Simple Dynamic model for pricing and hedging of heterogeneous CDOs

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Outline

- Top-down (aggregate loss) vs. bottom-up models.
- Local Intensity (LI) Model.
- Calibration of the LI model to the market of CDO tranches.
- Multi-name model.
- Calibration of multi-name model, numerical examples.
- Tranche deltas, recovering idiosyncratic risk.
- Generalization to heterogeneous recoveries.
- Stochastic recoveries.

Top-down vs. bottom-up approaches

- ✓ CDO tranche is a derivative of the aggregate portfolio loss, L.
- ✓ Value of a CDO tranche:

$$V_{\text{tr}} = V_{\text{prot}} - V_{fee} = \sum_{0 \le t_i \le T} \alpha_i \, \mathsf{E}[(L_{t_i} - k)^+ - (L_{t_i} - K)^+]$$

k, K - attachment/detachment points

- ✓ Most of the models used now in practice are static:
  - Constructed on top of the underlying portfolio members.
  - No modeling of the dynamics of the portfolio loss.
- ✓ Dynamic aggregate (top-down) loss models.
  - Simplicity.
  - Obscured relation to the underlying portfolio members.
- ✓ Dynamic models built on the top of portfolio members.
  - Too complicated to solve.

Local Intensity model

Intensity of default transitions is a deterministic function of the loss and time:

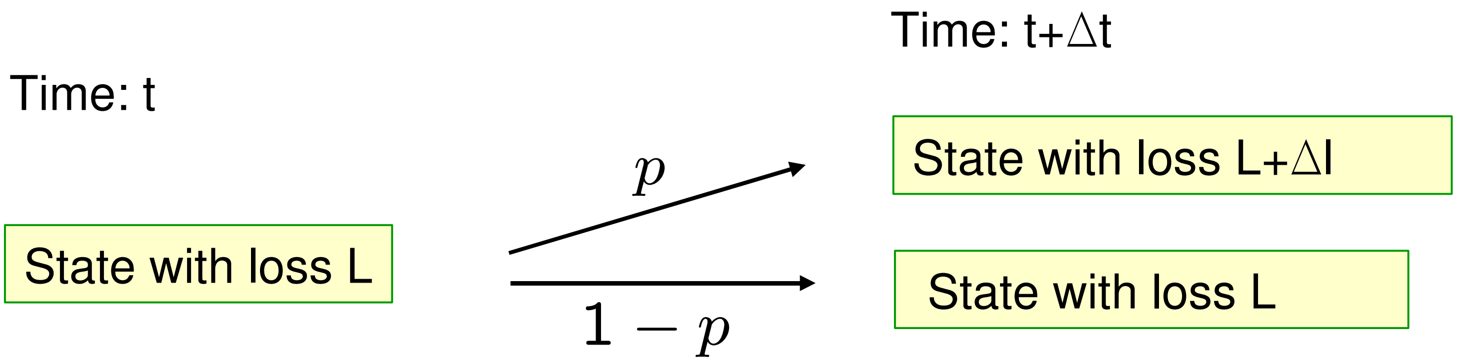
$$\lambda_t = \Lambda(L_t, t)$$

J. Sidenius, V. Piterbarg, and L. Andersen (2005)

P. Schönbucher (2005)

M. van der Voort (2006)

Instantaneous transition probability: $p = \Lambda(L, t)\Delta t$



Binomial tree for LI model

Loss given default:

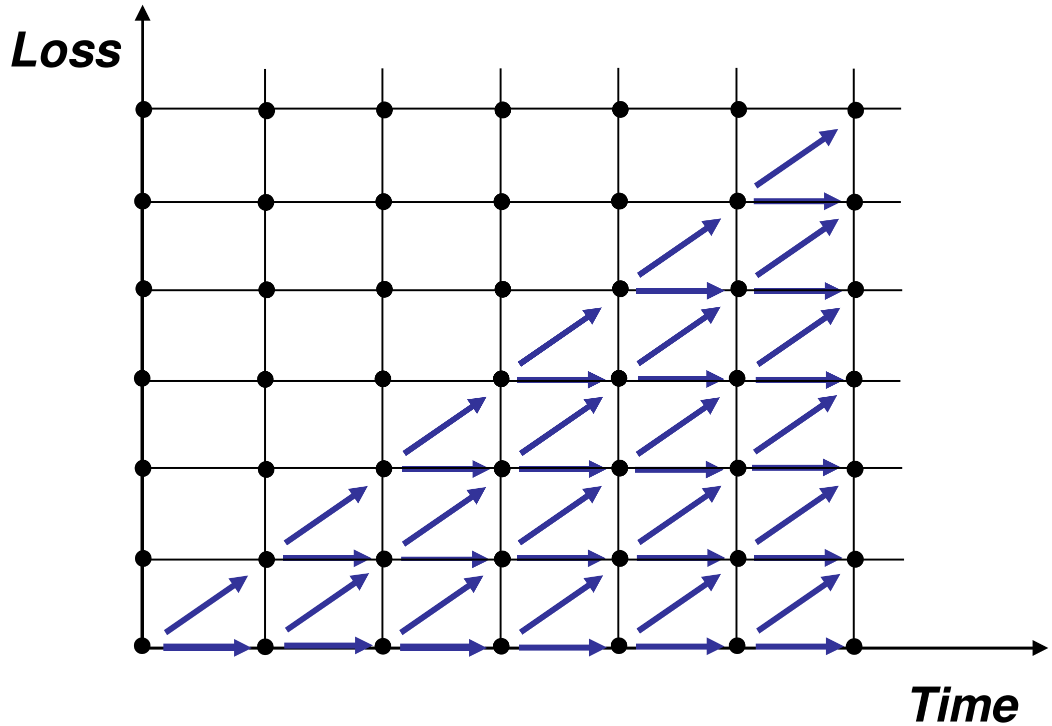
$$h = A(1 - R)$$

A - tranche notional

R - recovery coefficient

Loss values:

$$L = 0,\ h,\ 2h, \ldots, N_0 h$$



Loss PDF:

$$P(L, t_{i+1}) \;=\; [1 - \Delta_i \Lambda(L,t)]\, P(L,t_i) + 1_{L \ge h}\, \Delta_i \Lambda(L-h, t_i)\; P(L-h, t_i)$$

$$\Delta_i = t_{i+1} - t_i$$

Calibration of LI model to CDO tranches

CDO tranches can be replicated via portfolio of the stop-loss options:

$$V_{\text{tr}} = \sum_{0 \le t_i \le T} \alpha_i \, \mathsf{E}[(L_{t_i} - k)^+ - (L_{t_i} - K)^+]$$

→ Tranche values are determined completely by the loss PDF.

Market data on CDO tranches is not enough complete
for unique determination of the local intensity surface.

Calibration procedure

Assume a certain functional form for the local intensity surface and perform a parametric fit.

Arguments	Values

LI parameters → *LI surface* → *Loss PDF* → *Tranche values*

Possible functional forms of LI surface

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1. Number of survived assets can be factored out prior to LI parameterization.

Number of survived assets

Local Intensity per asset

$$\Lambda(N,t) = (N_0 - N)\ \lambda(N,t)$$

2. N - independent part of $\lambda(N,t)$ can be found from index spreads:

Additive:

$$\lambda(N,t) = \alpha_0(t) + \alpha(N,t)$$
$$\alpha(0,t) = 0$$

Multiplicative:

$$\lambda(N,t) = \alpha_0(t)\,\alpha(N,t)$$
$$\alpha(0,t) = 1$$

$$\alpha_0(t) = \frac{1}{N_0 - E[N]}\left(\frac{d}{dt}E[N] - \sum_{N=0}^{N_0}(N_0 - N)\,\alpha(N,t)\,P(N,t)\right)$$

in case of additive;
similarly for multiplicative

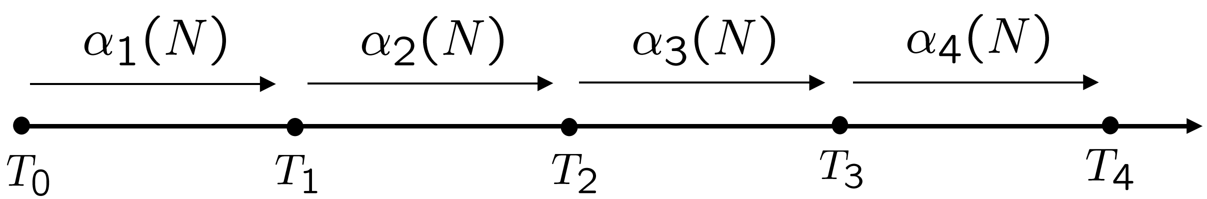
Index quotes → Expected number of defaults, $E[N,t]$ → Function $\alpha_0(t)$

Piecewise constant dependence on time ***(M. Arnsdorf and I. Halperin, 2007)***

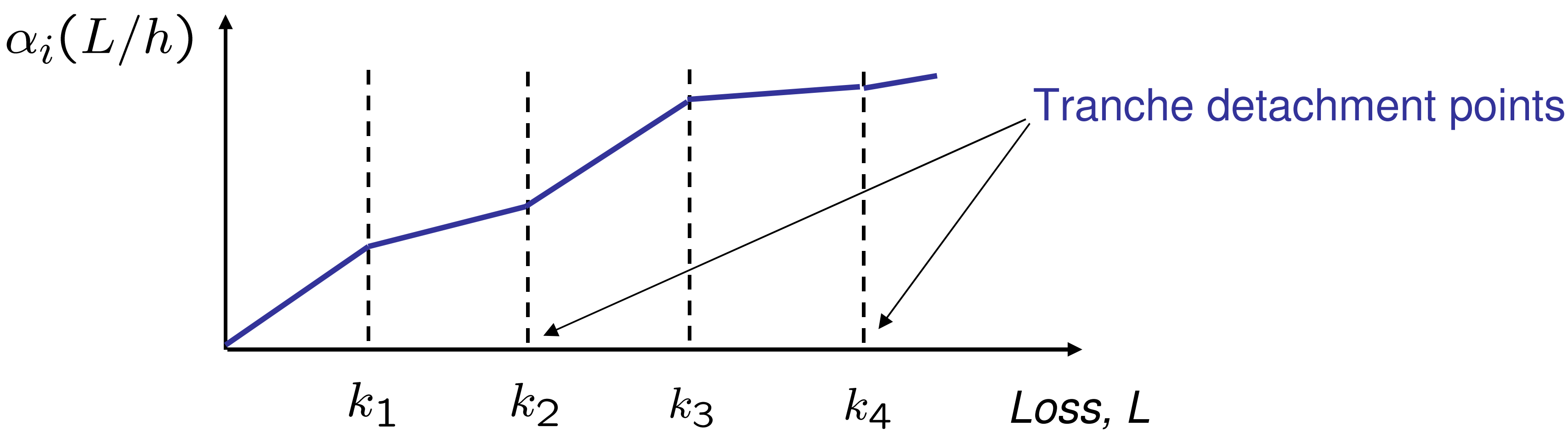
Tranche maturities:

$$T_0, T_1, \ldots T_M \qquad \alpha(N,t) = \begin{cases} \alpha_1(N), & 0 \le t < T_1, \\ \alpha_2(N), & T_1 \le t < T_2, \\ \vdots & \\ \alpha_M(N), & T_{M-1} \le t < T_M. \end{cases}$$

Bootstrap:



Piecewise linear dependence on the loss



Numerical results on calibration to CDO tranches

Fit to the tranches of iTraxx Europe Series 6, March 15, 2007.

	3y		5y		7y		10y	
Tranches	Model	Market	Model	Market	Model	Market	Model	Market
0-3%	0.04218	0.04218	0.1189	0.1188	0.27007	0.27	0.42	0.42
3-6%	3.8	4	54.621	54.5	131.1	131	350.56	350.5
6-9%	1.96	2.4	14.82	14.75	37.54	37.5	94.09	94
9-12%	1.12	0.9	6.24	6.25	17.19	17.25	40.82	41
12-22%	0.42	0.4	3.08	2.5	6.5	6	14.11	13.8

Market data from M. Arnsdorf and I. Halperin (2007).

Exact fit to the index by construction!

Multi-name model

Model is defined via individual default intensities

$$\lambda_k(t) = a_k(t) + b_k(t)Y(N_t, t)$$

Y(N,t) is calibrated to tranches, similarly to the local intensity in LI model

$a_k(t), b_k(t)$ are calibrated to individual CDSs

Possible specifications: *Additive:* $b_k(t) = const$

Multiplicative: $a_k(t) = 0$

Multiplicative form is preferable because it ensures positivity of all intensities.

Consider first the case of homogeneous recoveries: $R_k = R$

Basket loss, L, is related to the number of defaults, N: $L = hN$ h - loss given default

Applicability limits

Suggested model is a special case of a general default contagion model where default intensities are deterministic functions of the basket state

$$\lambda_k = \lambda_k(\mathbf{n}, t)$$

$\mathbf{n}-$ vector of default indicators

- No market risk
- Not a good model for pricing dynamic sensitive instruments (tranche options, etc)
- Suitable for hedging purposes

Alternative approach: stochastic intensities, and no explicit default contagion

$$\lambda_k(t) = X_k(t) + a_k Y(t)$$

$X_k(t), Y(t)-$ idiosyncratic and systemic stochastic variables

Affine case: A. Mortensen (2006), A. Eckner (2007)

Similar model with better calibration capability: S.Inglis, A. Lipton, I. Savescu, A. Seep

Markovian projection onto default contagion model

General intensity based model: default intensities are adaptive stochastic processes

$$\lambda_k = \lambda_k^{(a)}(T)$$

"projected" intensity $\lambda_k(\mathbf{n}, t) = \mathsf{E}\,[\,\lambda_k^a(t)\,|\,\mathbf{n}(t) = \mathbf{n}\,]$

Probability density satisfies forward Kolmogorov equation

$$\frac{d}{dt}\,p(\mathbf{n}, t) = \sum_k \hat{\mathcal{L}}_k \lambda_k(\mathbf{n}, t)\,p(\mathbf{n}, t) - \lambda_k(\mathbf{n}, t)\,p(\mathbf{n}, t)$$

$$\hat{\mathcal{L}}_k\; f(n_1, \ldots, n_k = 1, \ldots, n_{N_0}) = f(n_1, \ldots, n_k = 0, \ldots, n_{N_0}),$$
$$\hat{\mathcal{L}}_k\; f(n_1, \ldots, n_k = 0, \ldots, n_{N_0}) = 0.$$

Proof:

$$p(\mathbf{m}, t+\delta t) = \mathsf{P}[\mathbf{n}(t+\delta t) = \mathbf{m}] = \sum_{\mathbf{n}} \mathsf{P}[\,\mathbf{n}(t+\delta t) = \mathbf{m}\,|\,\mathbf{n}(t) = \mathbf{n}\,]\,\mathsf{P}[\mathbf{n}(t) = \mathbf{n}]$$

Consider transition probability $\mathbf{n} \to \mathbf{m}$ in the limit $\delta t \to 0$

$$m_i = n_i, \quad i \neq k$$
$$m_i = n_i + 1, \quad i = k$$

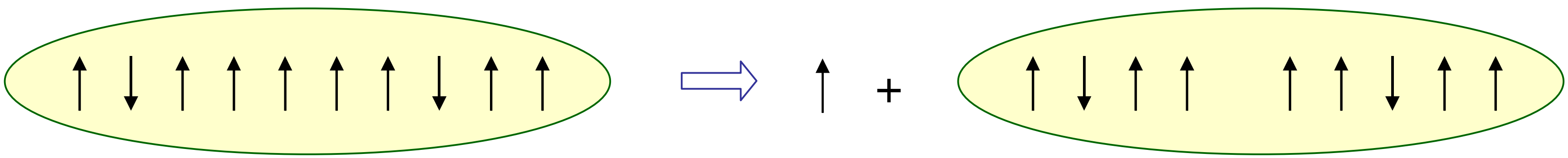
$$\lim_{\delta t \to +0} \frac{1}{\delta t} \mathsf{P}[\,\mathbf{n}(t+\delta t) = \mathbf{m}\,|\,\mathbf{n}(t) = \mathbf{n}\,] = \mathsf{E}\,[\,\lambda_k^a(t)\,|\,\mathbf{n}(t) = \mathbf{n}\,]$$

Solution

Probability that k-th asset has survived and that there are N defaults in the basket

$$P_k(N,t) = P[\,d_k(t) = 0, N_t = N\,]$$

“Extract” k-th asset from the basket



Forward Kolmogorov equation for $P_k(N,t)$

$$\frac{d}{dt}P_k(N,t) = \Lambda_k(N-1,t)P_k(N-1,t) - [\Lambda_k(N,t)+\lambda_k(N,t)]\,P_k(N,t)$$

$\Lambda_k(N,t)$ – intensity of defaults in the “reduced” basket

$$\Lambda_k(N) = \frac{1}{\delta_t}\sum_{p\neq k} \mathsf{E}[\,d_p(t+\delta_t) - d_p(t)\,|\,d_k(t) = 0, N_t = N],\qquad \delta_t \to 0$$

Probability density of defaults in the whole basket

$$P_B(N,t) = \frac{1}{N_0 - N} \sum_{k=1}^{N_0} P_k(N,t)$$

Intensity of defaults in the basket

$$\Lambda_B(N,t) = \frac{\sum_{k=1}^{N_0} \lambda_k(N,t) P_k(N,t)}{P_B(N,t)}$$

Change of the basket probability distribution under idiosyncratic shift of survival probability of k-th asset

$$\delta_k P_B(N,t) = \delta_k [P_k(N,t) - P_k(N-1,t)]$$

Approximate solution

Direct calculation of $\Lambda_k(N,t)$ is computationally expensive because the configuration space is huge: 2^125

Mean field approach (inspired by the condensed matter theory)

$$\Lambda_k(N,t) \approx \Lambda_B(N,t)$$

We will take:

$$\Lambda_k(N,t) = \left(1 - \frac{1}{N_0 - N}\right)\Lambda_B(N,t)$$

Formally becomes exact in the limit $N_0 \to \infty$.

Systematic, controllable, approximation in $1/(N_0 - N)$.

Resulting system of equations for forward propagation

Suppose that $P_k(N,t)$ is known at time t_i

1. Find default number PDF

$$P_B(N,t) = \frac{1}{N_0 - N} \sum_{k=1}^{N_0} P_k(N,t)$$

2. Find basket intensity:

$$\Lambda_B(N,t) = \frac{\sum_{k=1}^{N_0} \lambda_k(N,t) P_k(N,t)}{P_B(N,t)}$$

3. Make use of mean field approximation

$$\Lambda_k(N,t) = \left(1 - \frac{1}{N_0 - N}\right) \Lambda_B(N,t)$$

4. Find $P_k(N,t)$ on the next time step from forward Kolmogorov equation

$$\frac{P_k(N,t_{i+1}) - P_k(N,t_i)}{\Delta_i} = \Lambda_k(N-1,t_i) P_k(N-1,t_i) - [\Lambda_k(N,t_i) + \lambda_k(N,t_i)]\, P_k(N,t_i)$$

Self-consistency check

Default number PDF and basket intensity must be related via

$$\frac{d}{dt}P_B(N,t) = \Lambda_B(N-1,t)P_B(N-1,t) - \Lambda_B(N,t)P_B(N,t)$$

This is indeed the case for the choice

$$\Lambda_k(N,t) = \left(1 - \frac{1}{N_0 - N}\right)\Lambda_B(N,t)$$

General condition:

$$\sum_k \Lambda_k(N,t)P_k(N,t) = (N_0 - N - 1)\sum_{k=1}^{N_0} \lambda_k(N,t)P_k(N,t)$$

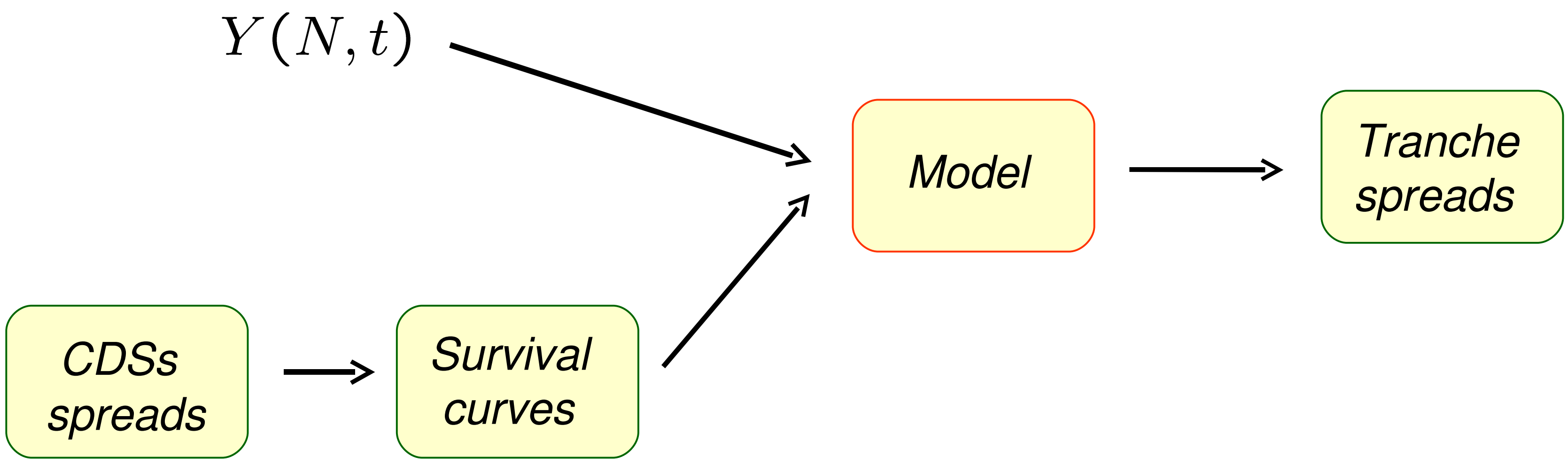
Another possible approximation choice:

$$\Lambda_k(N,t) = \Lambda_B(N,t) - \lambda_k(t)$$

Calibration

Model: $$\lambda_k(t) = b_k(t)\, Y(N_t, t)$$

Given $Y(N,t)$, $b_k(N,t)$ are found automatically



Calibration to tranches is a combination of bootstrap and iterative fitting (exactly as in the case of LI model).

Coefficients $b_k(t)$

Unconditioned intensity implied by the k-th survival curve is known

$$\lambda_k(t_i) = \frac{1}{p_k(t_i)} \frac{p_k(t_i) - p_k(t_{i+1})}{t_{i+1} - t_i}$$

Calibration condition can be stated as

$$\lambda_k(t) = \mathsf{E}[\,\lambda_k(N,t)\,] = b_k(t)\,\mathsf{E}[\,Y(N,t)\,]$$

which can be resolved at each time step

$$b_k(t) = \frac{\lambda_k(t_i)}{E[Y(N,t)]} = \lambda_k(t_i)\,p_k(t_i)\left(\sum_{N=1}^{N_0-1} Y(N,t)\,P_k(N,t_i)\right)^{-1}$$

Numerical results on calibration

Dow Jones CDX.NA.IG.7, Jan 12, 2007

Market:

	5y	7y	10y
0-3%	0.2303	0.42905	0.53021
3-7%	71.8	201.3	477.1
7-10%	13.32	41.93	104.3
10-15%	5.33	17.54	49.1
15-30%	2.64	6.88	15.2

Model,
homogeneous basket:
(CDSs are set to index)

	5y	7y	10y
0-3%	0.2303	0.42901	0.53022
3-7%	71.83	201.43	477.11
7-10%	13.36	41.97	104.4
10-15%	5.35	17.61	49.1
15-30%	2.9	7.46	15.6

Model,
heterogeneous basket:

	5y	7y	10y
0-3%	0.2303	0.42905	0.53022
3-7%	71.8	201.5	477.15
7-10%	13.34	42.0	104.4
10-15%	5.28	17.7	49.14
15-30%	2.82	7.6	15.3

Numerical results on calibration

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ITRAXX 9, Apr 17, 2008

Market:

	5y	7y	10y
0-3%	0.3422	0.3995	0.4512
3-6%	370.3	464.5	590.1
6-9%	245.9	273.6	337.5
9-12%	166.03	190.5	218.2
12-22%	85.2	92.6	106.9
22-100%	31.2	36.0	37.2

Model,
homogeneous basket:
(CDSs are set to index)

	5y	7y	10y
0-3%	0.3420	0.4007	0.45106
3-6%	370.0	462.4	589.8
6-9%	244.7	276.8	337.1
9-12%	167.2	189.8	218.0
12-22%	83.3	93.2	106.4
22-100%	30.1	34.9	36.8

Model,
heterogeneous basket:

	5y	7y	10y
0-3%	0.3420	0.4009	0.4516
3-6%	370.0	461.6	588.9
6-9%	244.3	277.3	337.9
9-12%	167.5	189.6	217.8
12-22%	82.8	94.2	107.0
22-100%	29.8	34.6	36.6

Hedge ratios

Goal: hedge a tranche against market fluctuations of CDS spreads

$$\delta_k = \frac{\text{DV01 of Tranche}}{\text{DV01 of k-th CDS}}$$

DV01 – dollar value change per 1 bp spread shift

Approach in Gaussian Copula model:

$$X_k = \sqrt{1-\beta_k^2}\,\xi_k + \beta_k\,Z$$

Default of k-th asset takes place if

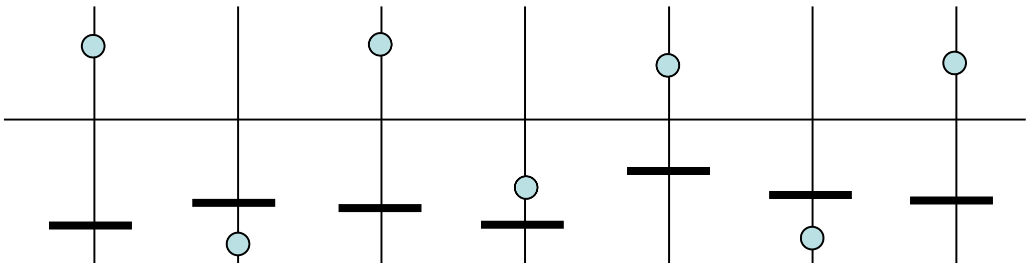
$$X_k < b_k(t)$$

The barrier, b, is calibrated to the asset default probability

$$N(b_k(t)) = d_k(t)$$

Hedge ratios in static Copula models

Mechanical analogy for Gaussian Copula model



Default indicator: $d_1=0$ $d_2=1$ $d_3=0$ $d_4=0$ $d_5=0$ $d_6=1$ $d_7=0$

Recipe for obtaining delta of k-th asset:

1. Shift k-th CDS on 1 bp and obtain perturbed survival probability of k-th asset
2. Find the corresponding perturbed value of the barrier b
3. Calculate change in the value of the tranche
4. Find delta as DV01 of Tranche / DV01 of k-th CDS

This is an idiosyncratic procedure, only k-th assets is perturbed

$$\sum_{d_k=0,1} P(d_1, .., d_k, ..d_{N_0}) = const$$

Hedging via dynamic model

Model: $$\lambda_k(t) = a_k(t) + b_k(t) Y(N_t, t)$$

Default contagion effect:

default intensities jump at a default in the basket:

$$\Delta\ \lambda_k(t) = b_k(t) \left[Y(N_t, t) - Y(N_t + 1, t) \right]$$

Coefficients $a_k(t)$ do not enter !

Recipe for finding “contagious” deltas:

1. Shift k-th CDS on 1 bp and obtain perturbed survival probability of k-th asset
2. Calibrate model to shifted CDSs adjusting coefficients a_k only.
3. Calculate change in the value of the tranche
4. Find delta = DV01 of Tranche / DV01 of k-th CDS

Numerical results for "contagious" deltas

Dow Jones CDX.NA.IG.7, Jan 12, 2007

Market tranche spreads:

	5y	7y	10y
0-3%	0.23026	0.429	0.53021
3-7%	71.8	201.3	477.1
7-10%	13.3	41.9	104.3
10-15%	5.3	17.5	49.1
15-30%	2.6	6.9	15.2

Deltas in case of homogeneous portfolio:

Multi-name dynamic

	5y	7y	10y
0-3 %	0.763	0.403	0.162
3-7 %	0.197	0.379	0.412
7-10 %	0.0259	0.0731	0.143
10-15 %	0.0154	0.0481	0.121
15-30 %	0.023	0.0564	0.111

Gaussian Copula

	5y	7y	10y
0-3 %	0.718	0.427	0.181
3-7 %	0.204	0.360	0.449
7-10 %	0.0348	0.0743	0.141
10-15 %	0.0250	0.0546	0.110
15-30 %	0.033	0.062	0.106

Numerical results for “contagious” deltas

ITRAXX 9, Apr 17, 2008

Market tranche spreads:

	5y	7y	10y
0-3%	0.3422	0.3995	0.4512
3-6%	370.34	464.47	590.1
6-9%	245.87	273.63	337.45
9-12%	166.03	190.5	218.2
12-22%	85.2	92.6	106.9
22-100%	31.2	36.0	37.2

Deltas in case of homogeneous portfolio:

Multi-name dynamic

	5y	7y	10y
0-3 %	0.244	0.186	0.148
3-6 %	0.141	0.156	0.164
6-9 %	0.0931	0.102	0.116
9-12 %	0.0678	0.0703	0.080
12-22 %	0.1287	0.130	0.145
22-100 %	0.3486	0.386	0.393

Gaussian Copula

	5y	7y	10y
0-3 %	0.233	0.180	0.136
3-6 %	0.146	0.152	0.153
6-9 %	0.0989	0.104	0.114
9-12 %	0.0720	0.0768	0.0851
12-22 %	0.135	0.146	0.165
22-100 %	0.343	0.375	0.394

Change of default number PDF under idiosyncratic shift of k-th CDS spread:

$$\delta_k P_B(N,t) = \delta_k[P_k(N,t) - P_k(N-1,t)]$$

Proof:

$$P_k(N,t) = P[\, d_k(t) = 0, N_t = N\,]$$

$$P_k^d(N,t) = P[\, d_k(t) = 1, N_t = N\,]$$

Default number PDF: $P_B(N,t) = P_k(N,t) + P_k^d(N,t)$

Change in default number PDF: $\delta_k P_B(N,t) = \delta_k P_k(N,t) + \delta_k P_k^d(N,t)$

Idiosyncratic constraint:

$$\sum_{d_k=0,1} P(d_1, .., d_k, ..d_{N_0}) = const \quad \Rightarrow \quad \delta_k[P_k(N,t) + P_k^d(N+1,t)] = 0$$

Numerical scheme for calculation of idiosyncratic deltas

1. Shift k-th CDS on 1 bp and obtain perturbed survival probability of k-th asset

2. Rescale k-th intensity

$$\lambda_k(N,t) \to \lambda'_k(N,t) = \kappa(t)\lambda_k(N,t)$$

Scaling function $\kappa(t)$ is found via matching perturbed k-th survival curve

3. Perturbed value of $P_k(N,t)$ is found from:

$$\frac{P'_k(N,t_{i+1}) - P'_k(N,t_i)}{\Delta_i} = \Lambda_k(N-1,t_i)P'_k(N-1,t_i) - [\Lambda_k(N,t_i) + \kappa(t)\lambda_k(N,t_i)]\, P'_k(N,t_i)$$

$\Lambda_k(N,t)$ should not be perturbed !

Numerical results for idiosyncratic deltas

Dow Jones CDX.NA.IG.7, Jan 12, 2007

Market tranche spreads:

	5y	7y	10y
0-3%	0.23026	0.429	0.53021
3-7%	71.8	201.3	477.1
7-10%	13.3	41.9	104.3
10-15%	5.3	17.5	49.1
15-30%	2.6	6.9	15.2

Deltas in case of homogeneous portfolio:

Multi-name dynamic

	5y	7y	10y
0-3 %	0.837	0.563	0.221
3-7 %	0.182	0.411	0.632
7-10 %	0.0221	0.0497	0.113
10-15 %	0.0124	0.0432	0.089
15-30 %	0.0030	0.00397	0.030

Gaussian Copula

	5y	7y	10y
0-3 %	0.718	0.427	0.181
3-7 %	0.204	0.360	0.449
7-10 %	0.0348	0.0743	0.141
10-15 %	0.0250	0.0546	0.110
15-30 %	0.033	0.062	0.106

Numerical results for idiosyncratic deltas

ITRAXX 9, Apr 17, 2008

Market tranche spreads:

	5y	7y	10y
0-3%	0.3422	0.3995	0.4512
3-6%	370.34	464.47	590.1
6-9%	245.87	273.63	337.45
9-12%	166.03	190.5	218.2
12-22%	85.2	92.6	106.9
22-100%	31.2	36.0	37.2

Deltas in case of homogeneous portfolio:

Multi-name dynamic

	5y	7y	10y
0-3 %	0.441	0.337	0.227
3-6 %	0.142	0.211	0.245
6-9 %	0.0848	0.103	0.153
9-12 %	0.131	0.157	0.169
12-22 %	0.085	0.083	0.112
22-100 %	0.152	0.164	0.180

Gaussian Copula

	5y	7y	10y
0-3 %	0.233	0.180	0.136
3-6 %	0.146	0.152	0.153
6-9 %	0.0989	0.104	0.114
9-12 %	0.0720	0.0768	0.0851
12-22 %	0.135	0.146	0.165
22-100 %	0.343	0.375	0.394

Numerical results on deltas: Comparison with Gaussian Copula.

Let's consider an artificially simplified setup:

1. Take Gaussian Copula model with constant correlation.
2. Take a set of survival curves of the form $p_k(t) = \exp(-\lambda_k t)$
3. Generate tranche quotes for attachment points 0-3, 3-7, 7-10, 10-15, 15-30 %
4. Calibrate dynamic model to generated tranche quotes.
5. Find tranche deltas in both models.

Two cases:

I. Homogeneous:
All curves are set to 50 bp spread

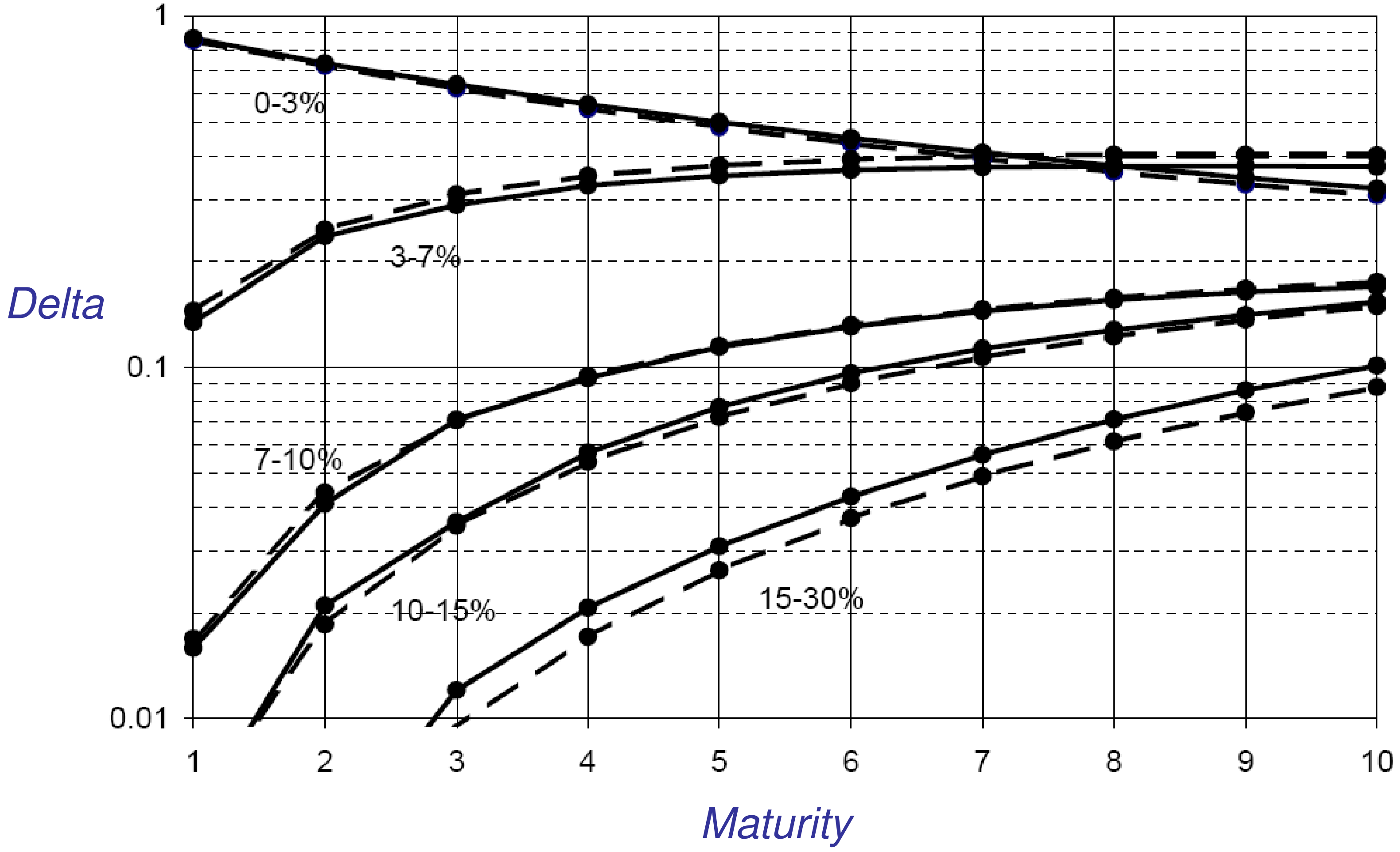
II. Heterogeneous:
Spreads: 6.25, 12.5, 25, 50, 100, 200, 400 bp
All other CDSs are at 50 bp

Homogeneous portfolio

Gaussian Copula correlations: =20%

Solid line: Dynamic

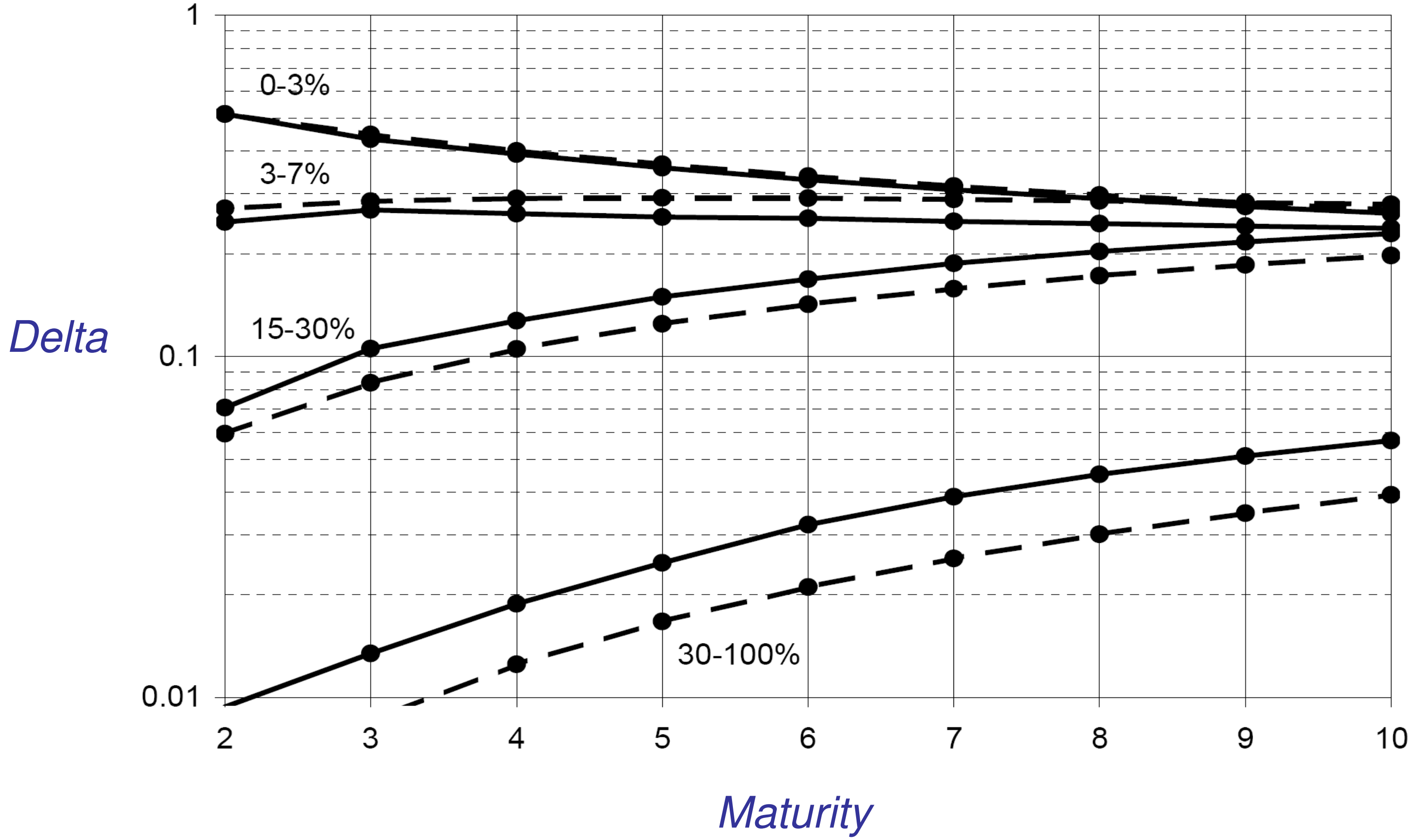
Dashed line: Gaussian Copula



Homogeneous portfolio

Gaussian Copula correlations: =40%

Solid line: Dynamic
Dashed line: Gaussian Copula



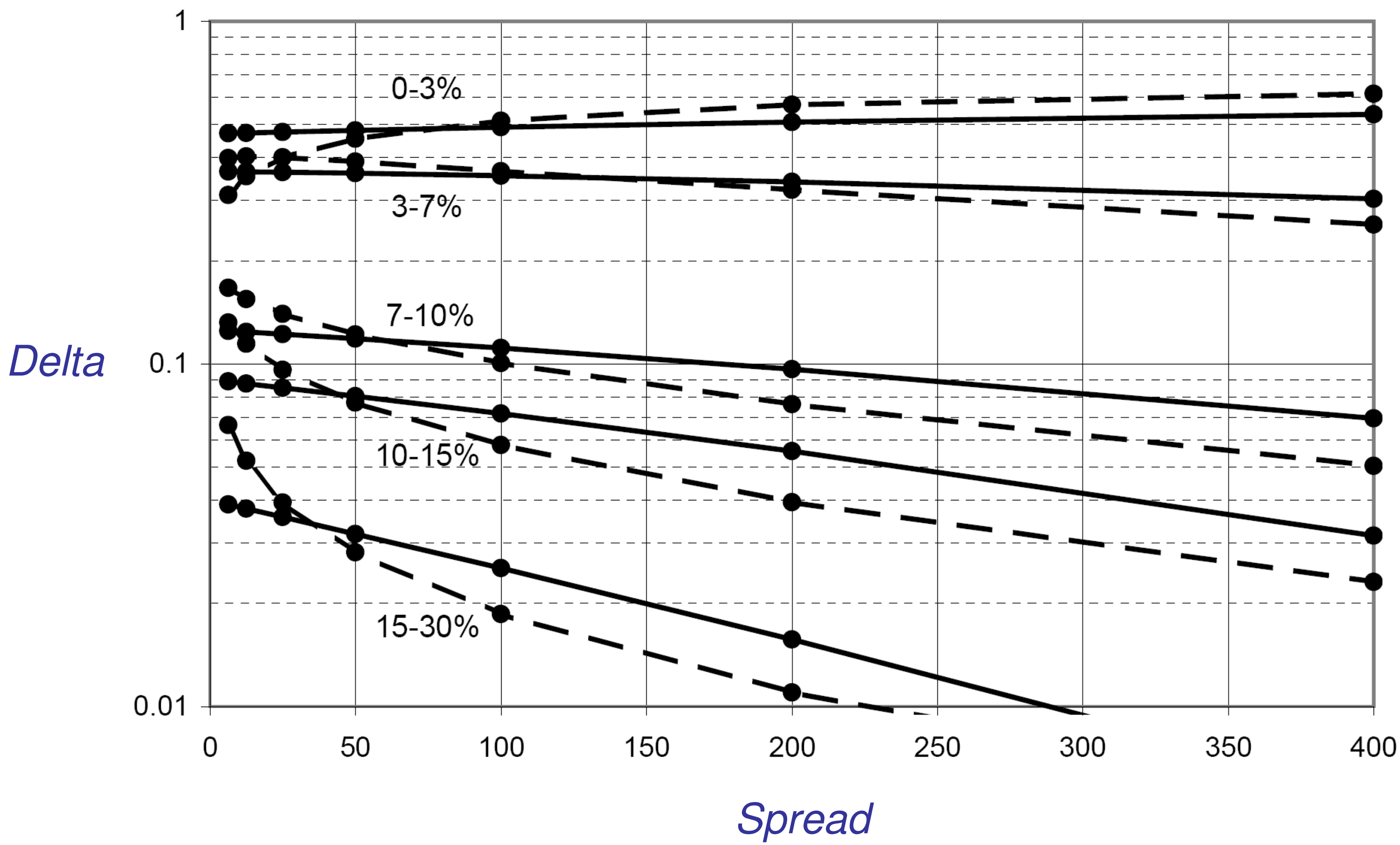
Heterogeneous portfolio

Maturity: 5y

Gaussian Copula correlations: =20%

Solid line: Dynamic

Dashed line: Gaussian Copula



ITRAXX 9, Apr 17, 2008

DC – dynamic “contagious”, DI – dynamic idiosyncratic, CG – Gauss Copula

Maturity:	5y			7y			10y		
Method:	DC	DI	GC	DC	DI	GC	DC	DI	GC
CDS spreads:	29.3			33.4			32.4		
0-3%	0.2404	0.1740	0.1064	0.1613	0.1201	0.0753	0.1010	0.0835	0.0512
3-6%	0.1530	0.0570	0.0540	0.1548	0.0722	0.0595	0.1429	0.0846	0.0617
6-9%	0.1006	0.0292	0.0420	0.1072	0.0366	0.0419	0.1180	0.0529	0.0467
9-12%	0.0712	0.0653	0.0290	0.0749	0.0569	0.0296	0.0886	0.0657	0.0321
12-22%	0.1303	0.0275	-0.0035	0.1449	0.0345	-0.0031	0.1650	0.0425	0.0061
22-100%	0.3468	0.6740	0.7961	0.4144	0.7141	0.8248	0.4729	0.7240	0.8440
CDS spreads:	90.7			97			101.9		
0-3%	0.2571	0.4583	0.2064	0.1823	0.3449	0.1572	0.1347	0.2282	0.1165
3-6%	0.1553	0.1550	0.1333	0.1650	0.2199	0.1387	0.1626	0.2518	0.1417
6-9%	0.0994	0.0771	0.0968	0.1119	0.1094	0.0998	0.1217	0.1564	0.1104
9-12%	0.0679	0.1766	0.0734	0.0757	0.1594	0.0771	0.0864	0.1830	0.0853
12-22%	0.1224	0.0691	0.1483	0.1303	0.0944	0.1595	0.1524	0.1092	0.1786
22-100%	0.3211	0.1031	0.3657	0.3677	0.1306	0.3972	0.3893	0.1601	0.4075
CDS spreads:	200			207.5			213		
0-3%	0.2931	0.5560	0.3200	0.2392	0.4335	0.2577	0.1940	0.2975	0.2033
3-6%	0.1560	0.1767	0.2038	0.1828	0.2542	0.2131	0.1933	0.2947	0.2125
6-9%	0.0929	0.0759	0.1264	0.1117	0.1120	0.1372	0.1278	0.1660	0.1504
9-12%	0.0584	0.1440	0.0878	0.0693	0.1402	0.0973	0.0824	0.1695	0.1083
12-22%	0.1011	0.0520	0.1720	0.0956	0.0732	0.1960	0.1073	0.0889	0.2158
22-100%	0.2878	0.0185	0.0956	0.2893	0.0176	0.1046	0.2852	0.0294	0.1199

Generalization to heterogeneous recoveries

Possible approaches:

1. Make intensities to be functions of portfolio loss: $\lambda_k(t) = b_k(t)\, Y(L_t, t)$
2. Recover loss PDF within the developed scheme.

First approach turns out to be computationally expensive we choose the second

Joint PDF of N and L: $P(L, N, t)$

Probability density of loss: $P(L,t) = \sum_{N=1}^{N_0} P(L,N,t)$

Forward Kolmogorov equation for P(L,N,t):

$$\frac{d}{dt} P(L,N,t) = \sum_l t(l, N-1, t)\, P(L-l, N-1, t) - t(l, N, t)\, P(L, N, t)$$

Transition matrix: $t(l,N,t) = \frac{1}{P_B(N,t)} \sum_{k=1}^{N_0} \lambda_k(N,t)\, P_k(N,t)\, \mathbf{1}_{l=A(1-R_k)}$

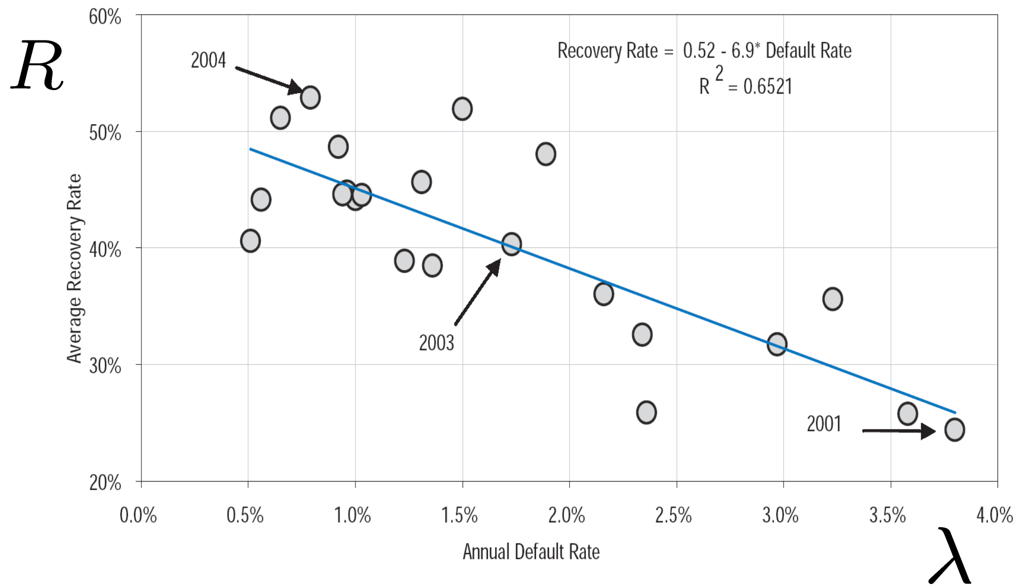
Correlations of recoveries and default intensities

It is an established fact that realized recoveries are correlated with default intensities

Hamilton, et al. "Default and Recovery Rates of Corporate Bond Issuers." Moody's Investor's Services, January 2005.

$$R_k(\lambda) = \max(a - b\lambda, 0)$$

Exhibit 10 – Correlation between Recovery Rates and Annual Default Rates, 1983-2004



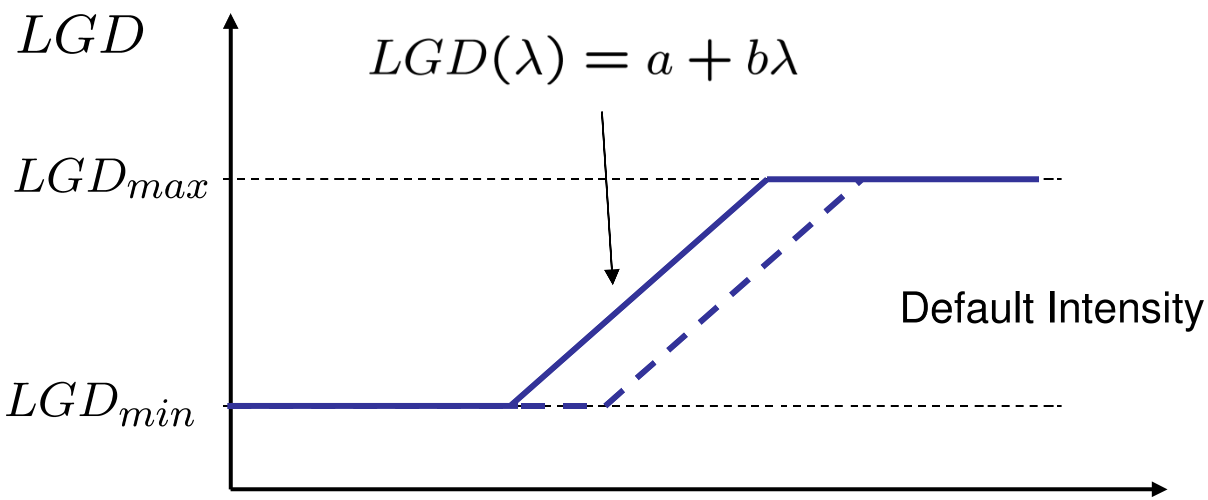
Problem:

Using intensity-dependent recovery will break calibration to single names.

$R_k(t)$ → *CDSs spreads* → *Survival curves* → *Model* → $R_k(\lambda_k, t)$

Dynamically adjusted recoveries

More general form for dependence of loss given default (LGD) on intensity



LGD used in CDS stripping:

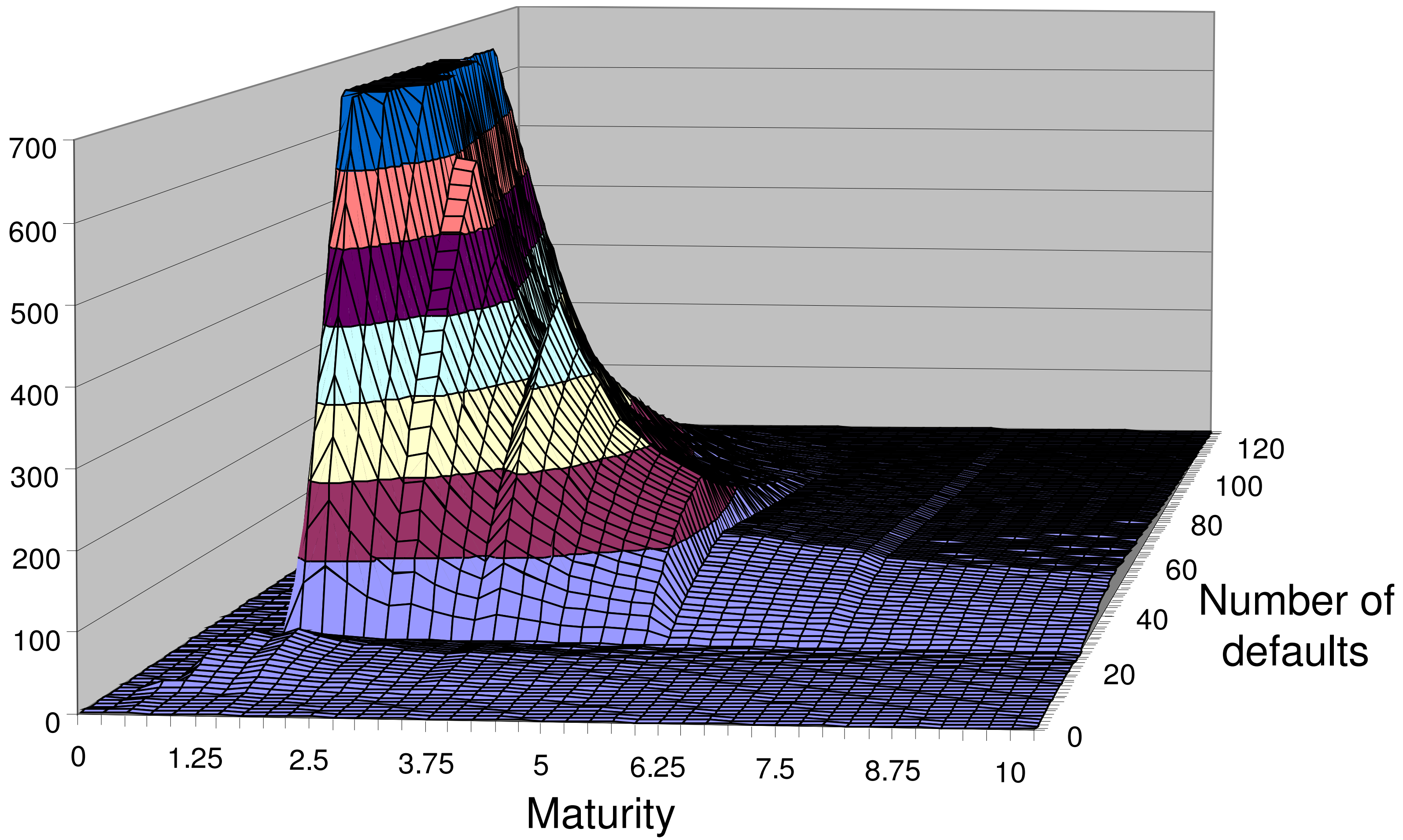
$$LGD_k = \frac{\mathsf{E}[\lambda_t \, LGD_k(\lambda_t, t)]}{\mathsf{E}[\lambda_t]}$$

Recipe:

Take LGD and b as given and adjust coefficient a on each calibration step.

Basket local intensity surface in the case of static recoveries

ITRAXX 9, Apr 17, 2008. Heterogeneous basket



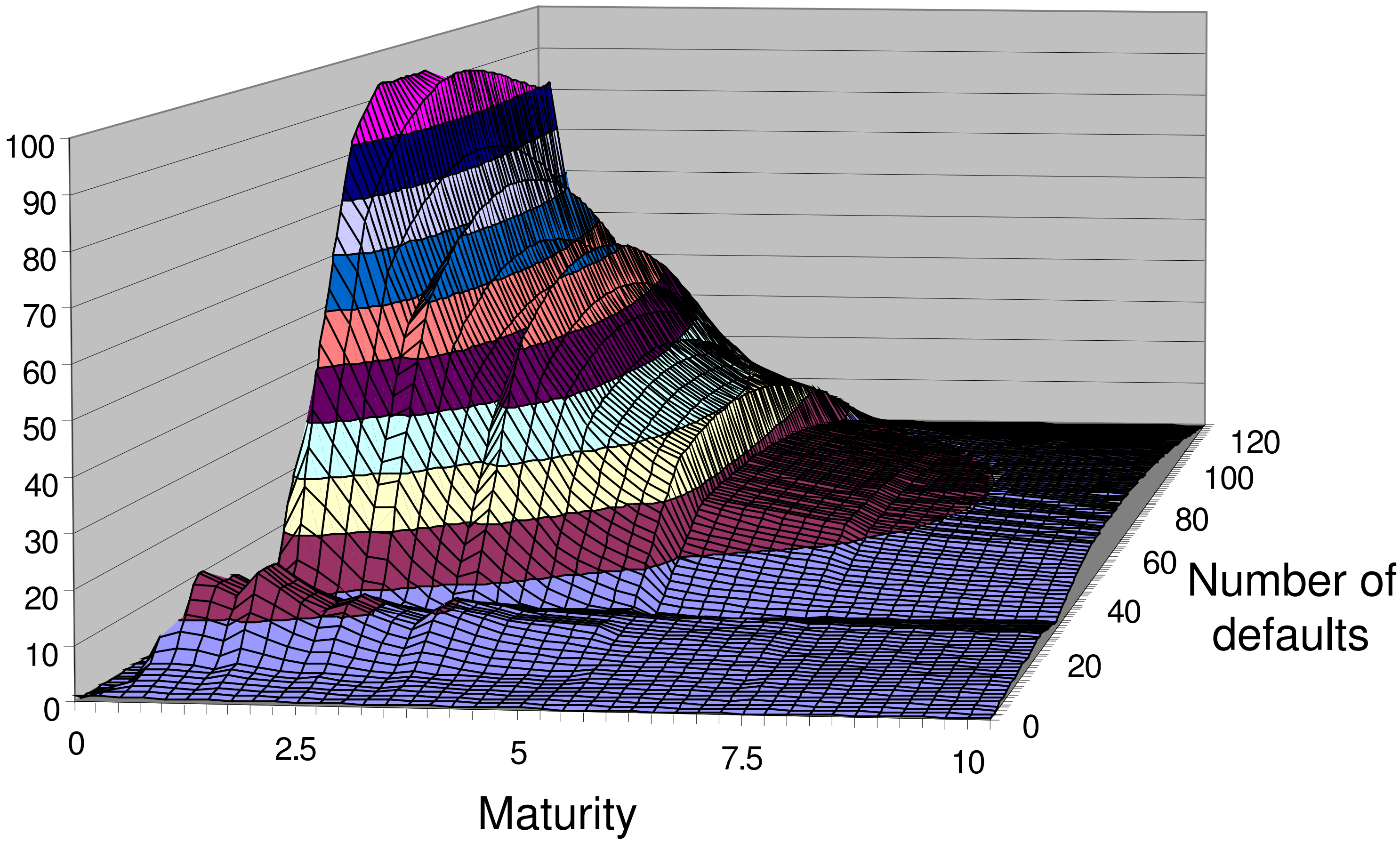
Basket default intensity is unreasonably high at low maturities in the region corresponding to the super senior tranche!

Basket local intensity surface in the case of dynamic recovery

ITRAXX 9, Apr 17, 2008. Heterogeneous basket

LGD vs. intensity slope: *b=2*

$R_{min}=0, R_{max}=1$

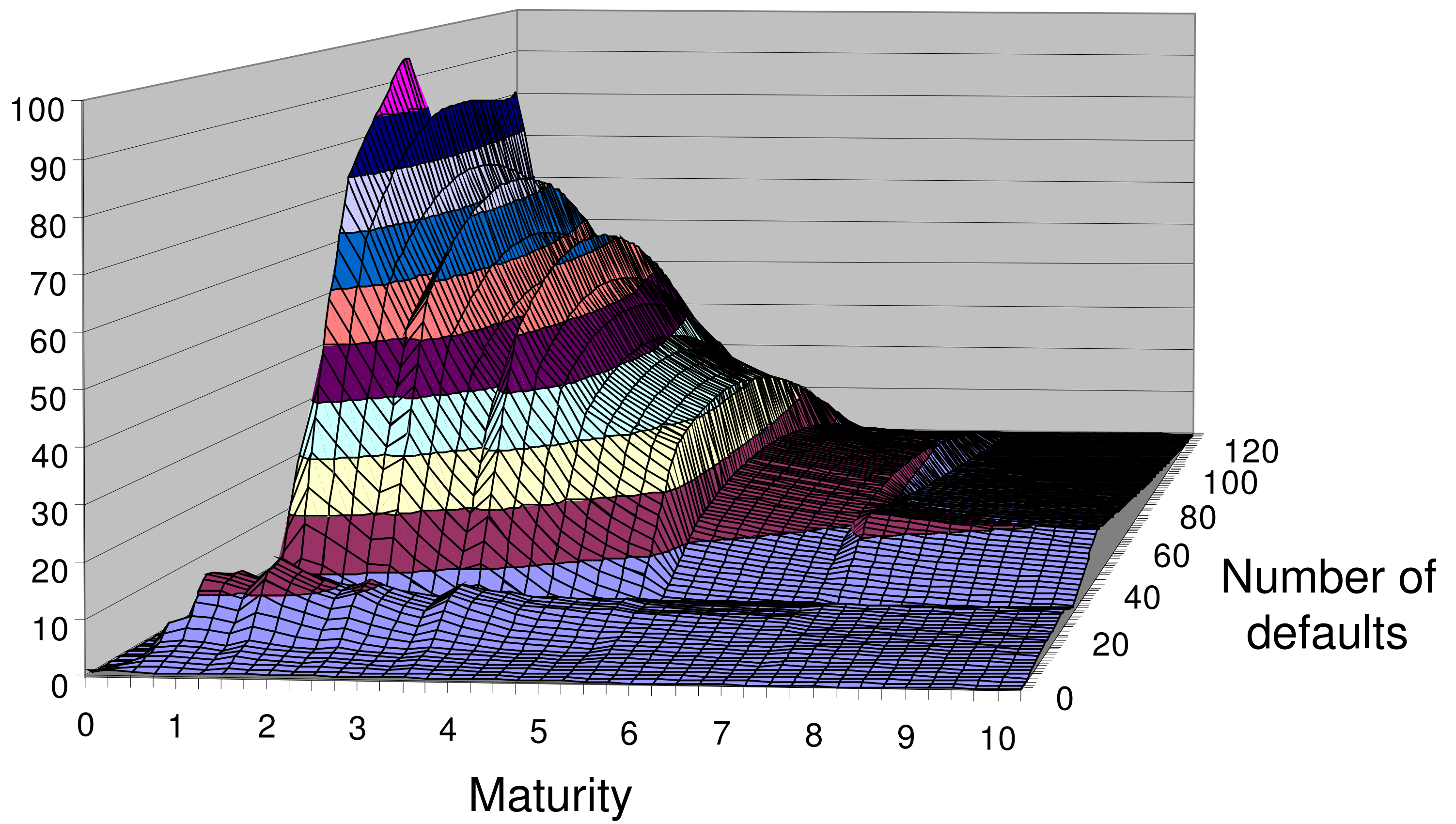


Basket local intensity surface in the case of dynamic recovery

ITRAXX 9, Apr 17, 2008. Heterogeneous basket

LGD vs. intensity slope, *b=4*

$R_{min}=0, R_{max}=1$



Numerical results on calibration

CDX NA IG 12, June 19, 2009

Market:

	5y	7y	10y
0-3%	0.6382	0.68895	0.733
3-7%	0.3418	0.40975	0.468
7-10%	0.1474	0.1959	0.2534
10-15%	0.0444	0.06225	0.0864
15-30%	-0.0069	-0.00825	-0.00835
30-100%	-0.0279	-0.03495	-0.0401

+100bp

Model
LGD vs. intensity slope, *b=2*
$R_{min}=0$, $R_{max}=0.8$

homogeneous basket:
(CDSs are set to index)

	5y	7y	10y
0-3%	0.6382	0.6889	0.7331
3-7%	0.3418	0.40991	0.4677
7-10%	0.1474	0.1959	0.2535
10-15%	0.0443	0.06188	0.08612
15-30%	-0.00706	-0.00788	-0.00824
30-100%	-0.0286	-0.0351	-0.0409

+100bp

Model,
heterogeneous basket:

	5y	7y	10y
0-3%	0.6382	0.689	0.7331
3-7%	0.34177	0.40967	0.4680
7-10%	0.14737	0.1960	0.2536
10-15%	0.04435	0.06213	0.08629
15-30%	-0.00706	-0.0081	-0.00838
30-100%	-0.0286	-0.0351	-0.0409

+100bp

Bespoke portfolio pricing

Model: $$\lambda_k(t) = b_k(t) Y(N_t, t)$$

Procedure:

1. Calibrate to CDSs and tranches of the reference portfolio.
2. Take Y(N,t) and calibrate coefficients $b_k(t)$ to CDSs of bespoke portfolio.

If number of assets in bespoke portfolio differs from that in reference portfolio

$$Y_b(N, t) = Y(N N_0 / N_b, t)$$

N_b – number of assets in the bespoke portfolio.

N_0 – number of assets in the reference portfolio.

Conclusion

- Simple bottom-up dynamic credit model is suggested
- Semi-analytic forward induction scheme is developed
- Simultaneous calibration to individual CDSs and tranches
- Tranche deltas and bespoke portfolio pricing

Future development:

- Improving approximation
- Incorporating stochastic component into asset default intensities