s$$

Forward intensity

The forward intensity $\lambda_t(\theta)$ of τ is given by $\lambda_t(\theta) = -\partial_\theta \ln G_t(\theta)$

$$G_t(\theta) = \exp(-\int_0^\theta \lambda_t(u)du)$$

We assume that $\mathbb{F}$ is a Brownian filtration. There exists $\Psi(t,\theta)$ such that

1. $G_t(\theta) = G_0(\theta) \exp\left(\int_0^t \Psi(s,\theta)dW_s - \frac{1}{2}\int_0^t \Psi^2(s,\theta)ds\right)$;
2. $\lambda_t(\theta) = \lambda_0(\theta) - \int_0^t \psi(s,\theta)dW_s + \int_0^t \psi(s,\theta)\Psi(s,\theta)ds$;
3. $G_t = \exp\left(-\int_0^t \lambda_s^{\mathbb{F}} ds + \int_0^t \Psi(s,s)dW_s - \frac{1}{2}\int_0^t \Psi^2(s,s)ds\right)$;

where $\Psi(t,\theta) = \int_0^\theta \psi(t,u)du$

$G_t(\theta) := \mathbb{P}(\tau > \theta|\mathcal{F}_t) = \int_\theta^\infty f_t(u)du$

Example: "Cox-like" construction. Here

- λ is a non-negative $\mathbb{F}$-adapted process, $\Lambda_t = \int_0^t \lambda_s ds$
- Θ is a given r.v. independent of $\mathcal{F}_\infty$ with unit exponential law
- V is a $\mathcal{F}_\infty$ -measurable non-negative random variable
- $\tau = \inf\{t : \Lambda_t \geq \Theta V\}$.

For any θ and t,

$$G_t(\theta) = \mathbb{P}(\tau > \theta | \mathcal{F}_t) = \mathbb{P}(\Lambda_\theta < \Theta V | \mathcal{F}_t) = \mathbb{P}\left(\exp - \frac{\Lambda_\theta}{V} \geq e^{-\Theta} \middle| \mathcal{F}_t \right).$$

Let us denote $\exp(-\Lambda_t/V) = 1 - \int_0^t \psi_s ds$, with

$$\psi_s = (\lambda_s/V) \exp - \int_0^s (\lambda_u/V)\, du,$$

and define $\gamma_t(s) = \mathbb{E}\left(\psi_s | \mathcal{F}_t\right)$. Then, $f_t(s) = \gamma_t(s)/\gamma_0(s)$.

Backward construction of the density

Let $\varphi(\cdot, \alpha)$ be a family of densities on $\mathbb{R}^+$, depending of some parameter and $X \in \mathcal{F}_\infty$ a random variable. Then

$$\int_0^\infty \varphi(u, X)du = 1$$

and we can choose

$$f_t(u) = \mathbb{E}(f_\infty(u)|\mathcal{F}_t) = \mathbb{E}(\varphi(u, X)|\mathcal{F}_t)$$

$\mathbb{G}$-martingale characterization

A càdlàg process $Y^{\mathbb{G}}$ is a **$\mathbb{G}$-martingale** if and only if there exist an $\mathbb{F}$-adapted càdlàg process Y and an $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-optional process $Y_t(.)$ such that

$$Y_t^{\mathbb{G}} = Y_t 1_{\{\tau > t\}} + Y_t(\tau) 1_{\{\tau \leq t\}}$$

and that

- $(Y_t G_t + \int_0^t Y_s(s) f_s(s) ds,\ t \geq 0)$ is an $\mathbb{F}$-local martingale;
- $(Y_t(\theta) f_t(\theta), t \geq \theta)$ is an $\mathbb{F}$-martingale.

Girsanov theorem

Let $Z^{\mathbb{G}}_t = z_t 1_{\{\tau > t\}} + z_t(\tau) 1_{\{\tau \le t\}}$ be a positive $\mathbb{G}$-martingale with $Z^{\mathbb{G}}_0 = 1$ and let $Z^{\mathbb{F}}_t = z_t G_t + \int_0^t z_t(u) f_t(u) du$ be its $\mathbb{F}$ projection.

Let $\mathbb{Q}$ be the probability measure defined on $\mathcal{G}_t$ by $d\mathbb{Q} = Z^{\mathbb{G}}_t d\mathbb{P}$.

Then, $f^{\mathbb{Q}}_t(\theta) = f_t(\theta) \frac{z_t(\theta)}{Z^{\mathbb{F}}_t}$, and:

(i) the $\mathbb{Q}$-conditional survival process is defined by $G^{\mathbb{Q}}_t = G_t \dfrac{z_t}{Z^{\mathbb{F}}_t}$

(ii) the $(\mathbb{F}, \mathbb{Q})$-intensity process is $\lambda^{\mathbb{F},\mathbb{Q}}_t = \lambda^{\mathbb{F}}_t \dfrac{z_t(t)}{z_{t-}}$, dt- a.s.;

(iii) $L^{\mathbb{F},\mathbb{Q}}$ is the $(\mathbb{F}, \mathbb{Q})$-local martingale

$$L^{\mathbb{F},\mathbb{Q}}_t = L^{\mathbb{F}}_t \frac{z_t}{Z^{\mathbb{F}}_t} \exp \int_0^t (\lambda^{\mathbb{F},\mathbb{Q}}_s - \lambda^{\mathbb{F}}_s) ds$$

The change of probability measure generated by the two processes

$$z_t = (L_t^{\mathbb{F}})^{-1}, \qquad z_t(\theta) = \frac{f_\theta(\theta)}{f_t(\theta)}$$

provides a model where the immersion property holds true, and where the intensity processes does not change

Several Defaults

We introduce the *conditional joint survival process* $G_t(u,v)$ by setting, for every u, v, t,

$$G_t(u,v) = \mathbb{P}(\tau_1 > u, \tau_2 > v \,|\, \mathcal{F}_t).$$

We assume that the conditional joint density $f_t(u,v) = \partial_{12} G_t(u,v)$ with respect to u and v exists: $G_t(u,v)$ can be represented as follows

$$G_t(u,v) = \int_u^\infty dx \int_v^\infty dy \; f_t(x,y).$$

The process

$$M_t^1 = H_t^1 - \int_0^{t\wedge\tau_1\wedge\tau_2} \widetilde{\lambda}_u^1 \, du - \int_{t\wedge\tau_1\wedge\tau_2}^{t\wedge\tau_1} \lambda^{1|2}(u, \tau_2) \, du,$$

is a $\mathbb{G}$-martingale, where

$$\widetilde{\lambda}_t^i = -\frac{\partial_i G_t(t,t)}{G_t(t,t)}, \quad \lambda^{1|2}(t,s) = -\frac{f_t(t,s)}{\partial_2 G_t(t,s)}$$

Toy model:

$$\widetilde{\lambda}_t^i = -\frac{\partial_i G(t,t)}{G(t,t)}, \quad \lambda^{1|2}(t,s) = -\frac{f(t,s)}{\partial_2 G(t,s)}$$

CDS price

Let

$$V_t = \widetilde{V}_t \mathbb{1}_{t<\tau_1\wedge\tau_2} + \widehat{V}_t(\tau_2)\mathbb{1}_{\tau_2<t<\tau_1}$$

The dynamics of the price of a CDS are

$$\begin{aligned} dV_t &= (1-H_t^1)\left(\kappa - \delta(t)\Big((1-H_t^2)\widetilde{\lambda}_t^1 + H_t^2\lambda_t^{1|2}(\tau_2)\Big)\right)dt \\ & -V_{t-}dM_t^1 + (1-H_t^1)(V_t^{1|2}(t) - V_{t-})dM_t^2 \\ &+ (1-H_t^1)((1-H_t^2)\sigma_t^1 + H_t^2\sigma_t^{1|2}(\tau_2))d\widehat{W}_t \end{aligned}$$

Toy model

$$\begin{aligned} dV_t &= (1-H_t^1)\left(\kappa - \delta(t)\Big((1-H_t^2)\widetilde{\lambda}_t^1 + H_t^2\lambda_t^{1|2}(\tau_2)\Big)\right)dt \\ & -V_{t-}dM_t^1 + (1-H_t^1)(V_t^{1|2}(t) - V_{t-})dM_t^2 \end{aligned}$$

$$\widetilde{V}_t = \frac{1}{G_t(t,t)} \left(- \int_t^T \delta(u) \partial_1 G_t(u,t)\, du - \kappa \int_t^T G_t(u,t)\, du \right).$$

Toy Model

$$\widetilde{V}_t = \frac{1}{G(t,t)} \left(- \int_t^T \delta(u) \partial_1 G(u,t)\, du - \kappa \int_t^T G(u,t)\, du \right).$$

$$V_t^{1|2}(s) = \frac{1}{\partial_2 G_t(t,s)} \left(- \int_t^T \delta(u) f_t(u,s)\, du - \kappa \int_t^T \partial_2 G_t(u,s)\, du \right).$$

Toy model

$$V_t^{1|2}(s) = \frac{1}{\partial_2 G(t,s)} \left(- \int_t^T \delta(u) f(u,s)\, du - \kappa \int_t^T \partial_2 G(u,s)\, du \right).$$

Volatility

From PRT, there exists g such that

$$G_t(u,v) = G_0(u,v) + \int_0^t g_s(u,v)\, dW_s,$$

The volatility is given by

$$\sigma_t^1 \quad = \quad -\frac{1}{G_t(t,t)} \left(\int_t^T \big(\delta(u)\, \partial_1 g_t(u,t) + \kappa g_t(u,t)\big)\, du + g_t(t,t)\widetilde{V}_t \right)$$

$$\sigma_t^{1|2}(t,s) \quad = \quad \frac{-1}{\partial_2 G_t(t,s)} (\int_t^T \delta(u) \partial_{12} g_t(u,s) du + \kappa \int_t^T \partial_2 g_t(u,s)\, du + \widehat{V}_t \partial_2 g_t(t,s))$$

and the $\mathbb{G}$-Brownian motion $\widehat{W}$ satisfies

$$\widehat{W}_{t\wedge\tau_1} = W_{t\wedge\tau_1} - \int_0^{t\wedge\tau_1\wedge\tau_2} \frac{g_s(s,s)}{G_s(s,s)} ds) - \int_{t\wedge\tau_1\wedge\tau_2}^{t\wedge\tau_1} \frac{\partial_2 g_s(s,\tau_2)}{\partial_2 G_s(s,\tau_2)}\, ds$$

THANK YOU FOR YOUR ATTENTION