Unified Credit-Equity Modeling

Rafael Mendoza-Arriaga

Based on joint research with: Vadim Linetsky and Peter Carr

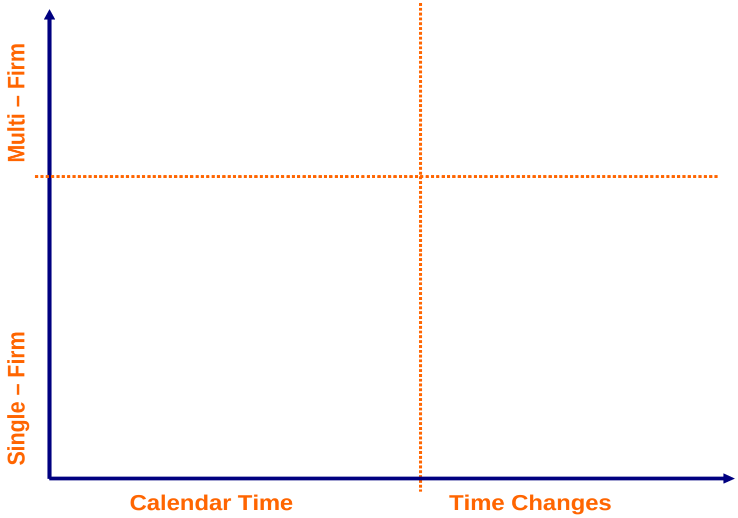
The University of Texas at Austin
McCombs School of Business (IROM)

Recent Advancements in the Theory and Practice of Credit Derivatives

Nice, France

September 28-30, 2009

Research Projects



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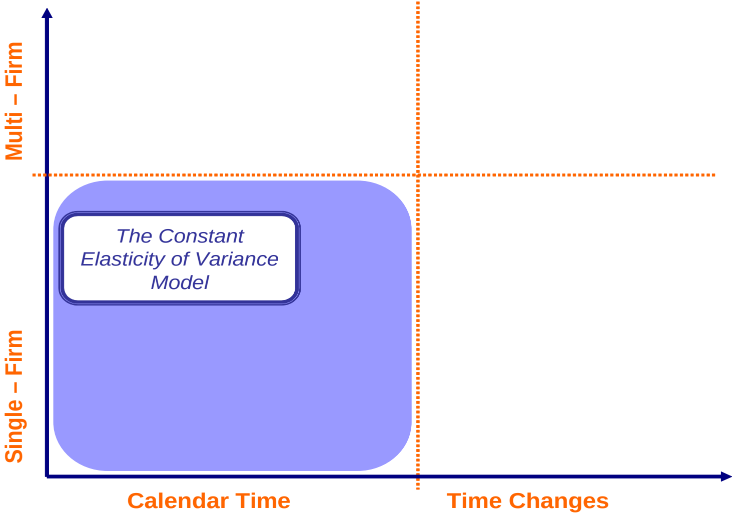

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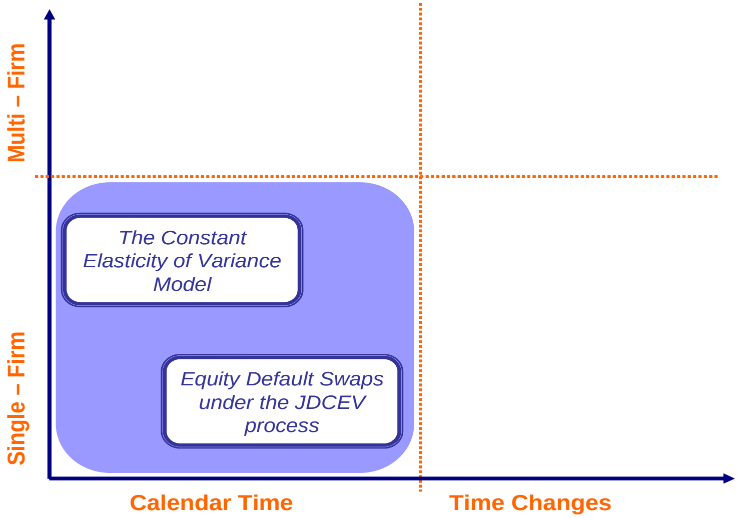

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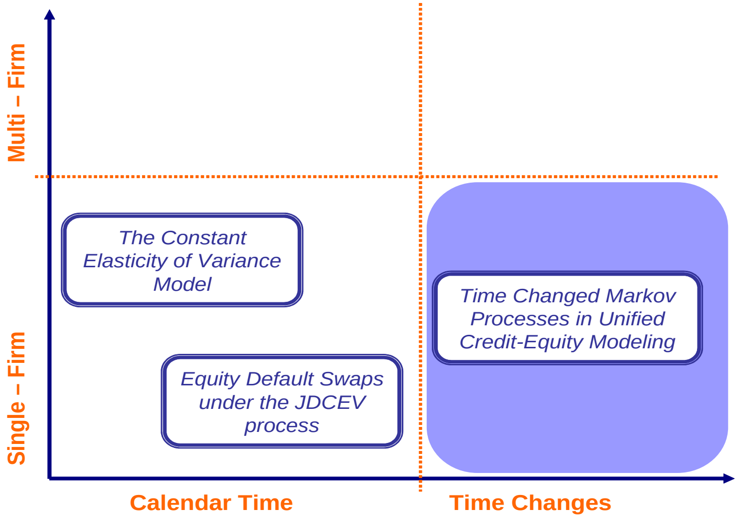

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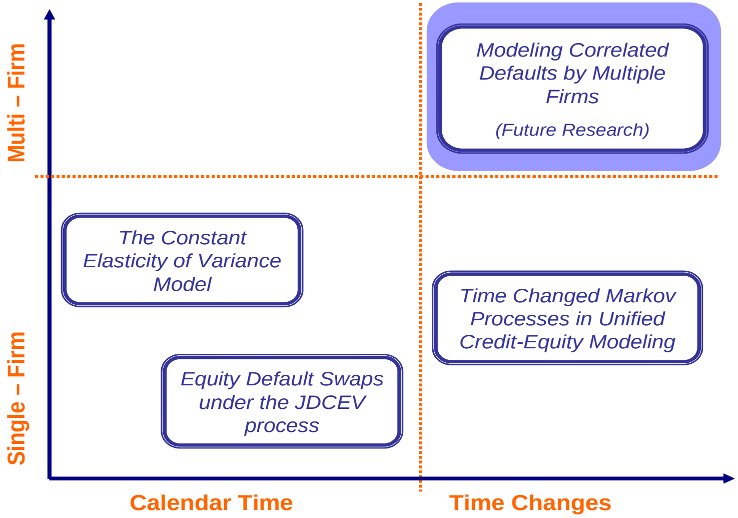
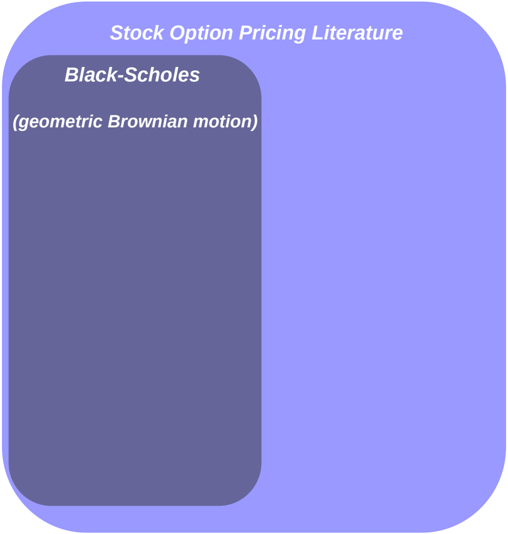


Literature Review



Literature Review

Stock Option Pricing Literature

Black-Scholes

(geometric Brownian motion)

- ***Infinite lifetime process***

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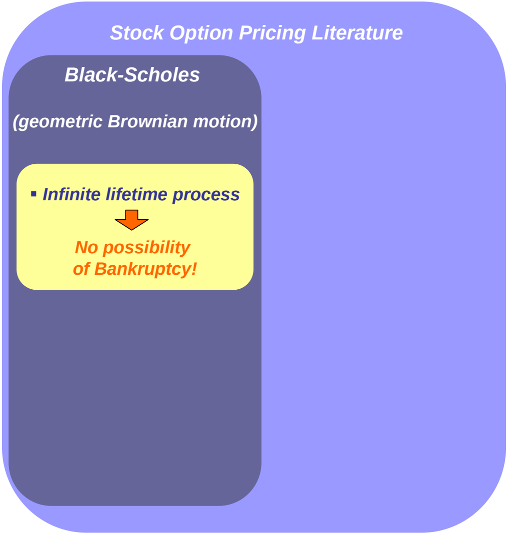

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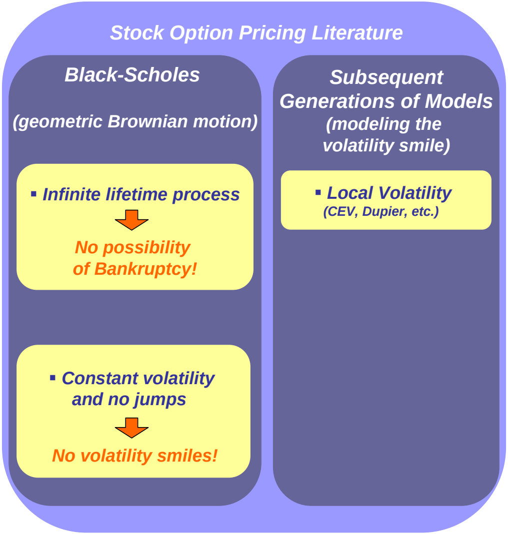

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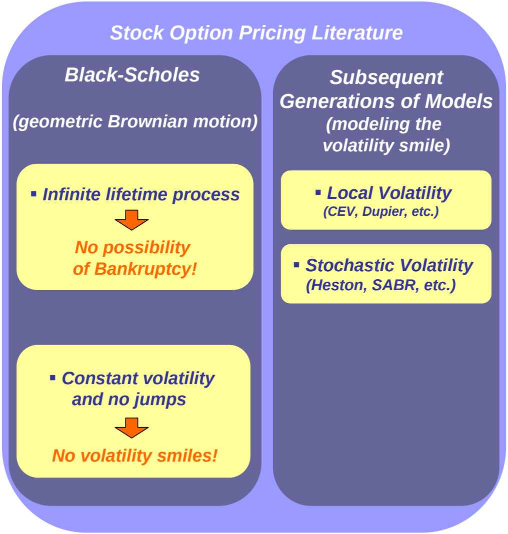

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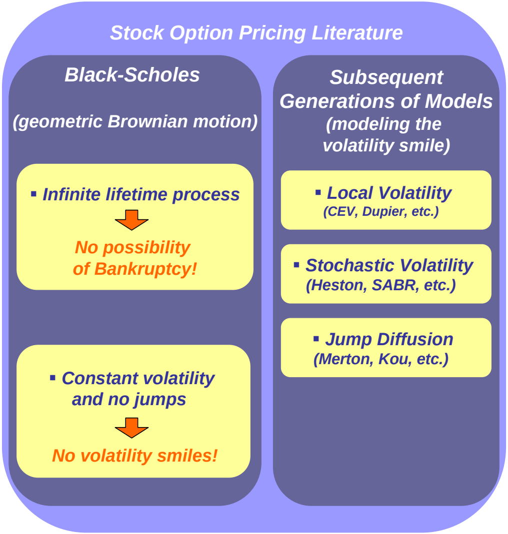

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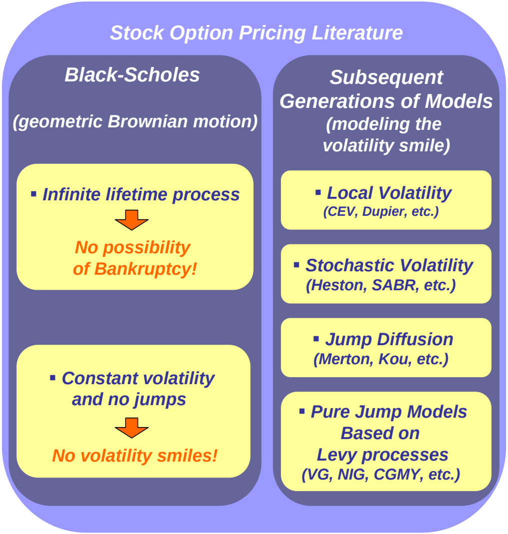

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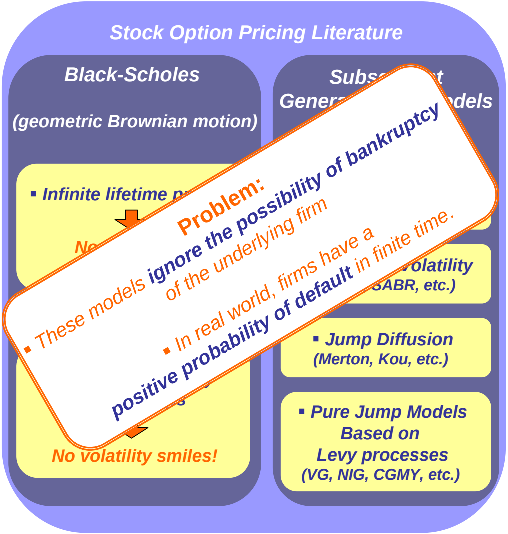

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- *Stochastic Volatility* (Heston, SABR, etc.)
- *Jump Diffusion* (Merton, Kou, etc.)
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Credit Risk Literature

Reduced Form Framework

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Modeling Focus:

Credit Default Events, Credit Spreads, Credit Derivatives, etc

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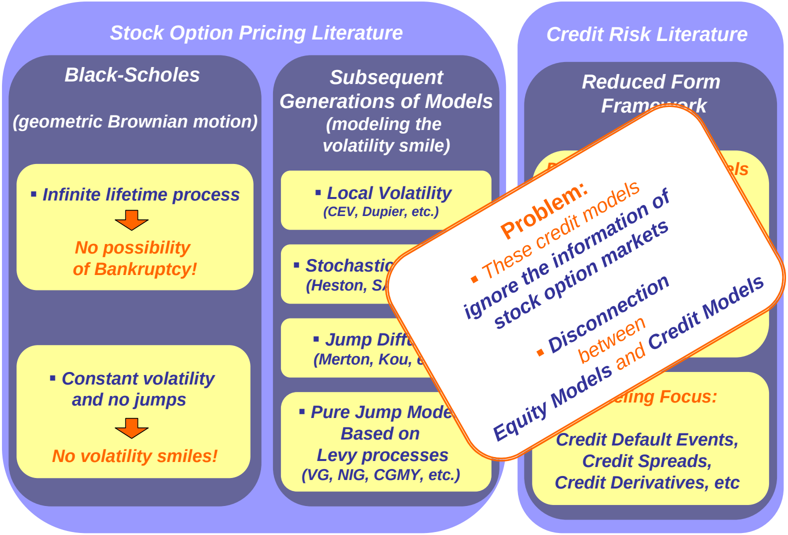

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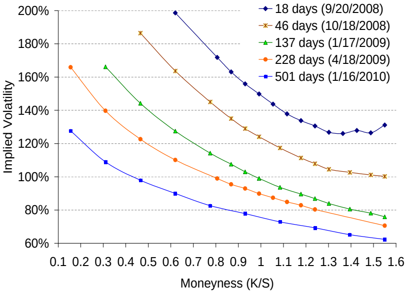
Unified Credit-Equity Modeling

Motivating Example

2 weeks before bankruptcy (9/02/2008) Lehman Brothers (LEH) stock price price was $16.13



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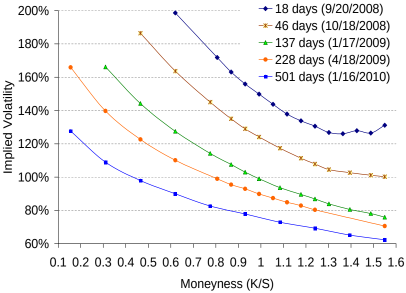

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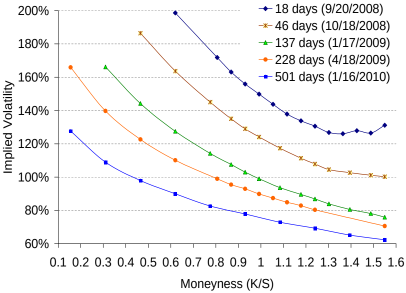

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Open Interest on Put contracts with strike prices $K = 2.5$ USD

- Maturing on 4/18/2009 (228 days) were 1529 contracts
- Maturing on 1/16/2010 (501 days) were 2791 contracts

The Case for the Next Generation of Unified Credit-Equity Models

- Put options provide default protection. Deep out-of-the-money puts are essentially credit derivatives which close the link between equity and credit products.

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- Put options provide default protection. Deep out-of-the-money puts are essentially credit derivatives which close the link between equity and credit products.
- Pricing of equity derivatives should take into account the possibility of bankruptcy of the underlying firm.
- Possibility of default contributes to the implied volatility skew in stock options.

Research Goals

Unified Credit –Equity Framework

Credit and equity derivatives on the same firm should be modeled within a unified framework

- Consistent pricing *across Credit and Equity assets*
- Consistent risk management and hedging

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Our Goal is to develop analytically tractable unified credit-equity models to improve pricing, calibration, and hedging

- Analytical tractability is desirable for fast computation of prices and Greeks, and calibration.

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  - As stock price rises ⇛ arrival rate of large jumps decrease

Our Contributions (cont.)

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- Lévy subordinator time change induces jumps with state-dependent Levy measure, including the possibility of a jump-to-default (stock drops to zero).
- Time integral of an activity rate process induces stochastic volatility in the diffusion dynamics, the Levy measure, and default intensity.

Unifying Credit-Equity Models

The Jump to Default Extended Diffusions (JDED)

Before moving on to use time changes to construct models with jumps and stochastic volatility, we review the Jump-to-Default Extended Diffusion framework (JDED)

Jump to Default Extended Diffusions (JDED)

Defaultable Stock Price

$$S_t = \begin{cases} \tilde{S}_t, & \zeta > t \\ 0, & \zeta \le t \end{cases}$$

(ζ default time)

We assume *absolute priority:* the stock holders do not receive any recovery in the event of default

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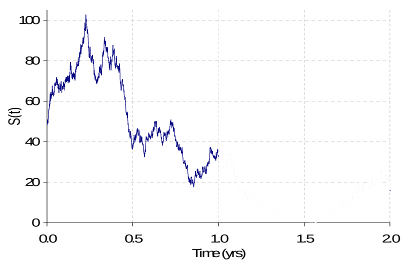

Model the pre-default stock dynamics under an EMM $\mathbb{Q}$ as:

$$d\tilde{S}_t = [\ \mu\ +\ h(\tilde{S}_t)\ ]\tilde{S}_t dt + \sigma(\tilde{S}_t)\,\tilde{S}_t dB_t$$

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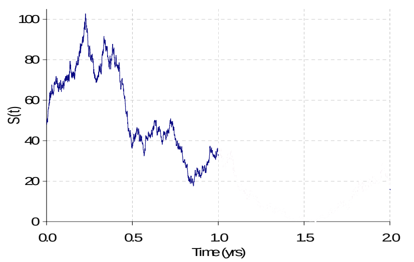

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$\Rightarrow \mu = r - q$. *Drift*: short rate r minus the dividend yield q

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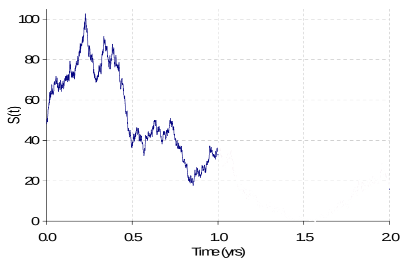

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$\Rightarrow$ $\sigma(S)$. State dependent *volatility*

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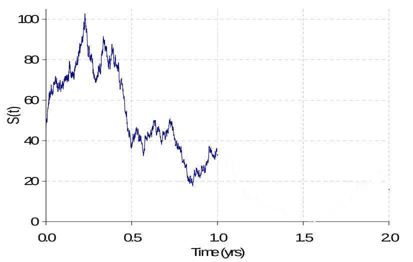

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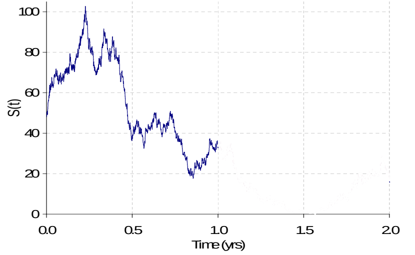

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$\Rightarrow$ $h(S)$. State dependent *default intensity*

- Compensates for the *jump-to-default* and ensures the discounted martingale property

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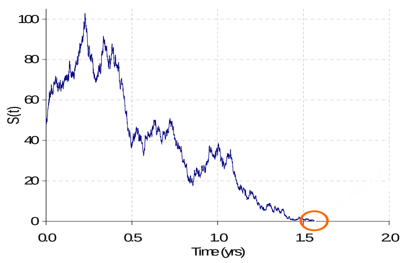

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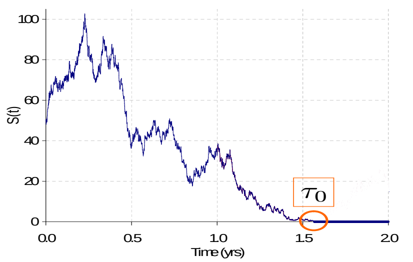

If the diffusion $\tilde{S}_t$ can hit zero:

$\Rightarrow$ Bankruptcy at the first hitting time of zero,

$$\tau_0 = \inf\left\{t : \tilde{S}_t = 0\right\}$$

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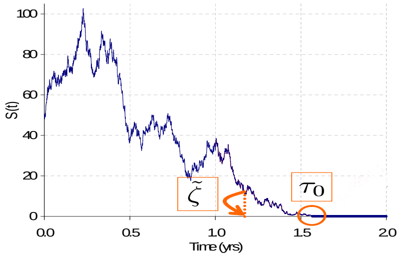

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$$\tilde{\zeta} = \inf\left\{ t \in [0, \tau_0] : \int_0^t h(\tilde{S}_u) \geq e \right\}, \quad e \approx Exp(1)$$

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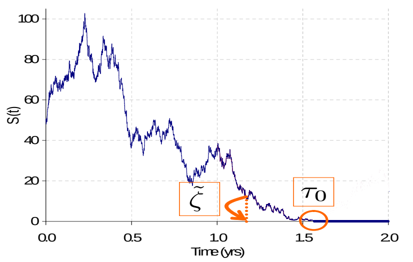

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$\Rightarrow$ At time $\tilde{\zeta}$ the stock price S_t jumps to zero and the firm defaults on its debt

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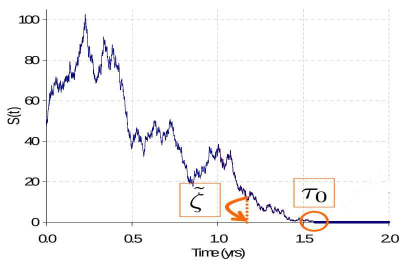

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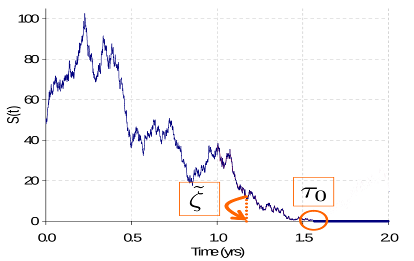

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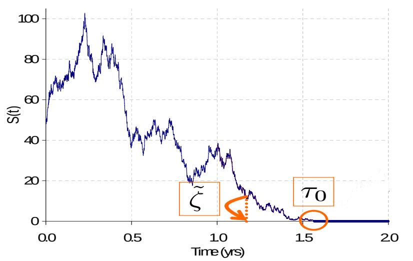

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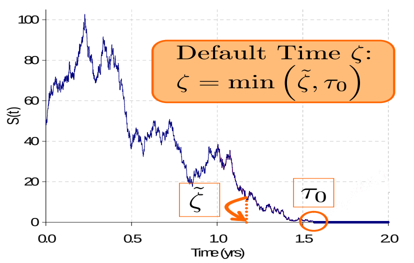

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$$\zeta = \min\left(\tilde{\zeta}, \tau_0\right)$$

Contingent Claims

Risk Neutral Survival Probability *(no default by time T)*

$$Q(S, t; T) = \mathbb{E}\left[\mathbf{1}_{\{\zeta > T\}}\right]$$

$$= \mathbb{E}\left[e^{-\int_t^T h(S_u)du}\ \mathbf{1}_{\{\tau_0 > T\}}\right]$$

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1. No jump-to-default before maturity T,
2. Diffusion does not hit zero before maturity T.

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Defaultable Zero Coupon Bond (at time t)

$$B(S,t;T) = \underbrace{e^{-r(T-t)}Q(S,t;T)}_{\substack{\text{Disc. Dollar if} \\ \text{No Default occurs} \\ \text{prior to maturity}}}$$

Recall: $Q(S,t;T)$ is the risk neutral survival probability

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$$B(S,t;T) = \underbrace{e^{-r(T-t)}Q(S,t;T)}_{\substack{\text{Disc. Dollar if}\\ \text{No Default occurs}\\ \text{prior to maturity}}} + \underbrace{e^{-r(T-t)}R\left[1-Q(S,t;T)\right]}_{\substack{\text{Disc. recovery } R\in[0,1]\\ \text{if Default occurs}\\ \text{before maturity}}}$$

Recall: $Q(S,t;T)$ is the risk neutral survival probability
R is a fraction of a dollar paid at maturity.

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$$B(S,t;T) = \underbrace{e^{-r(T-t)}Q(S,t;T)}_{\substack{\text{Disc. Dollar if} \\ \text{No Default occurs} \\ \text{prior to maturity}}} + \underbrace{e^{-r(T-t)}R\left[1-Q(S,t;T)\right]}_{\substack{\text{Disc. recovery } R\in[0,1] \\ \text{if Default occurs} \\ \text{before maturity}}}$$

Recall: $Q(S,t;T)$ is the risk neutral survival probability
R is a fraction of a dollar paid at maturity.

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Call Option

$$C(S,t;K,T) = e^{-r(T-t)}\mathbb{E}\left[e^{-\int_t^T h(S_u)du}(S_T - K)^+ \mathbf{1}_{\{\tau_0 > T\}}\right]$$

Contingent Claims

Put Payoff (Strike Price $K > 0$)

$$\underbrace{(K - S_T)^+ \mathbf{1}_{\{\zeta > T\}}}_{\substack{\text{Put Payoff} \\ \text{given no default} \\ \text{by time } T}}$$

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$$+ Ke^{-r(T-t)}\left[1 - Q(S, t; T)\right]$$

NOTE. A *default claim* is embedded in the Put Option

Jump-to-Default Extended Constant Elasticity of Variance (JDCEV) Model

The JDCEV process (Carr and Linetsky (2006))

$$dS_t = [\mu + h(S_t)]S_t\,dt + \sigma(S_t)S_t\,dB_t, \quad S_0 = S > 0$$

$\underline{\sigma(S) = aS^{\beta}}$ — CEV Volatility (Power function of S)

$\underline{h(S) = b + c\,\sigma^2(S)}$ — Default Intensity (Affine function of Variance)

- $a > 0$ ⇒ volatility scale parameter (fixing ATM volatility)
- $\beta < 0$ ⇒ volatility elasticity parameter
- $b \geq 0$ ⇒ constant default intensity
- $c \geq 0$ ⇒ sensitivity of the default intensity to variance

For $c = 0$ and $b = 0$ the JDCEV reduces to the standard CEV process

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The model is consistent with:

- leverage effect $\Rrightarrow S \Downarrow \rightarrow \sigma(S) \Uparrow$
- stock volatility–credit spreads *linkage* $\Rrightarrow \sigma(S) \Uparrow \leftrightarrow h(S) \Uparrow$

An Application of Jump to Default Extended Diffusions (JDED)

Equity Default Swaps under the JDCEV Model

Equity Default Swaps (EDS)

- Credit-Type Instrument to bring protection in case of a Credit Event

  Credit Events:

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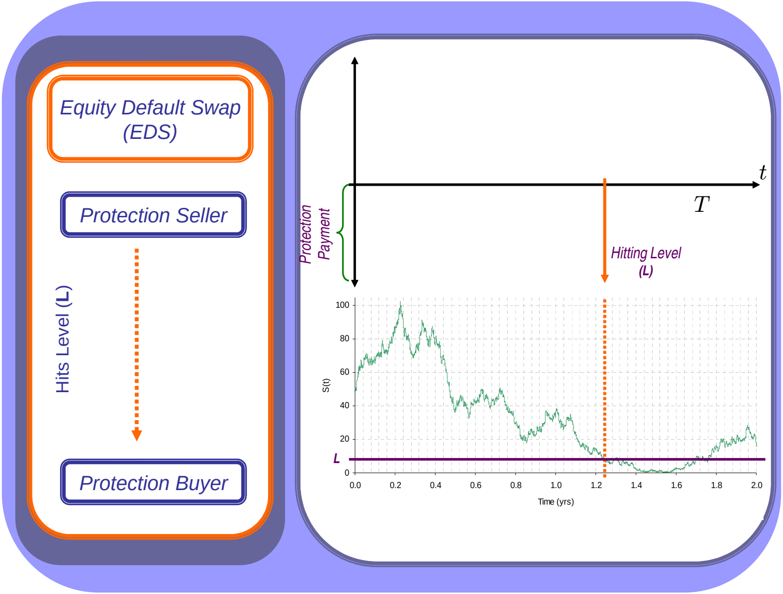
  - Protection Seller pays a recovery amount $(1 - \mathfrak{r})$ for each dollar of principal at credit event time, if the event occurs prior to Maturity.

Equity Default Swaps (EDS)



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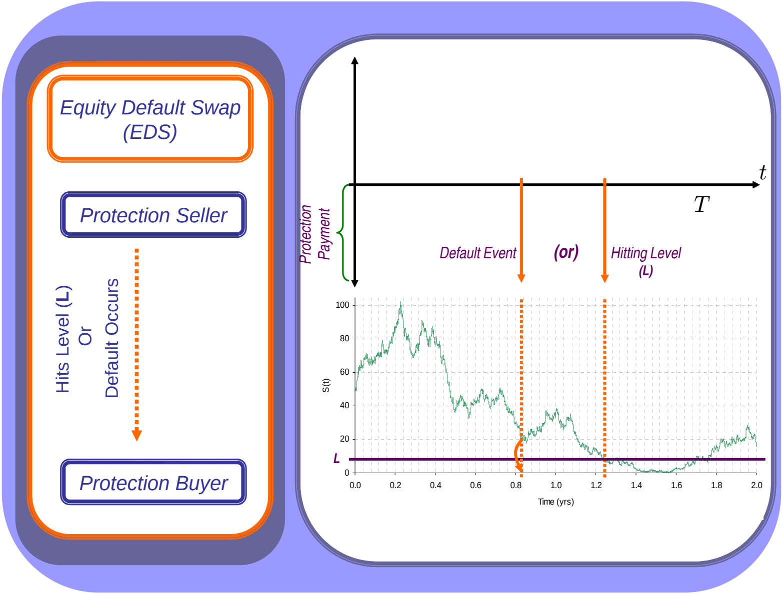

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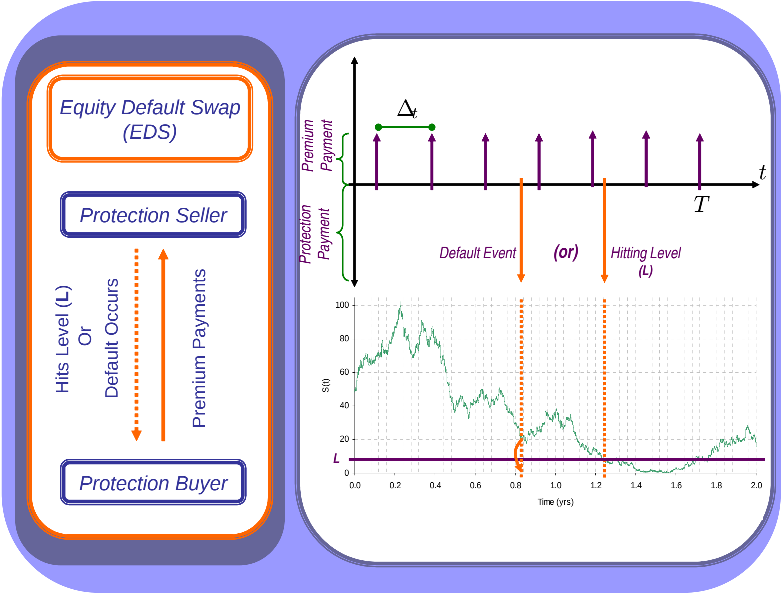

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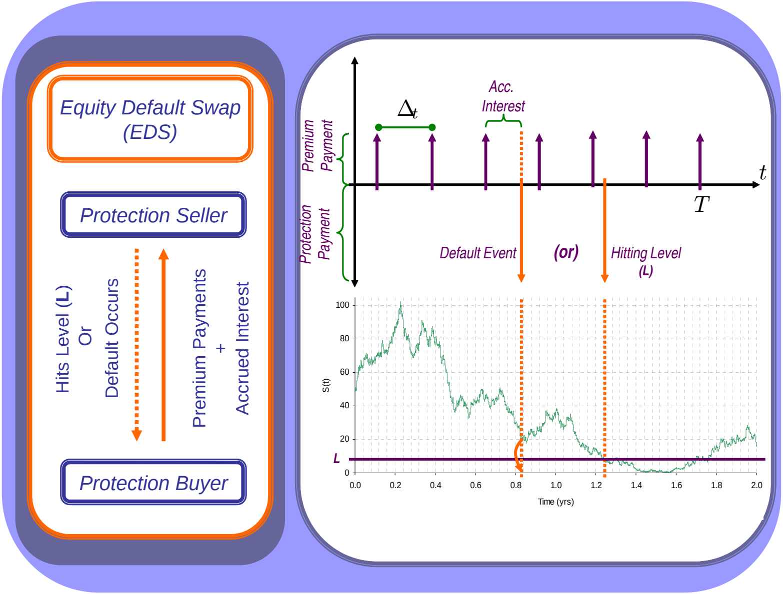

Equity Default Swaps (EDS): Balance Equation

We want to obtain the EDS rate ϱ^* that balances out:

$$\varrho^* = \{\varrho \,|\text{PV(Protection Payment)=PV(Premium Payments + Accrued Interest)}\}$$

Define: Credit Event Time $\Rrightarrow$ $T_L^\Delta = \min\{\textit{first hitting time to L, Default Time}\}$

PV(Protection Payment) $\quad (1-\mathfrak{r}) \cdot \mathbb{E}\left[e^{-r \cdot T_L^\Delta} \mathbf{1}_{\{T_L^\Delta \leq T\}}\right]$

PV(Premium Payments) $\quad \varrho \cdot \Delta_t \cdot \sum_{i=1}^N e^{-r \cdot t_i} \mathbb{E}\left[\mathbf{1}_{\{T_L^\Delta \geq t_i\}}\right]$

PV(Accrued Interests) $\quad \varrho \cdot \mathbb{E}\left[e^{-r \cdot T_L^\Delta}\left(T_L^\Delta - \Delta_t \cdot \underline{\left[\frac{T_L^\Delta}{\Delta_t}\right]}\right) \mathbf{1}_{\{T_L^\Delta \leq T\}}\right]$

Δ_t	Time Interval
$\mathfrak{r}$	Recovery
T	Maturity
T_L^Δ	Credit Event Time
ϱ	EDS rate
r	Risk Free Rate

Details

Equity Default Swaps (EDS)

Advantages of EDS over CDS

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  (as in the case of firms with high yield debt)
- EDS closes the gap between equity and credit instruments since it is structurally similar to the credit default swap.

Time-Changing the Jump to Default Extended Diffusions (JDED)

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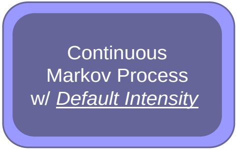
- Our contribution is to *construct far-reaching extensions* by introducing *jumps and stochastic volatility* by means of time-changes

Time-Changing the Jump to Default Extended Diffusions (JDED)

"Time Changes of Markov Processes in Credit-Equity Modeling"

General Panorama



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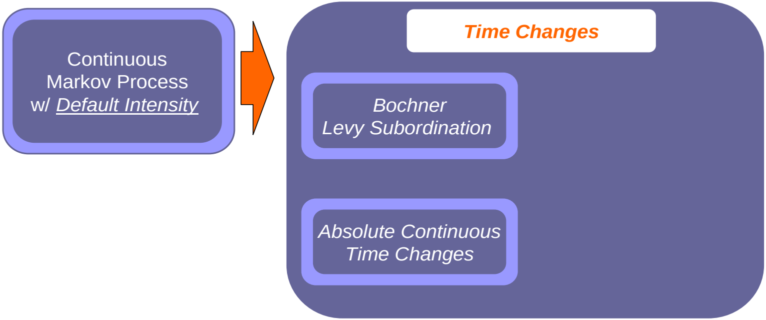

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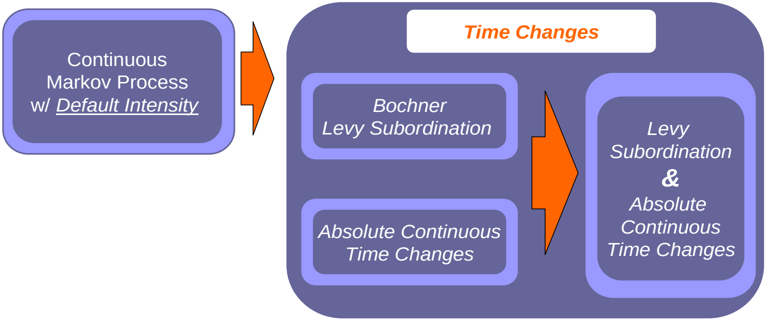

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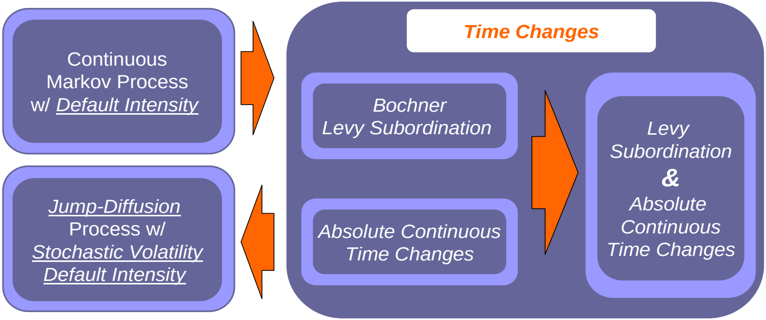

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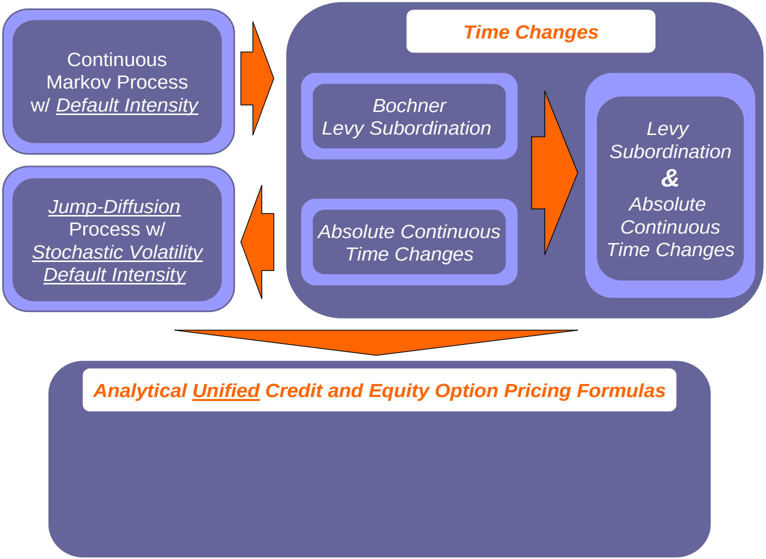

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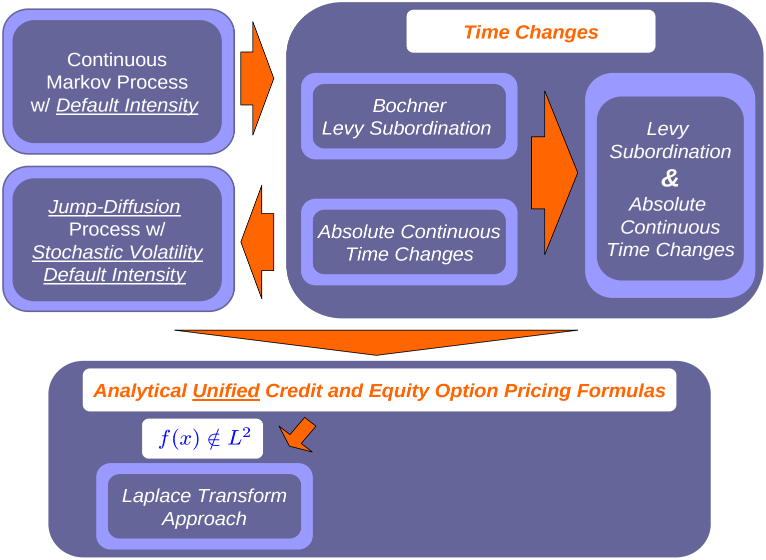

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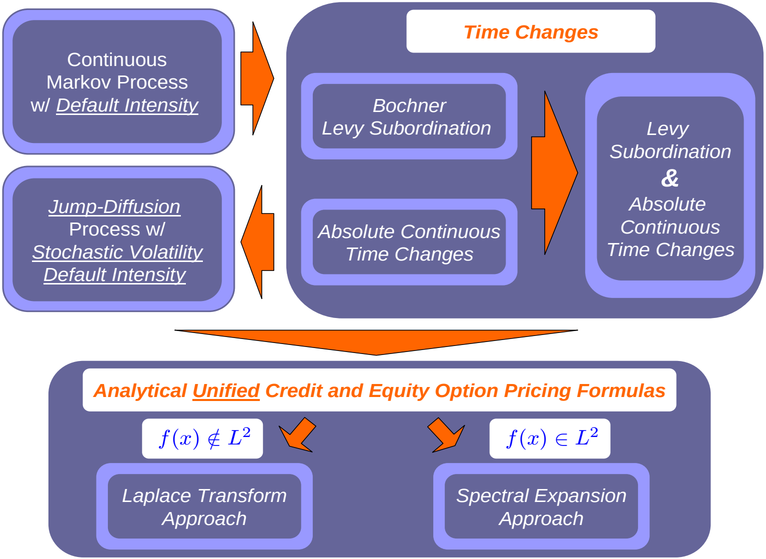

Time-Changed Process $Y_t = X_{T_t}$

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$$Y_t = X_{T_t}$$

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Random Clock $\{T_t, t \geq 0\}$

Non-decreasing RCLL process starting at $T_0 = 0$ and $\mathbb{E}[T_t] < \infty$.

- We are interested in T.C. with analytically tractable Laplace Transform (LT):

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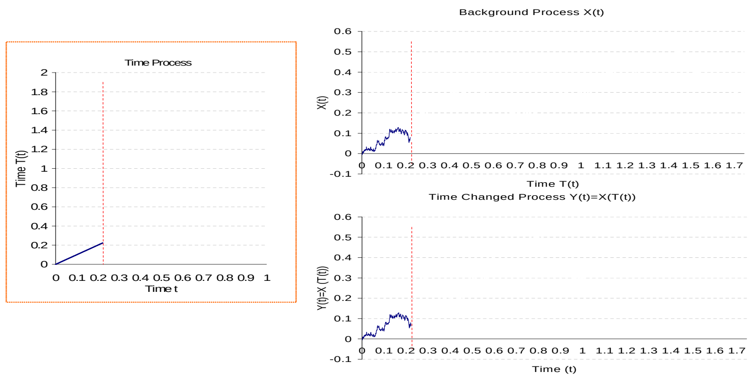
3. Composite Time Changes $\Rrightarrow$ induce jumps & stochastic volatility

Illustration of Lévy Subordinators

$Y = X_{T_t}$ where $X_t = B_t$ and $T_t = t +$ *Compound Poisson Process with Exponential Jumps*



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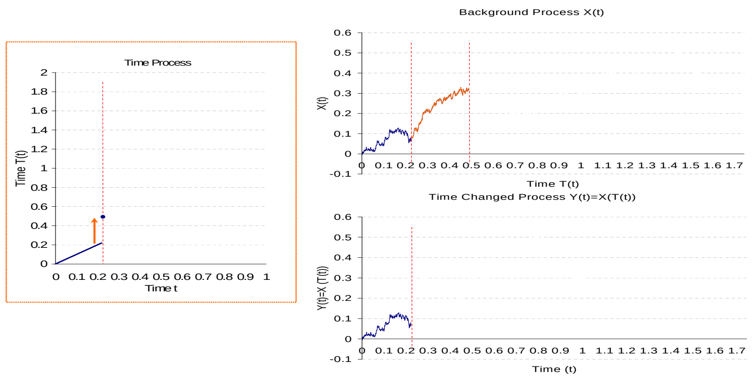

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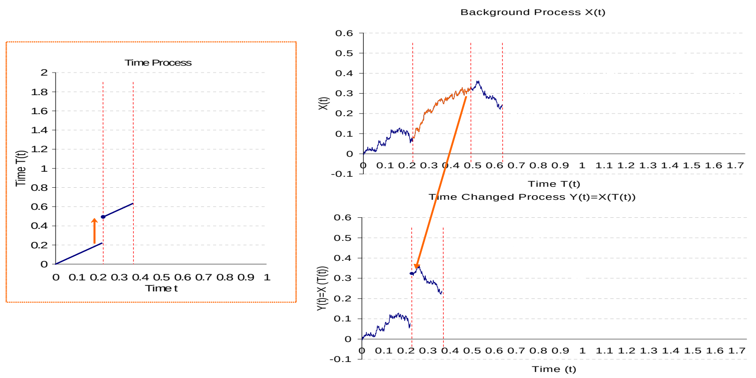

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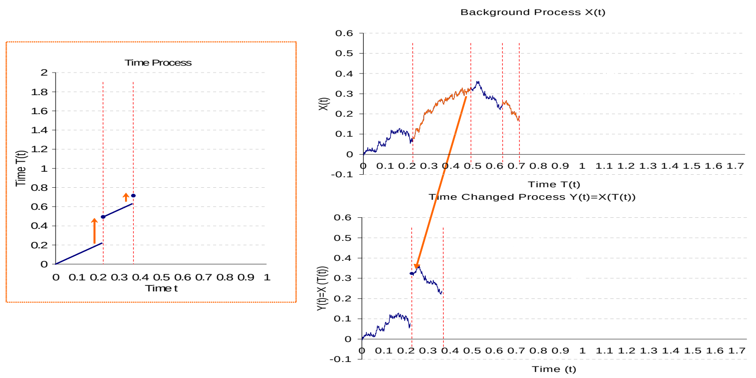

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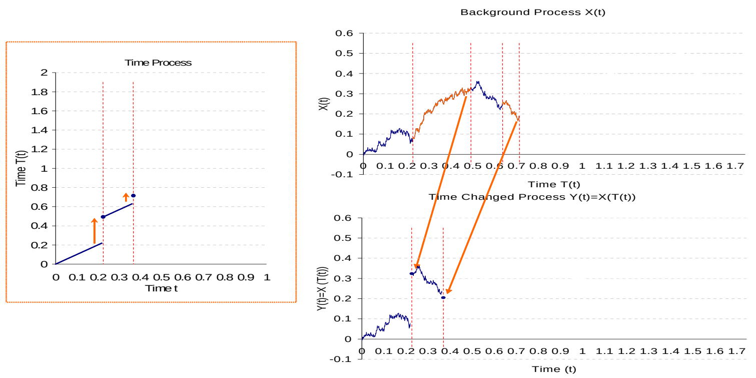

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Examples of Lévy Subordinators

Three Parameter Lévy measure:

$$\nu(ds) = Cs^{-Y-1}e^{-\eta s}ds$$

where $C > 0, \ \eta > 0, \ Y < 1$ ▸ Details

- C changes the time scale of the process (simultaneously modifies the intensity of jumps of all sizes)
- Y controls the small size jumps
- η defines the decay rate of big jumps

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Lévy-Khintchine formula

$$\mathcal{L}(t,\lambda) = e^{-\phi(\lambda)t}$$

where
$$\phi(\lambda) = \begin{cases} \gamma\lambda - C\Gamma(-Y)[(\lambda+\eta)^Y - \eta^Y], & Y \neq 0 \\ \gamma\lambda + C\ln(1+\lambda/\eta), & Y = 0 \end{cases}$$

Absolutely Continuous Time Changes

Absolutely Continuous Time Changes (A.C)

An A.C. Time change is the time integral of some positive function $V(z)$ of a Markov process $\{Z_t, t \geq 0\}$,

$$T_t = \int_0^t V(Z_u)du$$

We are interested in cases with Laplace Transform in closed form:

$$\mathcal{L}_z(t, \lambda) = \mathbb{E}_z \left[e^{-\lambda \int_0^t V(Z_u)du} \right]$$

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Example: The Cox-Ingersoll-Ross (CIR) process:

$$dV_t = \kappa(\theta - V_t)dt + \sigma_V \sqrt{V_t}dW_t$$

with $V_0 = v > 0$, rate of mean reversion $\kappa > 0$, long-run level $\theta > 0$, and volatility $\sigma_V > 0$.

Absolutely Continuous Time Changes

The Laplace Transform of the Integrated CIR process:

$$\mathcal{L}_v(t,\lambda) = \mathbb{E}_v\left[e^{-\lambda\int_0^t V_u du}\right] = A(t,\lambda)e^{-B(t,\lambda)v}$$

$$A = \left(\frac{2\varpi e^{(\varpi+\kappa)t/2}}{(\varpi+\kappa)(e^{\varpi t}-1)+2\varpi}\right)^{\frac{2\kappa\theta}{\sigma_V^2}},\ B = \frac{2\lambda(e^{\varpi t}-1)}{(\varpi+\kappa)(e^{\varpi t}-1)+2\varpi},\ \varpi = \sqrt{2\sigma_V^2\lambda+\kappa^2}$$

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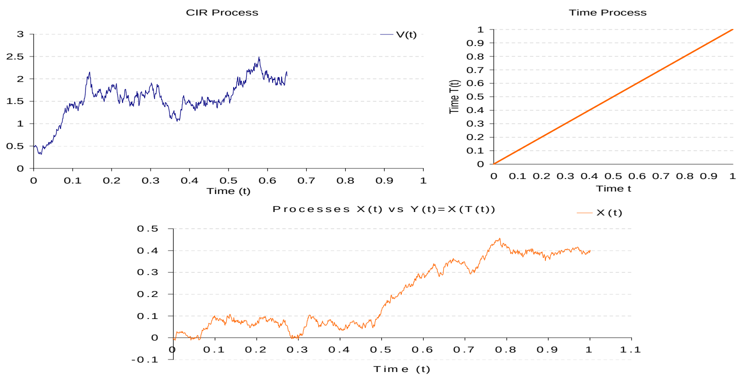
This is the Zero Coupon Bond formula under the CIR interest rate $r_t = \lambda V_t$.

Illustration of Absolutely Continuous Time Changes

CIR parameters $\kappa = 7$, $\theta = 2$, $V_0 = 0.5$ and $\sigma_v = \sqrt{2}$



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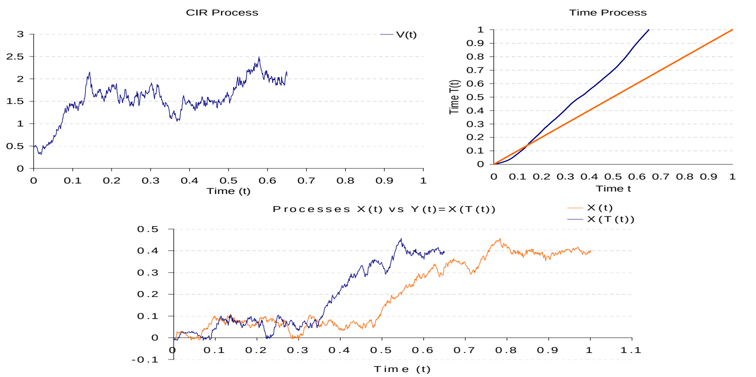

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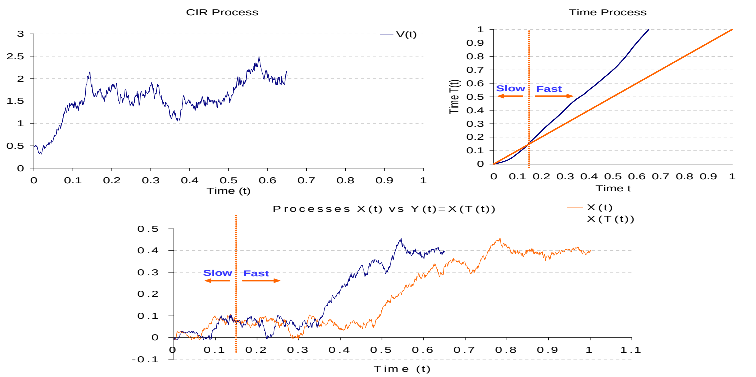

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Composite Time Changes

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A Composite Time Change induces both *jumps* and *stochastic volatility*

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Laplace Transform of the Composite Time Change

It is obtained by first conditioning w.r.t. the A.C. time change

$$\mathbb{E}[e^{-\lambda T_t}] = \mathbb{E}[e^{-T^2_t \phi(\lambda)}] = \mathcal{L}_z(t, \phi(\lambda))$$

Quick Summary

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1. A Jump-to-Default Extended Diffusion process:

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1. A Jump-to-Default Extended Diffusion process:

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2. A time-changed process $Y_t = X_{T_t}$ with the Laplace transform for the time change T_t given in closed form,

$$\mathbb{E}\left[e^{-\lambda T_t}\right]=\mathcal{L}\left(t,\lambda\right)$$

How do we evaluate contingent claims written on the time-changed process Y_t?

$$\mathbb{E}\left[f\left(Y_t\right)\mathbf{1}_{\{\zeta>T_t\}}\right]$$

Contingent Claims for the Time-Changed Process

Valuing contingent claims written on $Y_t = X_{T_t}$

$$\mathbb{E}\left[\mathbf{1}_{\{\zeta > T_t\}} f\left(Y_t\right)\right] = \mathbb{E}\left[\mathbb{E}_x\left[\mathbf{1}_{\{\zeta > T_t\}} f\left(X_{T_t}\right) \middle| T_t\right]\right]$$

Conditioning since X_t and T_t are independent

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We employ two methodologies to evaluate the expectations and do the pricing in closed form:

1. Resolvent Operator: general methodology.
2. Spectral Representation: for square-integrable payoffs.

Resolvent Operator

Resolvent Operator:

The Laplace Transform of the Expectation Operator:

$$(\mathcal{R}_\lambda f)(x) := \int_0^\infty e^{-\lambda t} \mathbb{E}_x \left[\mathbf{1}_{\{\zeta > t\}} f\left(X_t\right)\right] dt$$

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Bromwich Laplace Inversion

$$\mathbb{E}_x\left[\mathbf{1}_{\{\zeta>t\}} f\left(X_t\right)\right] = \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{\lambda t}(\mathcal{R}_\lambda f)(x) d\lambda$$

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- NOTE. The time t enters in this expression only through the exponential $e^{\lambda t}$

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Details

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Eigenfunction Expansion (when the spectrum of $\mathcal{G}$ is discrete):

$$\mathbb{E}_x\left[\mathbf{1}_{\{\zeta>t\}} f(X_t)\right] = \sum_{n=1}^{\infty} e^{-\lambda_n t} c_n \varphi_n(x)$$

where $c_n = \langle f, \varphi \rangle$ are the expansion coefficients and, λ_n are the eigenvalues, $\varphi_n(x)$ the eigenfunctions solving $\mathcal{G}\varphi_n(x) = \lambda_n \varphi_n(x)$

NOTE. The time t enters in this expression only through the exponential $e^{-\lambda_n t}$

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A new class of Credit-Equity Models with state-dependent jumps, S.V. and default intensity

Model Architecture for the Defaultable Stock

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$$dX_t = [\mu + h(X_t)]X_t\,dt + \sigma(X_t)X_t\,dB_t, \;\; X_0 = x > 0,$$
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- $\rho \Rrightarrow$ Compensation Parameter (discounted martingale) ▸ Details
- $\tau_d \Rrightarrow$ Default Time
  - If $\zeta = \min(\tau_0, \tilde{\zeta})$ is the lifetime of X, then
$$\tau_d = \inf\{t \geq 0 : \zeta \leq T_t\}$$
  - At τ_d the stock drops to zero (Bankruptcy)

Survival Probability and Defaultable Zero Bonds

Survival Probability

$$\mathbb{Q}(\tau_d > t) = \mathbb{Q}(\zeta > T_t)$$

$$= \sum_{n=0}^{\infty} \mathcal{L}\left(t, (b + \omega n)\right) \frac{\Gamma(1+c/|\beta|)\Gamma(n+1/(2|\beta|))}{\Gamma(\nu+1)\Gamma(1/(2|\beta|))n!}$$

$$\times A^{\frac{1}{2|\beta|}} x e^{-Ax^{-2\beta}} {}_1F_1(1 - n + c/|\beta|, \nu + 1, Ax^{-2\beta})$$

▸ Details

Where ${}_1F_1(a, b, z)$ is the *Kummer Confluent Hypergeometric* function; and $\omega = 2|\beta|(\mu + b)$, $\nu = \frac{1+2c}{2|\beta|}$, and $A = \frac{\mu+b}{a^2|\beta|}$.

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Defaultable Zero Coupon Bond

$$B_R(x, t) = e^{-rt}\mathbb{Q}(\tau_d > t) + Re^{-rt}[1 - \mathbb{Q}(\tau_d > t)]$$

Recovery fraction $R \in [0, 1]$

Put Options

Put Option

$$P(x;K,t) = P_D(x;K,t) + P_0(x;K,t),$$

where:

$$P_D(x;K,t) = Ke^{-rt}[1 - \mathbb{Q}(\tau_d > t)], \qquad \text{Default before t}$$

$$P_0(x;K,t) = e^{-rt}\mathbb{E}_x\left[(K - e^{\rho t}X_{T_t})^+\mathbf{1}_{\{\tau_d > t\}}\right] \qquad \text{No Default before t}$$

$$= e^{-(r-\rho)t}\sum_{n=1}^{\infty}\mathcal{L}(t,\lambda_n)c_n\varphi_n(x)$$

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$$P_0(x; K, t) = e^{-rt}\mathbb{E}_x\left[(K - e^{\rho t}X_{T_t})^{+}\mathbf{1}_{\{\tau_d > t\}}\right] \qquad \text{No Default before t}$$

$$= e^{-(r-\rho)t}\sum_{n=1}^{\infty}\mathcal{L}(t, \lambda_n)c_n\varphi_n(x)$$

- The default claim $P_D(x; K, t)$ is directly calculated from the Survival Probability $\mathbb{Q}(\tau_d > t)$ previously computed
- The claim, $P_0(x; K, t)$, is calculated by means of the Spectral Expansion since $f(x) = (K - x)^{+} \in L^2((0, \infty), \mathfrak{m})$

Put Options

Put claim conditional on *no default event* before maturity

$$P_0(x;K,t) = e^{-(r-\rho)t} \sum_{n=1}^{\infty} \mathcal{L}(t,\lambda_n) c_n \varphi_n(x)$$

where $k = K\,e^{-\rho t}$ and,

Eigenvalues $\Rrightarrow \lambda_n = \omega n + 2c(\mu + b) + b$, with $\omega = 2|\beta|(\mu + b)$

Eigenfunctions $\Rrightarrow \varphi_n(x) = A^{\frac{\nu}{2}} \sqrt{\frac{(n-1)!(\mu+b)(2c+1)}{\Gamma(\nu+n)}} x e^{-Ax^{-2\beta}} L_{n-1}^{(\nu)}(Ax^{-2\beta})$

Expansion Coefficients $\Rrightarrow c_n = \frac{A^{\nu/2+1} k^{2c+1-2\beta} \sqrt{\Gamma(\nu+n)}}{\Gamma(\nu+1)\sqrt{(\mu+b)(2c+1)(n-1)!}}$

$$\times \left\{ \frac{|\beta|}{c+|\beta|}\, {}_2F_2 \left(\begin{matrix} 1-n, & \frac{c}{|\beta|}+1 \\ \nu+1, & \frac{c}{|\beta|}+2 \end{matrix} ; Ak^{-2\beta} \right) - \frac{\Gamma(\nu+1)(n-1)!}{\Gamma(\nu+n+1)} L_{n-1}^{\nu+1}\left(Ak^{-2\beta}\right) \right\},$$

${}_2F_2$: generalized hypergeometric function, $L_n^{(\nu)}$: generalized Laguerre polynomials.

Numerical Examples

- Assume a background process $\{X_t, t > 0\}$ following a JDCEV, and a composite time change Inverse Gaussian Process & CIR:

Parameters

JDCEV	S	50	*CIR*	V	1
	a	10		θ	1
	β	-1		σ_V	1
	c	0.5		κ	4
	b	0.01	*IG*	γ	0
	r	0.05		η	8
	q	0		C	$2\sqrt{2/\pi}$

Note that $\gamma = 0$, thus the time changed process is a pure jump process!

Infinitesimal Generator of the Time Changed Process $(Y_t = X_{T_t}, V_t)$

$$\mathcal{G}f(x,v) =$$

$$\gamma v\left(\frac{1}{2}a^2x^{2\beta+2}\frac{\partial^2 f}{\partial x^2}(x,v) + (b+ca^2x^{2\beta})x\frac{\partial f}{\partial x}(x,v) - (b+c^2a^2x^{2\beta})f(x,v)\right)$$

$$+v\left(\int_{(0,\infty)}(f(y,v)-f(x,v))\,\pi(x,y)dy - k(x)\,f(x,v)\right)$$

$$+\frac{\sigma_V^2}{2}v\frac{\partial^2 f}{\partial v^2}(x,v) + \kappa(\theta - v)\frac{\partial f}{\partial v}(x,v)$$

- State dependent jump measure $\pi_(x,y) = \int_{(0,\infty)} p(s;x,y)\nu(ds)$
- Additional killing rate $k(x) = \int_{(0,\infty)} P_s(x,\{0\})\nu(ds)$

Details

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$$\mathcal{G}f(x,v) =$$

$$\gamma v \underbrace{\left(\frac{1}{2}a^2 x^{2\beta+2}\frac{\partial^2 f}{\partial x^2}(x,v) + (b + ca^2x^{2\beta})x\frac{\partial f}{\partial x}(x,v) - (b + c^2a^2x^{2\beta})f(x,v)\right)}_{\mathcal{G}_x f(x,v) \text{ JDCEV's infinitesimal generator}}$$

$$+v\underbrace{\left(\int_{(0,\infty)} \left(f(y,v) - f(x,v)\right)\pi(x,y)dy - k\left(x\right)f\left(x,v\right)\right)}_{\int_{(0,\infty)}(\mathcal{P}_s f - f)\nu(ds) \text{ Subordination component}}$$

$$+\underbrace{\frac{\sigma_V^2}{2}v\frac{\partial^2 f}{\partial v^2}(x,v) + \kappa(\theta - v)\frac{\partial f}{\partial v}(x,v)}_{\mathcal{G}_v f(x,v) \text{ CIR's infinitesimal generator}}$$

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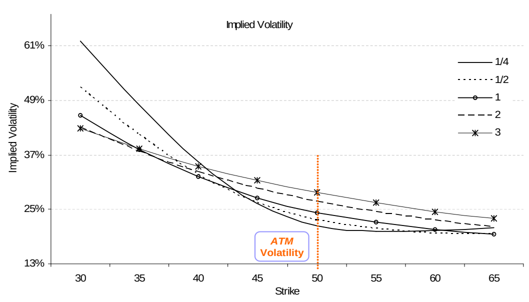
Details

Numerical Examples (Cont.)

Implied Volatility

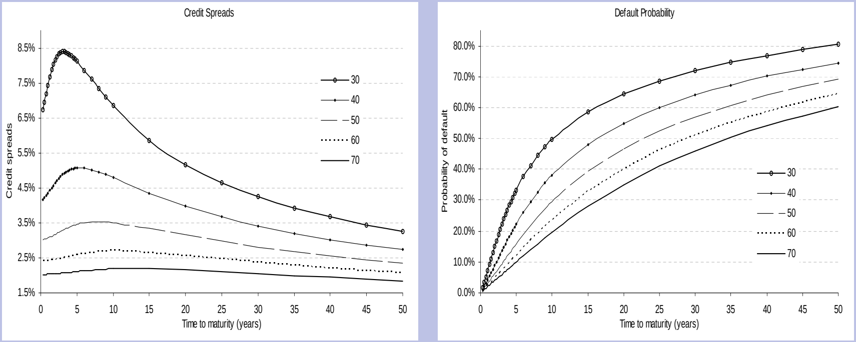


Time/Strike	*30*	*35*	*40*	*45*	*50*	*55*	*60*	*65*
1/4	62.04	47.94	35.52	26.19	21.41	20.09	20.28	20.88
1/2	51.94	41.47	32.72	26.39	22.64	20.72	19.84	19.46
1	45.74	38.24	32.14	27.53	24.30	22.12	20.65	19.64
2	43.03	37.68	33.23	29.61	26.72	24.45	22.66	21.25
3	42.80	38.34	34.55	31.34	28.64	26.39	24.52	22.96

Implied volatility smile/skew curves as functions of the strike price.

Numerical Examples (Cont.)

Credit Spreads and Default Probability



Credit spreads and default probabilities as functions of time to maturity for current stock price levels $S = 30,\ 40,\ 50,\ 60,\ 70.$

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- Our Stock price is a jump-diffusion process with stochastic volatility and default intensity,

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- Our Stock price is a jump-diffusion process with stochastic volatility and default intensity,
- The Default intensity explicitly depends on the stock price and volatility
- The leverage effect is introduced in the diffusion and in jumps components - as the stock falls, the diffusion volatility and arrival rates of large jumps increase

Summary of Features and Practical Benefits of Our Modeling Framework (cont.)

- Stochastic volatility affects the diffusion and jump components

Finish

Summary of Features and Practical Benefits of Our Modeling Framework (cont.)

- Stochastic volatility affects the diffusion and jump components
- Unified credit-equity framework ⇛ *consistency in the pricing and hedging* of credit and equity derivatives

▸ Finish

Summary of Features and Practical Benefits of Our Modeling Framework (cont.)

- Stochastic volatility affects the diffusion and jump components
- Unified credit-equity framework $\Rightarrow$ *consistency in the pricing and hedging* of credit and equity derivatives
- We obtain *analytical solutions* $\Rightarrow$ *faster computation* of prices and Greeks, and faster calibration

Finish

Questions?

Thank you!

Appendix

Appendix

Lévy Subordinators

Lévy subordinator

Non-decreasing Lévy process $\{T_t, t \geq 0\}$ with *positive jumps and non-negative drift*

Laplace Transform (LT):

$$\mathcal{L}(t,\lambda) = \mathbb{E}[e^{-\lambda T_t}] = e^{-t\phi(\lambda)}$$

The Laplace exponent $\phi(\lambda)$ is given by the Lévy-Khintchine formula:

$$\phi(\lambda) = \gamma\lambda + \int_{(0,\infty)} \left(1 - e^{-\lambda s}\right) \nu(ds)$$

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- $\gamma \geq 0$ $\quad \Rrightarrow$ positive drift
- $\nu(ds)$ $\quad \Rrightarrow$ Lévy measure which satisfies $\int_{(0,\infty)} (s \wedge 1) \nu(ds) < \infty$
- *transition probability* $\pi_t(ds)$ is obtained by:

$$\int_{[0,\infty)} e^{-\lambda s} \pi(ds) = e^{-t\phi(\lambda)}$$

Examples of Lévy Subordinators (cont.)

- The processes T_t is a *Compound Poisson processes* with gamma distributed jump sizes if $Y < 0$
  - Compound Poisson process with exponential jumps ($Y = -1$)

$$\nu(ds) = \alpha\eta e^{-\eta s} ds, \quad \phi(\lambda) = \gamma\lambda + \frac{\alpha\lambda}{\lambda + \eta}$$

- *Tempered Stable Subordinators* ($Y \in (0, 1)$)
  - *Inverse Gaussian process* ($Y = 1/2$)
  - *Gamma process* ($Y \to 0$)

- The processes with $Y \in [0, 1)$ are of infinite activity.

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Martingale Property

- Intensity $h(S)$ has to be added in the drift of X to compensate for jump to zero, and ρ and μ are parameters to be selected to make the discounted time-changed process into a martingale:

$$\mathbb{E}[S_{t_2}|\mathcal{F}_{t_1}] = e^{(r-q)(t_2-t_1)} S_{t_1}, \ \ t_1 \leq t_2,$$

  where r and q are the risk-free rate and dividend yield.

- If T_t is a subordinator, then μ can be arbitrary and,

$$\rho = r - q + \phi(-\mu).$$

- If T_t is an A.C. time change, then

$$\mu = 0, \ \ \rho = r - q.$$

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Survival Probability

1. Condition w.r.t the Random Clock T_t

$$\mathbb{Q}(\tau_d > t) = \mathbb{Q}(\zeta > T_t) = \mathbb{E}\left[\mathbb{E}\left[e^{-\int_0^u \lambda(S_v)dv}\mathbf{1}_{\{T_0>u\}}\middle| T_t = u\right]\right]$$

2. Since the Function $f(x) = 1$ is NOT in $L^2(\mathcal{D}, m)$, we use the resolvent operator R_λ

$$\mathbb{Q}(\zeta > T_t) = \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \mathcal{L}(t, -\lambda)(\mathcal{R}_\lambda 1)(x)d\lambda,$$

3. The resolvent is available in closed form

$$\mathcal{R}_\lambda f(x) = \int_0^\infty G_\lambda(x, y)f(y)dy$$

$G_\lambda(x, y)$ is the *Resolvent Kernel* or *Green's Function*

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Survival Probability

4. $G_\lambda(x,y)$ is known in closed form ($\mu + b > 0$):

$$G_\lambda(x,y) = \frac{\Gamma(\nu/2 + 1/2 - k(\lambda))}{(\mu+b)\Gamma(1+\nu)y} \left(\frac{x}{y}\right)^{c+1/2-\beta} e^{A(y^{-2\beta} - x^{-2\beta})}$$

$$\times M_{k(\lambda),\frac{\nu}{2}}(A(x \wedge y)^{-2\beta}) W_{k(\lambda),\frac{\nu}{2}}(A(x \vee y)^{-2\beta})$$

where $\nu = \frac{1+2c}{2|\beta|}$, $k(\lambda) = \frac{\nu-1}{2} - \frac{\lambda}{2|\beta|(\mu+b)}$, $A = \frac{\mu+b}{a^2|\beta|}$ and,

$M_{k,m}(z)$ and $W_{k,m}(z)$ are the first and second Whittaker functions.

5. Using the Cauchy Residue Theorem to invert the Resolvent we obtain the Survival Probability

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Spectral Expansion

Assume $\exists \mathfrak{m}$ on D with full support (i.e. $\mathrm{SSup}(\mathfrak{m}) = D$) s.t. the (bounded) contraction semigroup $\mathcal{P}_t$ (e.g. $\mathcal{P}_t f(x) = \mathbb{E}_x[f(X_t)\mathbf{1}_{\{\zeta > t\}}]$) are symmetric on $\mathcal{H} = L^2(D, \mathfrak{m})$

$$\langle \mathcal{P}_t f, g \rangle_{\mathfrak{m}} = \int_D \mathcal{P}_t f \, g \, d\mathfrak{m} = \int_D f \, \mathcal{P}_t g \, d\mathfrak{m} = \langle f, \mathcal{P}_t g \rangle_{\mathfrak{m}}$$

- Then the infinitesimal generator $\mathcal{G}$ is (generally unbounded) self-adjoint operator in $\mathcal{H}$, i.e., $\mathcal{G}$ is symmetric,

$$\langle \mathcal{G} f, g \rangle_{\mathfrak{m}} = \langle f, \mathcal{G} g \rangle_{\mathfrak{m}}, \quad \forall f, g \in \mathrm{Dom}(\mathcal{G})$$

- The domains of $\mathcal{G}$ and its adjoint $\mathcal{G}^*$ coincide in $\mathcal{H}$, i.e. $\mathrm{Dom}(\mathcal{G}) = \mathrm{Dom}(\mathcal{G}^*) \subset \mathcal{H}$
- The infinitesimal operator $\mathcal{G}$ is non-positive in $\mathcal{H}$, i.e. $\langle \mathcal{G} f, f \rangle_{\mathfrak{m}} < 0$ for all $f \in \mathrm{Dom}(\mathcal{G})$.

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Spectral Representation Theorem

Spectral Representation Theorem

Let $\mathcal{H}$ be a separable real Hilbert space and let $\{\mathcal{P}_t, t \geq 0\}$ be a strongly continuous self-adjoint contraction semigroup in $\mathcal{H}$ with the non-positive self-adjoint infinitesimal generator $\mathcal{G}$. Then there exists a unique integral representation of $\{\mathcal{P}_t, t \geq 0\}$ of the form

$$\mathcal{P}_t f = e^{t\mathcal{G}} f = \int_{[0,\infty)} e^{-\lambda t} E(d\lambda) f, \ \ f \in \mathcal{H}, \ \ t \geq 0,$$

where E is the spectral measure of the negative $-\mathcal{G}$ of the infinitesimal generator $\mathcal{G}$ of $\mathcal{P}$ with the support of the spectral measure (the *spectrum* of $-\mathcal{G}$) $\mathrm{Supp}(E) \subset [0, \infty)$, namely,

$$-\mathcal{G} f = \int_{[0,\infty)} \lambda E(d\lambda) f, \ \ f \in \mathrm{Dom}(\mathcal{G}),$$

$$\mathrm{Dom}(\mathcal{G}) = \left\{ f \in \mathcal{H} : \int_{[0,\infty)} \lambda^2 (E(d\lambda) f, f) < \infty \right\}.$$

Hille and Phillips (1957, Theorem 22.3.1) and Reed and Simon (1980, Theorem VIII.6)

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Discrete Case Spectral Representation

Things simplify further when the generator has a purely discrete spectrum. Let $-\mathcal{G}$ be a self-adjoint non-negative operator with purely discrete spectrum $\sigma_d(-\mathcal{G}) \subset [0, \infty)$. Then the spectral measure can be defined by

$$E(B) = \sum_{\lambda \in B} P(\lambda),$$

where $P(\lambda)$ is the orthogonal projection onto the eigenspace corresponding to the eigenvalue $\lambda \in \sigma_d(-\mathcal{G})$. Then the spectral theorem for the self-adjoint semigroup takes the simpler form:

$$\mathcal{P}_t f = e^{t\mathcal{G}} f = \sum_{\lambda \in \sigma_d(-\mathcal{G})} e^{-\lambda t} P(\lambda) f, \;\; t \geq 0, \;\; f \in \mathcal{H},$$

$$-\mathcal{G} f = \sum_{\lambda \in \sigma_d(-\mathcal{G})} \lambda P(\lambda) f, \;\; f \in \mathrm{Dom}(\mathcal{G}).$$

(e.g. $P(\lambda) f = c(\lambda)\phi_\lambda = \langle f, \phi_\lambda \rangle_{\mathfrak{m}} \phi_\lambda$)

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Notes on Calibration and the Implied Measure (Cont & Tankov, 2004)

- Exponential Lévy and jump-diffusion models correspond to incomplete market models

  $\Rrightarrow$ No perfect hedges can be found

  $\Rrightarrow$ The (equivalent) martingale measure cannot be defined in a unique way

- Any arbitrage-free market prices of securities can be represented as discounted conditional expectations w.r.t. a risk-neutral measure $\mathbb{Q}$ under which discounted asset prices are martingales

  $\Rrightarrow$ *Model Calibration.* Find a risk-neutral model $\mathbb{Q}$ which matches the prices of the *observed market prices* $V_{\{i \in I\}}(S)$ of securities $i \in I$ at time $t = 0$,

$$\forall i \in I, \quad V_i(S) = e^{-rt_i}\mathbb{E}^{\mathbb{Q}}[f(S_{t_i})]$$

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Notes on Calibration and the Implied Measure (Cont & Tankov, 2004)

- Least Square Calibration.

$$\theta^* = \arg\min_{\mathbb{Q}_\theta \in \mathcal{Q}} \sum_{i \in I} \omega_i \left| V_i^\theta(S, t_i) - V_i(S) \right|^2$$

where $\mathcal{Q}$ is the set of martingale measures

$\Rrightarrow$ The objective functional is non-convex.

$\Rrightarrow$ Since the number of observable prices is finite there are multiple Lévy measures giving the same error level (multiple local minimum)

- To obtain a unique solution in a stable manner we need to introduce a penalty functional (regularization) F

$$\theta^* = \arg\min_{\mathbb{Q}_\theta \in \mathcal{Q}} \sum_{i \in I} \omega_i \left| V_i^\theta(S, t_i) - V_i(S) \right|^2 + \alpha F(\mathbb{Q}_\theta | \mathbb{P}_0)$$

where $\mathbb{P}_0$ is the historical measure at $t = 0$ and F is a *convex* function which penalizes the objective if $\mathbb{Q}$ deviates much from $\mathbb{P}_0$ and ensures uniqueness (v.g. F relative entropy)

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Jump measure and killing rate

$$\textit{(Jump measure)}\ \pi(x,y) = 2|\beta|AC\left(\frac{y}{x}\right)^{c-\frac{1}{2}}\ y^{-(2\beta+1)}$$

$$\times \int_{(0,\infty)} \frac{s^{-3/2}e^{\left(\frac{\omega\nu}{2}-\xi-\eta\right)s}}{e^{\omega s}-1} \exp\left\{-A\left(\frac{x^{-2\beta}e^{\omega s}+y^{-2\beta}}{e^{\omega s}-1}\right)\right\} I_\nu\left(\frac{A(xy)^{-\beta}}{\sinh(\omega s/2)}\right) ds.$$

and

$$\textit{(killing rate)}\ k(x) =$$

$$C\int_{(0,\infty)} \left(1 - \frac{\Gamma\left(\frac{c}{|\beta|}+1\right)(\tau(s))^{\frac{1}{2|\beta|}} e^{-\tau(s)-bs}\, {}_1F_1\left(\begin{matrix}\frac{c}{|\beta|}+1 \\ \nu+1\end{matrix};\tau(s)\right)}{\Gamma(\nu+1)}\right) s^{-3/2}e^{-\eta s} ds$$

where $\tau(s) := \frac{\omega\, x^{-2\beta}}{2|\beta|^2 a^2(1-e^{-\omega\, s})}$,

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Protection Payment under JDCEV

$$\textbf{PV(Protection Payment)} = (1-\mathfrak{r})\,\mathbb{E}\left[e^{-r\cdot T_L^\Delta}\mathbf{1}_{\{T_L^\Delta \le T\}}\right]$$

$$= (1-\mathfrak{r})\left\{\underbrace{\mathbb{E}\left[e^{-r\cdot T_L - \int_0^{T_L} h(X_u)du}\mathbf{1}_{\{T_L \le T\}}\right]}_{\text{Diffusion Term}} + \int_0^T e^{-r\cdot u}\underbrace{\mathbb{E}\left[e^{-\int_0^u h(X_v)dv}h(X_u)\,\mathbf{1}_{\{T_L > u\}}\right]du}_{\text{Jump Term}}\right\}$$

Recall that the first hitting time to L is given by $T_L = \inf\{t : X_t = L\}$, and that the first jump time to Δ is given by $\zeta = \inf\left\{t \in [0,\infty] : \int_0^t h(X_u)\,du \ge e\right\}$

The default intensity is the power function:

$$h(X_t) = b + ca^2 X_t^{2\beta}$$

Notice. Since e is an exponentially distributed r.v. with unit mean, then $\mathbb{P}[\zeta > t] = e^{-\int_0^t h(X_u)du}$ and $\mathbb{P}[\zeta < t] = \int_0^t h(X_v)\, e^{-\int_0^v h(X_u)du}dv$

Premium Payment under JDCEV

$$\textbf{PV(Premium Payment)} = \varrho \cdot \Delta_t \cdot \sum_{i=1}^{N} e^{-r \cdot t_i} \mathbb{E}\left[\mathbf{1}_{\{T_L^{\Delta} \geq t_i\}}\right]$$

$$= \varrho \cdot \Delta_t \cdot \sum_{i=1}^{N} e^{-r \cdot t_i} \quad \underbrace{\mathbb{E}\left[e^{-\int_0^{t_i} h(X_u)du}\mathbf{1}_{\{T_L \geq t_i\}}\right]}_{\text{NO jump to default \& NO hitting level}}$$

The premium is paid at times t_i conditional on No default and that the stock price did Not drop to level L by time t_i

The default intensity is the power function:

$h(X_t) = b + ca^2 X_t^{2\beta}$

Accrued Interests under JDCEV

$$\textbf{PV(Acc. Interest)} = \varrho \cdot \mathbb{E}\left[e^{-r\cdot T_L^{\Delta}} \left(T_L^{\Delta} - \Delta_t \cdot \left\lfloor \frac{T_L^{\Delta}}{\Delta_t} \right\rfloor \right) \mathbf{1}_{\{T_L^{\Delta} \le T\}} \right]$$

$$= \varrho \sum_{i=0}^{N-1} \mathbb{E}\left[e^{-r\cdot T_L^{\Delta}} \left(T_L^{\Delta} - \Delta_t \cdot i \right) \mathbf{1}_{\{T_L^{\Delta} \in (t_i, t_{i+1})\}} \right]$$

Expressed in terms of Diffusion and Jump components:

$$= \varrho \cdot \Bigg\{ \int_0^T u\, e^{-r\cdot u} \underbrace{\mathbb{E}\left[e^{-\int_0^u h(X_v)dv} h(X_u) \mathbf{1}_{\{T_L \ge u\}} \right]}_{\text{Jump Term}} du$$

$$+ \underbrace{\mathbb{E}\left[e^{-r\cdot T_L - \int_0^{T_L} h(X_u)du} T_L \mathbf{1}_{\{T_L \le T\}} \right]}_{\text{Diffusion Term}}$$

$$- \sum_{i=1}^{N-1} (i \cdot \Delta_t) \int_{t_i}^{t_{i+1}} e^{-r\cdot u} \underbrace{\mathbb{E}\left[e^{-\int_0^u h(X_v)dv} h(X_u) \mathbf{1}_{\{T_L \ge u\}} \right]}_{\text{Jump Term}} du$$

$$- \sum_{i=1}^{N-1} (i \cdot \Delta_t) \left(\underbrace{\mathbb{E}\left[e^{-r\cdot T_L - \int_0^{T_L} h(X_u)du} \mathbf{1}_{\{T_L \le t_{i+1}\}} \right]}_{\text{Diffusion Term}} - \underbrace{\mathbb{E}\left[e^{-r\cdot T_L - \int_0^{T_L} h(X_u)du} \mathbf{1}_{\{T_L \le t_i\}} \right]}_{\text{Diffusion Term}} \right) \Bigg\}$$

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Expectations to Solve: Jump Term and Diffusion Term

- Jump Term.

$$\mathbb{E}\left[e^{-\int_0^u h(X_v)dv} h\left(X_u\right) \mathbf{1}_{\{T_L > u\}}\right]$$

Since the default intensity is given by a power function, $h(X_t) = b + ca^2 X_t^{2\beta}$, we can solve, more generally, for a given p the expectation which we name truncated p-Moment

$$\mathbb{E}\left[e^{-\int_0^u h(X_v)dv} \left(X_u\right)^p \mathbf{1}_{\{T_L > u\}}\right]$$

- Diffusion Term.[1] This term can be seen as the Expected Discount (given no default) up to the first hitting time to level L

$$\mathbb{E}\left[e^{-r\cdot T_L - \int_0^{T_L} h(X_u)du} \mathbf{1}_{\{T_L \leq T\}}\right]$$

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Solving the Expectations: the truncated p-Moment

The truncated p-Moment for $L > 0$ and $\mu + b > 0$ is given by

$$\mathbb{E}_x\left[e^{-\int_0^t h(X_u)du}\mathbf{1}_{\{T_L>t\}}(X_t)^p\right]$$

$$= \sum_{n=0}^{\infty}\left(\frac{A^{\frac{1-2c-2p}{4|\beta|}-\frac{1}{2}}\left(\frac{1-p}{2|\beta|}\right)_n\Gamma\left(1+\frac{2c+p}{2|\beta|}\right)}{n!\,\Gamma(1+\nu)}x^{\frac{1}{2}-c+\beta}e^{-\frac{A}{2}x^{-2\beta}}e^{(p(\mu+b)-(b+\omega\, n))t}\right.$$

$$\left.\times\left[M_{\frac{\nu-1}{2}+n-\left(\frac{2c+p}{2|\beta|}\right),\frac{\nu}{2}}\left(Ax^{-2\beta}\right)-\frac{M_{\frac{\nu-1}{2}+n-\left(\frac{2c+p}{2|\beta|}\right),\frac{\nu}{2}}\left(A\,L^{-2\beta}\right)}{W_{\frac{\nu-1}{2}+n-\left(\frac{2c+p}{2|\beta|}\right),\frac{\nu}{2}}\left(A\,L^{-2\beta}\right)}W_{\frac{\nu-1}{2}+n-\left(\frac{2c+p}{2|\beta|}\right),\frac{\nu}{2}}\left(Ax^{-2\beta}\right)\right]\right)$$

$$+\sum_{n=1}^{\infty}\left(e^{-\left(\omega\left(\kappa_n-\frac{\nu-1}{2}\right)+\xi\right)t}x^{\frac{1}{2}-c+\beta}e^{-\frac{A}{2}x^{-2\beta}}\frac{M_{\kappa_n,\frac{\nu}{2}}\left(A\,L^{-2\beta}\right)W_{\kappa_n,\frac{\nu}{2}}\left(Ax^{-2\beta}\right)}{\Gamma(1+\nu)\left[\frac{d}{d\kappa}W_{\kappa,\frac{\nu}{2}}\left(A\,L^{-2\beta}\right)\right]\Big|_{\kappa=\kappa_n}}\right.$$

$$\times\left[A^{\frac{1-2c-2p}{4|\beta|}-\frac{1}{2}}\frac{\Gamma\left(1-\frac{1-p}{2|\beta|}\right)\Gamma\left(1+\frac{2c+p}{2|\beta|}\right)\Gamma\left(\frac{\nu-1}{2}-\kappa_n-\frac{2c+p}{2|\beta|}\right)}{\Gamma\left(\frac{1-\nu}{2}-\kappa_n\right)}\right.$$

$$-\frac{2|\beta|\,A^{\frac{1-\nu}{2}}L^{-2\beta-1+p}\Gamma(\nu)}{(2|\beta|-1+p)}\,{}_2F_2\left(\begin{matrix}1-\frac{1-p}{2|\beta|}, & \frac{1-\nu}{2}-\kappa_n\\ 2-\frac{1-p}{2|\beta|}, & 1-\nu\end{matrix};A\,L^{-2\beta}\right)$$

$$\left.\left.-\frac{2|\beta|\,A^{\frac{1+\nu}{2}}L^{2c+p-2\beta}\Gamma(-\nu)\Gamma\left(\frac{1+\nu}{2}-\kappa_n\right)}{(2|\beta|+2c+p)\Gamma\left(\frac{1-\nu}{2}-\kappa_n\right)}\,{}_2F_2\left(\begin{matrix}1+\frac{2c+p}{2|\beta|}, & \frac{1+\nu}{2}-\kappa_n\\ 2+\frac{2c+p}{2|\beta|}, & 1+\nu\end{matrix};A\,L^{-2\beta}\right)\right]\right)$$

where $\kappa_n = \left\{\kappa \middle|\, W_{\kappa,\frac{\nu}{2}}\left(A\,L^{-2\beta}\right)=0\right\}$

Solving the Expectations: the truncated p-Moment

The truncated p-Moment for $L = 0$ (CDS case) and $\mu + b > 0$ is given by

$$\mathbb{E}_x\left[e^{-\int_0^t h(X_u)du}\mathbf{1}_{\{T_L>t\}}(X_t)^p\right]$$
$$= \sum_{n=0}^{\infty} \frac{A^{\frac{1-2c-2p}{4|\beta|}-\frac{1}{2}}\left(\frac{1-p}{2|\beta|}\right)_n\Gamma\left(1+\frac{2c+p}{2|\beta|}\right)}{n!\,\Gamma(1+\nu)} x^{\frac{1}{2}-c+\beta} e^{-\frac{A}{2}x^{-2\beta}} e^{(p(\mu+b)-(b+\omega\, n))t}$$
$$\times M_{\frac{\nu-1}{2}+n-\left(\frac{2c+p}{2|\beta|}\right),\frac{\nu}{2}}\left(Ax^{-2\beta}\right)$$

where $\kappa_n = \left\{\kappa \middle|\, W_{\kappa,\frac{\nu}{2}}\left(A\,L^{-2\beta}\right) = 0\right\}$

Solving the Expectations: Diffusion Term

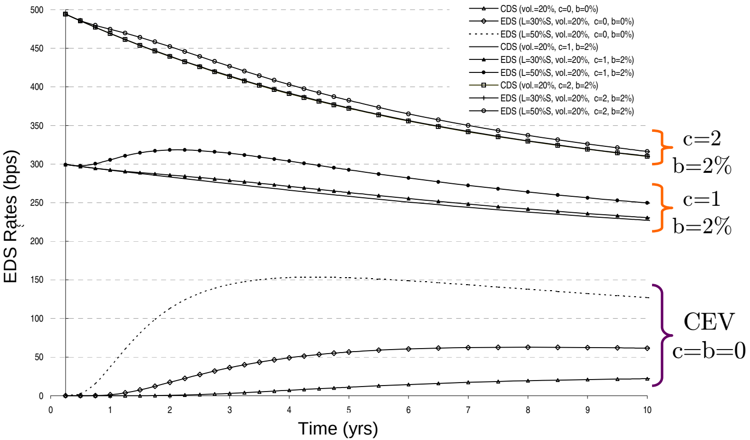
The Diffusion Term

$$\mathbb{E}_x\left[e^{-r\cdot T_L-\int_0^{T_L} h(X_u)du}\mathbf{1}_{\{T_L\le T\}}\right]=\left(\frac{x}{L}\right)^{\frac{1}{2}-c+\beta} e^{\epsilon\frac{A}{2}\left(x^{-2\beta}-L^{-2\beta}\right)}\times$$

$$\left[\frac{W_{\epsilon\frac{1-\nu}{2}-\frac{r+\xi}{\omega},\frac{\nu}{2}}\left(Ax^{-2\beta}\right)}{W_{\epsilon\frac{1-\nu}{2}-\frac{r+\xi}{\omega},\frac{\nu}{2}}\left(AL^{-2\beta}\right)}+\sum_{n=1}^{\infty}\frac{\omega\, e^{-\left(\omega\left(\kappa_n-\epsilon\frac{1-\nu}{2}\right)+r+\xi\right)T}}{\left(\omega\left(\kappa_n-\epsilon\frac{1-\nu}{2}\right)+r+\xi\right)}\frac{W_{\kappa_n,\frac{\nu}{2}}\left(Ax^{-2\beta}\right)}{\left[\frac{\partial}{\partial\kappa}W_{\kappa,\frac{\nu}{2}}\left(AL^{-2\beta}\right)\right]\Big|_{\kappa=\kappa_n}}\right]$$

Numerical Example 1: the effect of the sensitivity to variance "c"

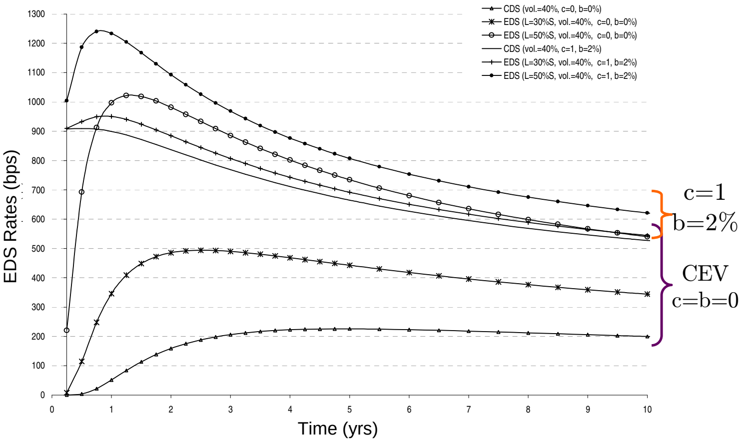
Default Intensity function: $h(X_t) = b + ca^2 X_t^{2\beta}$. We choose $a = \sigma / S_0^{\beta} = 10$



Δ_t	$\mathfrak{r}$	L	S_0	r	q	b	c	β	σ
0.25	0.5	$\{0, 15, 25\}$	50	0.05	0	$\{0, 0.02\}$	$\{0, 1, 2\}$	-1	0.20

Numerical Example 2: the effect of volatility "σ"

Default Intensity function: $h(X_t) = b + ca^2 X_t^{2\beta}$. We choose $a = \sigma / S_0^{\beta} = 20$



Δ_t	$\mathfrak{r}$	L	S_0	r	q	b	c	β	σ
0.25	0.5	{0, 15, 25}	50	0.05	0	{0, 0.02}	{0, 1}	-1	0.40

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