Modeling Counterparty Credit Exposure in the Presence of Margin Agreements

Michael Pykhtin

Counterparty Credit Risk Analytics
Bank of America

Recent Advancements in the Theory and Practice of Credit Derivatives

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Discussion Plan

- Margin agreements as a means of reducing counterparty credit exposure
- Collateralized exposure and the margin period of risk
- Semi-analytical method for calculating collateralized EE

Margin agreements as a means of reducing counterparty credit exposure

Introduction

- ***Counterparty credit risk*** is the risk that a counterparty in an ***OTC*** derivative transaction will default prior to the expiration of the contract and will be unable to make all contractual payments.
  - ***Exchange-traded*** derivatives bear no counterparty risk.
- The primary feature that distinguishes counterparty risk from lending risk is the uncertainty of the exposure at any future date.
  - **Loan**: exposure at any future date is the outstanding balance, which is certain (not taking into account prepayments).
  - **Derivative**: exposure at any future date is the replacement cost, which is determined by the market value at that date and is, therefore, uncertain.
- Counterparty risk is ***bilateral*** because
  - derivative values can be both positive and negative
  - both counterparties can default

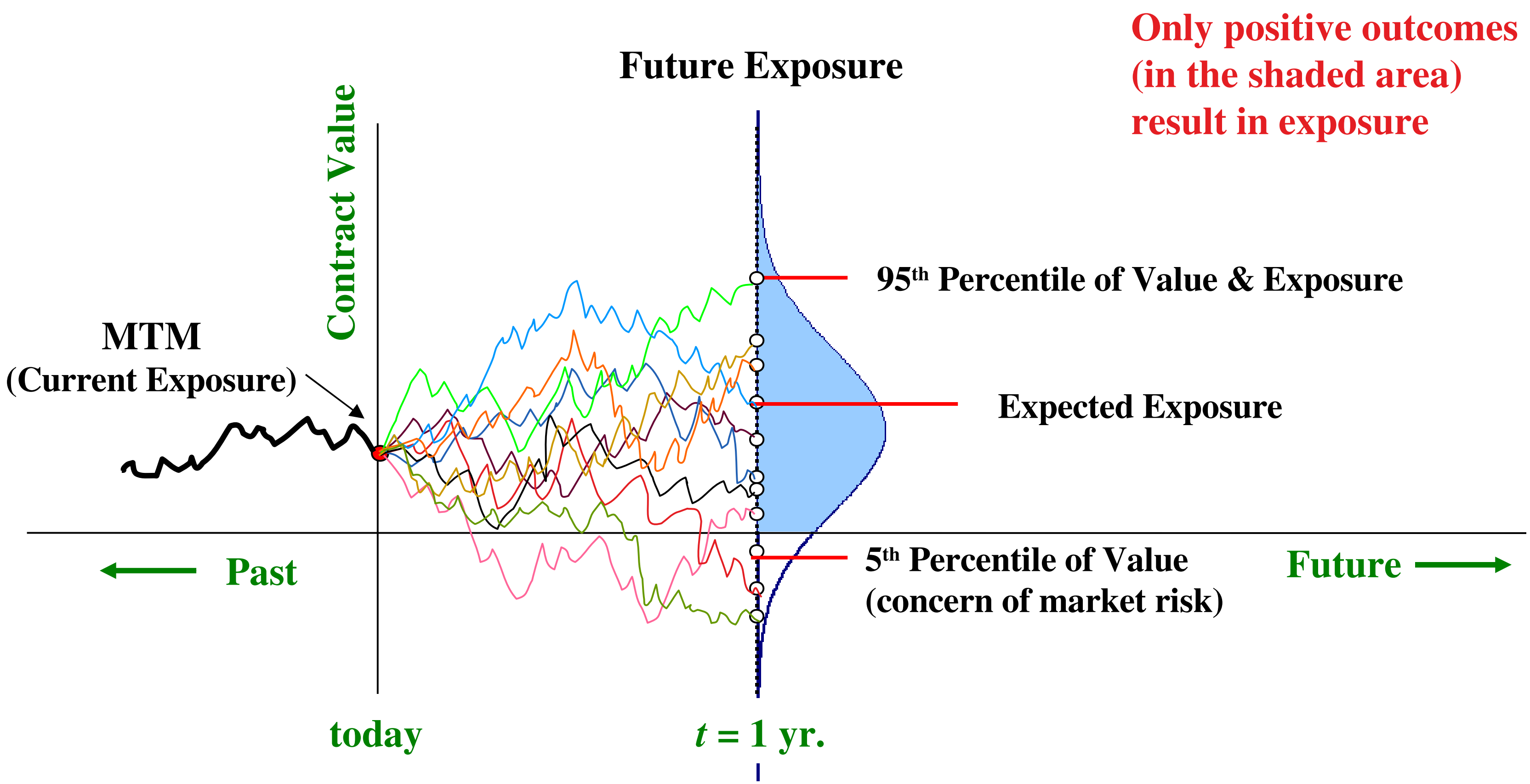
Exposure at Contract Level

- Assume that no netting or margin agreement is in place.
- Market value of contract *i* with a counterparty is known only for current date $t = 0$. For any future date *t*, this value $V_i(t)$ is uncertain and should be assumed random.
- If a counterparty defaults at time τ prior to contract maturity, economic loss is equal to the replacement cost of the contract
  - If $V_i(t) > 0$, we do not receive anything from defaulting counterparty, but have to pay $V_i(t)$ to another counterparty to replace the contract.
  - If $V_i(t) < 0$, we receive $|V_i(t)|$ from another counterparty, but have to forward this amount to the defaulting counterparty.
- Combining these two scenarios, we can specify ***contract-level exposure*** $E_i(t)$ at time *t* according to

$$E_i(t) = \max\{V_i(t), 0\}$$

Uncertainty of Future Exposure

- Future value and exposure are uncertain!



Exposure at Counterparty Level

- ***Counterparty-level exposure*** at future time *t* can be defined as the loss experienced by the bank if the counterparty defaults at time *t* under the assumption of no recovery
- If counterparty risk is not mitigated in any way, ***counterparty-level*** exposure equals the sum of ***contract-level*** exposures

$$E(t) = \sum_i E_i(t) = \sum_i \max\{V_i(t), 0\}$$

- If there are ***netting agreements***, derivatives with positive value at the time of default offset the ones with negative value within each netting set NS_k, so that ***counterparty-level exposure*** is

$$E(t) = \sum_k E_{\mathrm{NS}_k}(t) = \sum_k \max\left\{\sum_{i \in \mathrm{NS}_k} V_i(t), 0\right\}$$

  - Each non-nettable trade represents a netting set

Margin Agreements

- ***Margin agreements*** allow for further reduction of counterparty-level exposure.
- Margin agreement is a legally binding contract between two counterparties that requires one or both counterparties to post collateral under certain conditions:
  - A threshold is defined for one (unilateral agreement) or both (bilateral agreement) counterparties.
  - If the difference between the net portfolio value and already posted collateral exceeds the threshold, the counterparty must provide collateral sufficient to cover this excess (subject to minimum transfer amount).
- The threshold value depends primarily on the credit quality of the counterparty.

Exposure with Margin Agreements

- Assuming that some netting sets may be covered by margin agreements, we can write bank's exposure to the counterparty:

$$E_C(t) = \sum_{k} \max\left(\sum_{i \in \mathrm{NS}_k} V_i(t) - C_k(t), 0 \right)$$

where $C_k(t)$ is the market value of collateral available to the bank for netting set k at time t.

  – If NS_k is not covered by a margin agreement, then $C_k(t) \equiv 0$

- We assume the following sign convention:
  – $C_k(t) > 0$: at time t the bank holds collateral in the amount $|C_k(t)|$
  – $C_k(t) < 0$: at time t the bank has posted collateral in the amount $|C_k(t)|$
  – $C_k(t) = 0$: at time t the bank neither holds nor has posted collateral

Collateralized exposure and the margin period of risk

Unilateral Margin Agreement

- To simplify the notations, we will consider a single netting set:

$$E_C(t) = \max\{V(t) - C(t), 0\}$$

where $V(t)$ is the portfolio value for the netting set at time t :

$$V(t) = \sum_i V_i(t)$$

- Let's consider a ***unilateral*** margin agreement (in bank's favor) with threshold $H_{\text{cpt}} \geq 0$ and minimum transfer amount MTA.
- It is difficult to model collateral subject to MTA exactly because that would require ***daily*** simulation time points.
- In practice, the actual threshold H_{cpt} is often replaced with the effective threshold $H_{\text{cpt}}^{(\text{e})}$ defined as

$$H_{\text{cpt}}^{(\text{e})} = H_{\text{cpt}} + \text{MTA}$$

Naive Approach

- Collateral covers excess of portfolio value $V(t)$ over threshold $H_{\text{cpt}}^{(\text{e})}$

$$C(t) = \max\left\{ V(t) - H_{\text{cpt}}^{(\text{e})}, 0 \right\}$$

- Therefore, collateralized exposure is

$$E_C(t) = \max\left\{ V(t) - C(t), 0 \right\} = \begin{cases} 0 & \text{if} \quad V(t) \quad 0 \\ V(t) & \text{if} \quad 0 < V(t) \quad H_{\text{cpt}}^{(\text{e})} \\ H_{\text{cpt}}^{(\text{e})} & \text{if} \quad V(t) > H_{\text{cpt}}^{(\text{e})} \end{cases}$$

- Thus, ***any scenario*** of collateralized exposure is limited by the ***threshold*** from above and by ***zero*** from below.
- The problem with this approach is that it implicitly assumes that
  - collateral is delivered immediately
  - procedures of settling and replacing of trades start immediately when the required collateral is not posted

Margin Period of Risk

- Even with daily margin call frequency, there is a significant delay δt , known as the ***margin period of risk (MPR)***, between a margin call that the counterparty does not respond to and the start of the default procedures.
  - Margin calls can be disputed, and it may take several days for the bank to realize that the counterparty is defaulting rather than disputing the call
  - There is a grace period after the bank issues a notice of default. During this grace period the counterparty may still post collateral
- Thus, collateral available at time t is determined by portfolio value at time $t \quad \delta t$.
- While δt is not known with certainty, it is usually assumed to be a fixed number.
  - Assumed value of δt depends on margin call frequency and trade liquidity
  - Typical assumption for daily calls and liquid trades is $\delta t = 2$ weeks

Including MPR in the Model

- Suppose that at time $t-\delta t$ we have collateral $C(t-\delta t)$ and portfolio value is $V(t-\delta t)$
- Then, the amount $\Delta C(t)$ that should be posted by time t is

$$\Delta C(t) = \max\left\{ V(t-\delta t) - C(t-\delta t) - H_{\text{cpt}}^{(\text{e})}, -C(t-\delta t) \right\}$$

  - Negative $\Delta C(t)$ means that the bank will return collateral
- Collateral $C(t)$ available at time t is

$$C(t) = C(t-\delta t) + \Delta C(t) = \max\left\{ V(t-\delta t) - H_{\text{cpt}}^{(\text{e})}, 0 \right\}$$

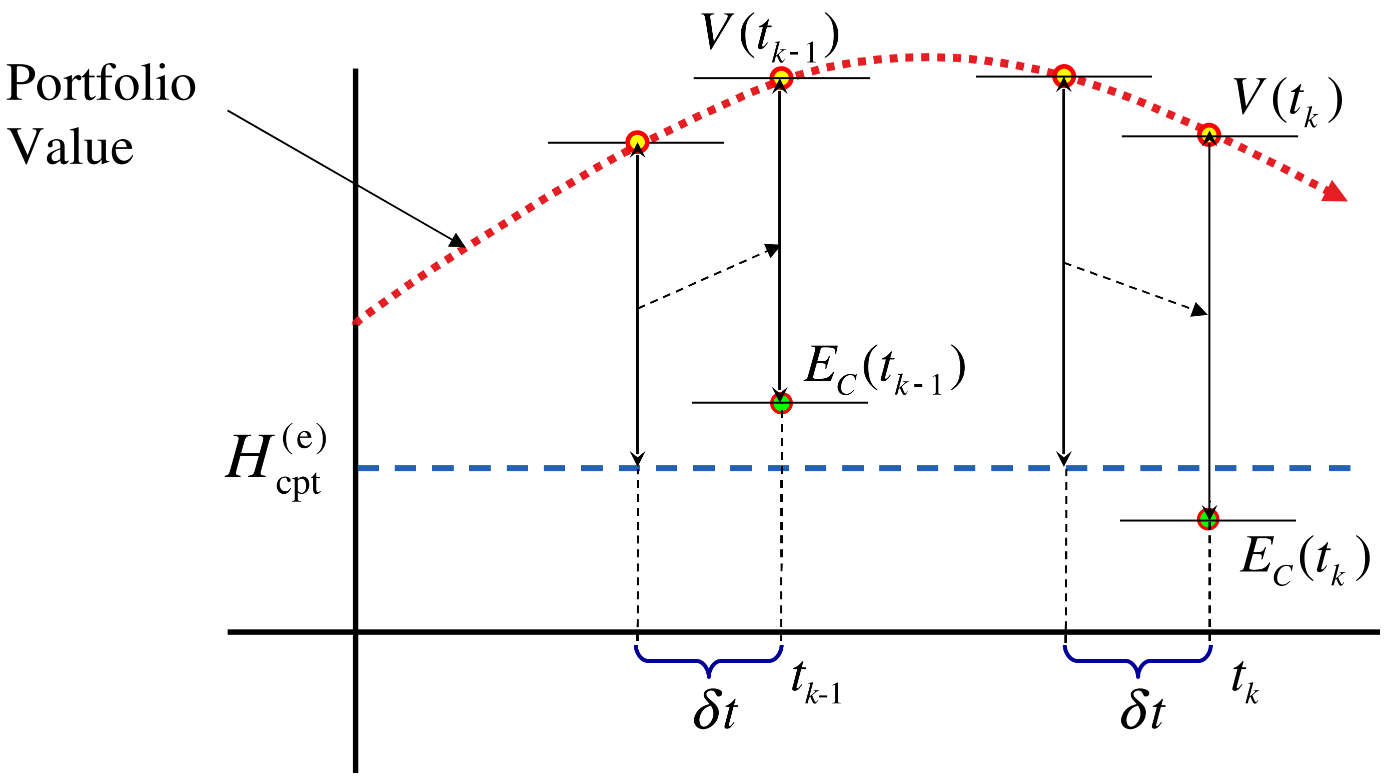
- For comparison, collateral under the "naive" model is

$$C_{\text{naive}}(t) = \max\left\{ V(t) - H_{\text{cpt}}^{(\text{e})}, 0 \right\}$$

- Thus, to determine collateralized exposure at time t, we need to simulate portfolio value both at $t-\delta t$ and at t.

Full Monte Carlo Method

- Simulating exposure for collateralized counterparty
  - Collateralized exposure can go above the threshold due to MPR



Bilateral Margin Agreement

- Under a ***bilateral*** margin agreement, both the counterparty and the bank have to post collateral.
- Two thresholds are defined: $H_{\text{cpt}} \quad 0$ and $H_{\text{bnk}} \quad 0$
  - H_{bnk} is negative because we value trades from the bank's perspective
  - Bank posts collateral when portfolio value falls below H_{bnk}
  - Recall that we treat collateral posted by bank as a negative amount
- Two effective thresholds are specified:

$$H_{\text{cpt}}^{(e)} = H_{\text{cpt}} + \text{MTA} \qquad H_{\text{bnk}}^{(e)} = H_{\text{bnk}} - \text{MTA}$$

- After effective thresholds are defined, the bilateral margin agreement is treated as if it had zero MTA.

Collateral and Exposure for Bilateral MA

- Collateral available to bank at time t is given by

$$C(t) = \max\left\{ V(t - dt) - H_{\text{cpt}}^{(\text{e})}, 0 \right\} + \min\left\{ V(t - dt) - H_{\text{bnk}}^{(\text{e})}, 0 \right\}$$

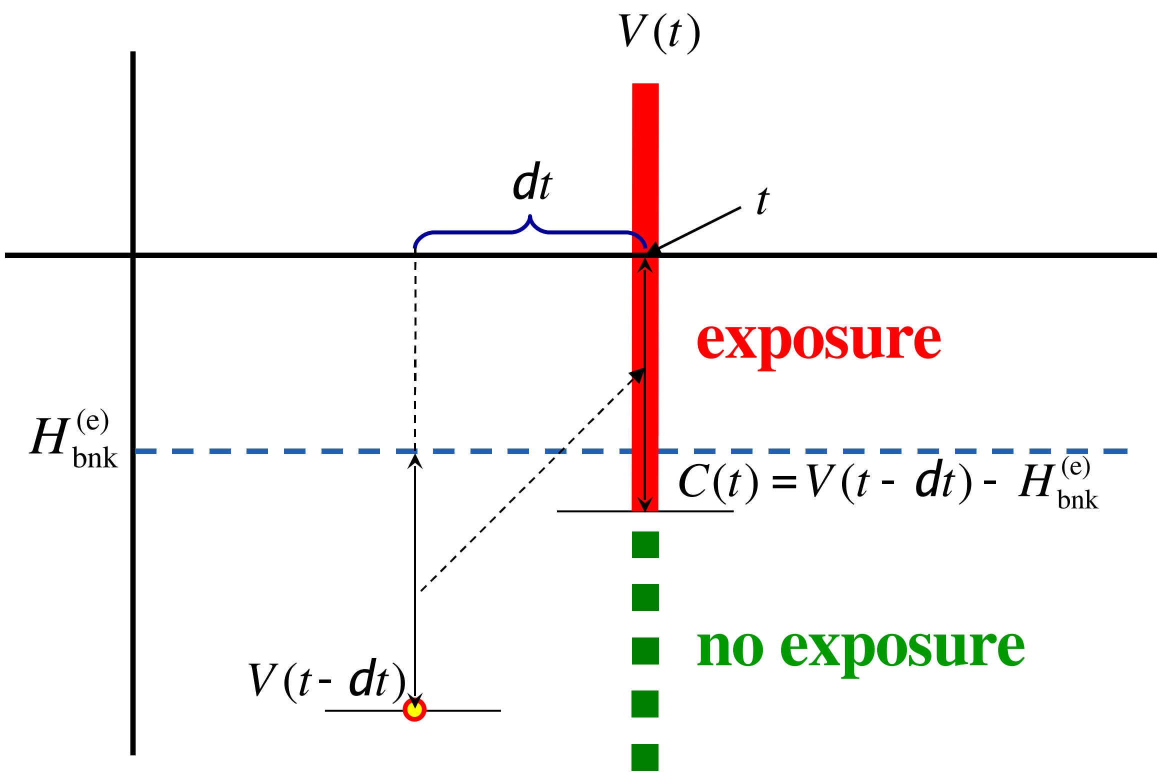
- The two terms above describe two types of future scenarios:
  - <u>First term</u>: the bank receives collateral $C(t) > 0$
  - <u>Second term</u>: the bank posts collateral $C(t) < 0$
- Note that both terms cannot be non-zero simultaneously!
- Bank's exposure to counterparty is still given by

$$E_C(t) = \max\left\{ V(t) - C(t), 0 \right\}$$

- If the counterparty defaults when the bank has posted collateral, is there any credit exposure for the bank?

Exposure from Posting Collateral

- When the bank posts collateral, it can experience loss if the portfolio value increases by more than $|H^{(e)}_{bnk}|$ over the MPR δt



$$E_C(t) = \max\{V(t) - C(t), 0\}$$

Semi-analytical method for collateralized EE

Portfolio Value at Primary Time Points

- Let us assume that we have run simulation ***only*** for primary time points t and obtained portfolio value distribution in the form of M quantities $V^{(j)}(t)$, where j (from 1 to M) designates different scenarios

- From the set $\{V^{(j)}(t)\}$ we can estimate the unconditional expectation $\mu(t)$ and standard deviation $\sigma(t)$ of the portfolio value, as well as any other distributional parameter

- Can we estimate collateralized EE profile ***without*** simulating portfolio value at the look-back time points $\{V^{(j)}(t - dt)\}$?

Collateralized EE Conditional on Scenario

- Collateralized EE can be represented as

$$\mathrm{EE}_C(t) = \mathrm{E}[\mathrm{EE}_C^{(j)}(t)]$$

where $\mathrm{EE}_C^{(j)}(t)$ is the collateralized EE ***conditional*** on $V^{(j)}(t)$:

$$\mathrm{EE}_C^{(j)}(t) = \mathrm{E}\left[\max\{V_C^{(j)}(t), 0\} \,\middle|\, V^{(j)}(t)\right]$$

where $V_C^{(j)}(t)$ is the ***collateralized portfolio value*** defined as

$$V_C^{(j)}(t) = V^{(j)}(t) - C^{(j)}(t)$$

- If we can calculate $\mathrm{EE}_C^{(j)}(t)$ analytically, the ***unconditional collateralized EE*** can be obtained as the simple average of $\mathrm{EE}_C^{(j)}(t)$ over all scenarios j :

$$\mathrm{EE}_C(t) = \frac{1}{M}\sum_{j=1}^{M} \mathrm{EE}_C^{(j)}(t)$$

If Portfolio Value Were Normal…

- Let us assume that portfolio value $V(t)$ at time t is normally distributed with expectation $\mu(t)$ and standard deviation $\sigma(t)$.
- Then, we can construct ***Brownian bridge*** from $V(0)$ to $V^{(j)}(t)$
- Conditionally on $V^{(j)}(t)$, $V^{(j)}(t-\delta t)$ has ***normal distribution*** with ***expectation***

$$a^{(j)}(t) = \frac{\delta t}{t} V(0) + \frac{t-\delta t}{t} V^{(j)}(t)$$

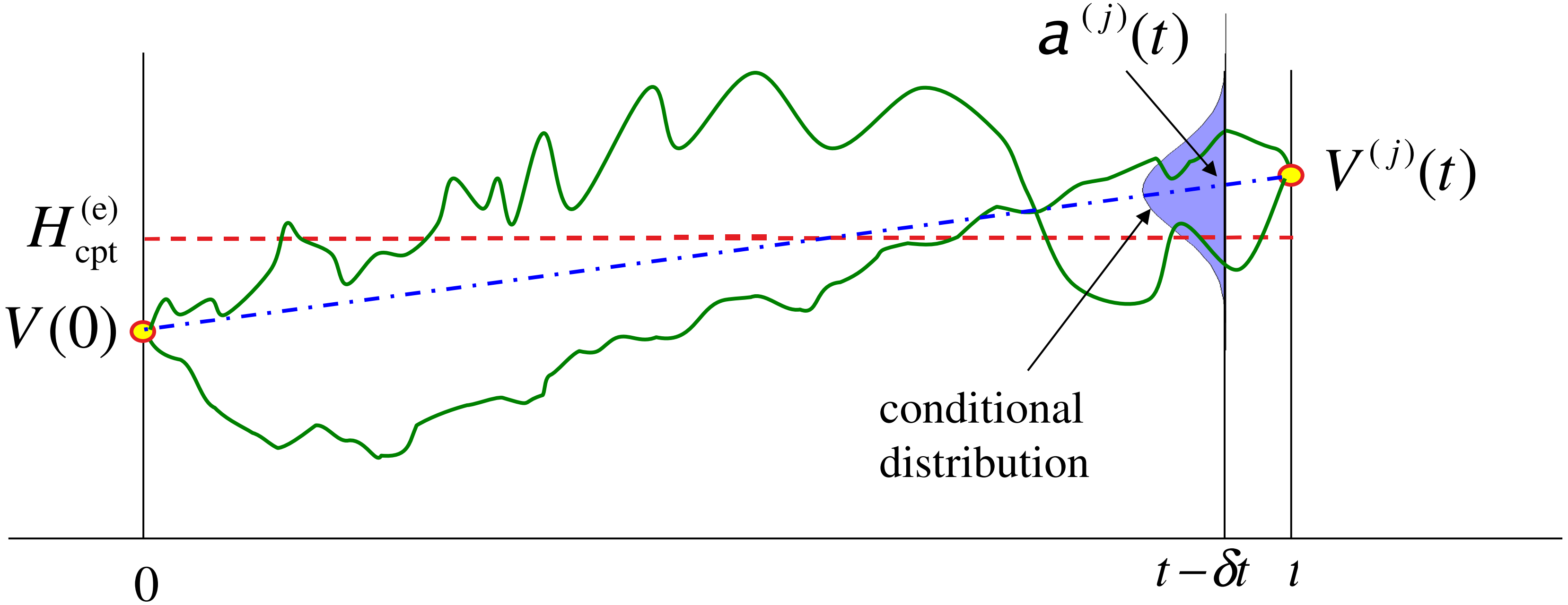
and ***standard deviation***

$$b(t) = \sigma(t) \sqrt{\frac{\delta t\,(t-\delta t)}{t^2}}$$

- ***Conditional collateralized EE*** can be obtained in closed form by integrating over a single normal variable!

Illustration: Brownian Bridge

- Brownian bridge from $V(0)$ to $V^{(j)}(t)$



- Conditionally on $V^{(j)}(t)$, the distribution of $V^{(j)}(t-dt)$ is normal with mean $a^{(j)}(t)$ and standard deviation $b(t)$

Arbitrary Portfolio Value Distribution

- We will keep the assumption that, conditionally on $V^{(j)}(t)$, the distribution of $V^{(j)}(t-dt)$ is normal, but will replace $\sigma(t)$ with a local quantity $\sigma_{\text{loc}}(t)$
- Let us describe portfolio value $V(t)$ at time t as

$$V(t) = v(t, Z)$$

  where $v(t, Z)$ is a monotonically increasing function of a standard normal random variable Z.

- Let us also define a ***normal equivalent*** portfolio value as

$$W(t) = w(t, Z) = m(t) + s(t)Z$$

- To obtain $\sigma_{\text{loc}}(t)$, we will scale $\sigma(t)$ by the ratio of probability densities of $W(t)$ and $V(t)$

Scaled Standard Deviation

- Let us denote probability density of quantity X via $f_X(\,)$ and scale the standard deviation according to

$$\sigma_{\mathrm{loc}}(t,Z) = \frac{f_{W(t)}[w(t,Z)]}{f_{V(t)}[v(t,Z)]}\sigma(t)$$

- Changing variables from $W(t)$ and $V(t)$ to Z, we have

$$f_{V(t)}[v(t,Z)] = \frac{\varphi(Z)}{v(t,Z)/\;Z} \qquad f_{W(t)}[w(t,Z)] = \frac{\varphi(Z)}{\sigma(t)}$$

- Substitution to the definition of $\sigma_{\mathrm{loc}}(t,Z)$ above gives

$$\sigma_{\mathrm{loc}}(t,Z) = \frac{v(t,Z)}{Z}$$

Estimating CDF

- Value of $Z^{(j)}$ corresponding to $V^{(j)}(t)$ can be obtained from

$$Z^{(j)} = \mathrm{F}^{-1}\left(F_{V(t)}[V^{(j)}(t)] \right)$$

- Let us sort the array $V^{(j)}(t)$ in the increasing order so that

$$V^{[j(k)]}(t) = V_{\text{sorted}}^{(k)}(t)$$

where $j(k)$ is the sorting index

- From the sorted array we can build a piece-wise constant CDF that jumps by $1/M$ as $V(t)$ crosses any of the simulated values:

$$F_{V(t)}[V^{[j(k)]}(t)] \quad \frac{1}{2}\frac{k-1}{M} + \frac{1}{2}\frac{k}{M} = \frac{2k-1}{2M}$$

Estimating Derivative

- Now we can obtain $Z^{(j)}$ corresponding to $V^{(j)}(t)$ as

$$Z^{[j(k)]} = \Phi^{-1} \quad \frac{2k-1}{2M}$$

- Local standard deviation $\sigma_{\text{loc}}^{(j)}(t)$ can be estimated as :

$$\sigma_{\text{loc}}^{[j(k)]}(t) \quad \sigma_{\text{loc}}(t, Z^{[j(k)]}) \quad \frac{V^{[j(k+\Delta k)]}(t) - V^{[j(k-\Delta k)]}(t)}{Z^{[j(k+\Delta k)]} - Z^{[j(k-\Delta k)]}}$$

- Offset $\quad k$ should not be too small (too much noise) or too large (loss of "locality"). This range seems to work very well:

$$20 \quad \Delta k \quad 0.05M$$

Back to the Bridge

- We assume that, conditionally on $V^{(j)}(t)$, $V^{(j)}(t - dt)$ has ***normal distribution*** with ***expectation***

$$a^{(j)}(t) = \frac{dt}{t} V(0) + \frac{t - dt}{t} V^{(j)}(t)$$

and ***standard deviation***

$$b^{(j)}(t) = s_{\text{loc}}^{(j)}(t) \sqrt{\frac{dt\,(t - dt)}{t^2}}$$

- ***Collateralized exposure*** depends on $dV^{(j)}(t) = V^{(j)}(t) - V^{(j)}(t - dt)$ which is also normal conditionally on $V^{(j)}(t)$ with the same standard deviation $b^{(j)}(t)$ and expectation $da^{(j)}(t)$ given by

$$da^{(j)}(t) = V^{(j)}(t) - a^{(j)}(t) = \frac{dt}{t} \; V^{(j)}(t) - V(0)$$

Unilateral MA: Conditional Exposure

- Collateral available at time t conditional on scenario j is

$$C^{(j)}(t) = \max\left\{ V^{(j)}(t - dt) - H_{\text{cpt}}^{(\text{e})}, 0 \right\}$$

- Conditional collateralized portfolio value at time t is

$$V_C^{(j)}(t) = V^{(j)}(t) - C^{(j)}(t) = \min\left\{ V^{(j)}(t), H_{\text{cpt}}^{(\text{e})} + dV^{(j)}(t) \right\}$$

- Conditional collateralized exposure at time t is

$$E_C^{(j)}(t) = \max\left(\min\left\{ V^{(j)}(t), H_{\text{cpt}}^{(\text{e})} + dV^{(j)}(t) \right\}, 0 \right)$$

$$= 1_{\left\{ V^{(j)}(t) > 0 \right\}} \min\left\{ V^{(j)}(t), [H_{\text{cpt}}^{(\text{e})} + dV^{(j)}(t)]^+ \right\}$$

Unilateral MA: Conditional EE

- Evaluating the conditional expectation, we obtain:

$$\mathrm{EE}_C^{(j)}(t) = 1_{\{V^{(j)}(t)>0\}} \Big\{ \quad H_{\mathrm{cpt}}^{(\mathrm{e})} + da^{(j)}(t) \quad \mathsf{F}\left(d_{\mathrm{cpt}}^{(2)}\right) - \mathsf{F}\left(d_{\mathrm{cpt}}^{(1)}\right)$$

$$+ b^{(j)}(t) \;\; f\left(d_{\mathrm{cpt}}^{(2)}\right) - f\left(d_{\mathrm{cpt}}^{(1)}\right) \quad + V^{(j)}(t)\mathsf{F}\left(d_{\mathrm{cpt}}^{(1)}\right)\Big\}$$

where $\mathsf{F}(\;)$ and $f(\;)$ are the CDF and the density of the standard normal distribution, respectively.

- Quantities $d_a^{(1)}$ and $d_a^{(2)}$ (where a can be either cpt or bnk) are defined according to

$$d_a^{(1)} = \frac{H_a^{(\mathrm{e})} + da^{(j)}(t) - V^{(j)}(t)}{b^{(j)}(t)} \qquad\qquad d_a^{(2)} = \frac{H_a^{(\mathrm{e})} + da^{(j)}(t)}{b^{(j)}(t)}$$

Bilateral MA: Conditional Exposure

- Collateral available at time t conditional on scenario j is

$$C^{(j)}(t) = \max\left\{ V^{(j)}(t - dt) - H_{\text{cpt}}^{(\text{e})}, 0 \right\} + \min\left\{ V^{(j)}(t - dt) - H_{\text{bnk}}^{(\text{e})}, 0 \right\}$$

- Conditional collateralized portfolio value at time t is

$$V_C^{(j)}(t) = \begin{cases} H_{\text{cpt}}^{(\text{e})} + dV^{(j)}(t) & \text{if } dV^{(j)}(t) < V^{(j)}(t) - H_{\text{cpt}}^{(\text{e})} \\ V^{(j)}(t) & \text{if } V^{(j)}(t) - H_{\text{cpt}}^{(\text{e})} \quad dV(t) \quad V^{(j)}(t) - H_{\text{bnk}}^{(\text{e})} \\ H_{\text{bnk}}^{(\text{e})} + dV^{(j)}(t) & \text{if } dV^{(j)}(t) > V^{(j)}(t) - H_{\text{bnk}}^{(\text{e})} \end{cases}$$

- Conditional collateralized exposure at time t is

$$E_C^{(j)}(t) = \max\left\{ V_C^{(j)}(t), 0 \right\}$$

Bilateral MA: Conditional EE

- Evaluating the conditional expectation, we obtain:

$$\mathrm{EE}_C^{(j)}(t) = 1_{\left\{V^{(j)}(t)>0\right\}}\ \mathrm{EE}_C^{(j+)}(t) + 1_{\left\{V^{(j)}(t)\ \ 0\right\}}\ \mathrm{EE}_C^{(j-)}(t)$$

where $\mathrm{EE}_C^{(j+)}(t)$ and $\mathrm{EE}_C^{(j-)}(t)$ are given by

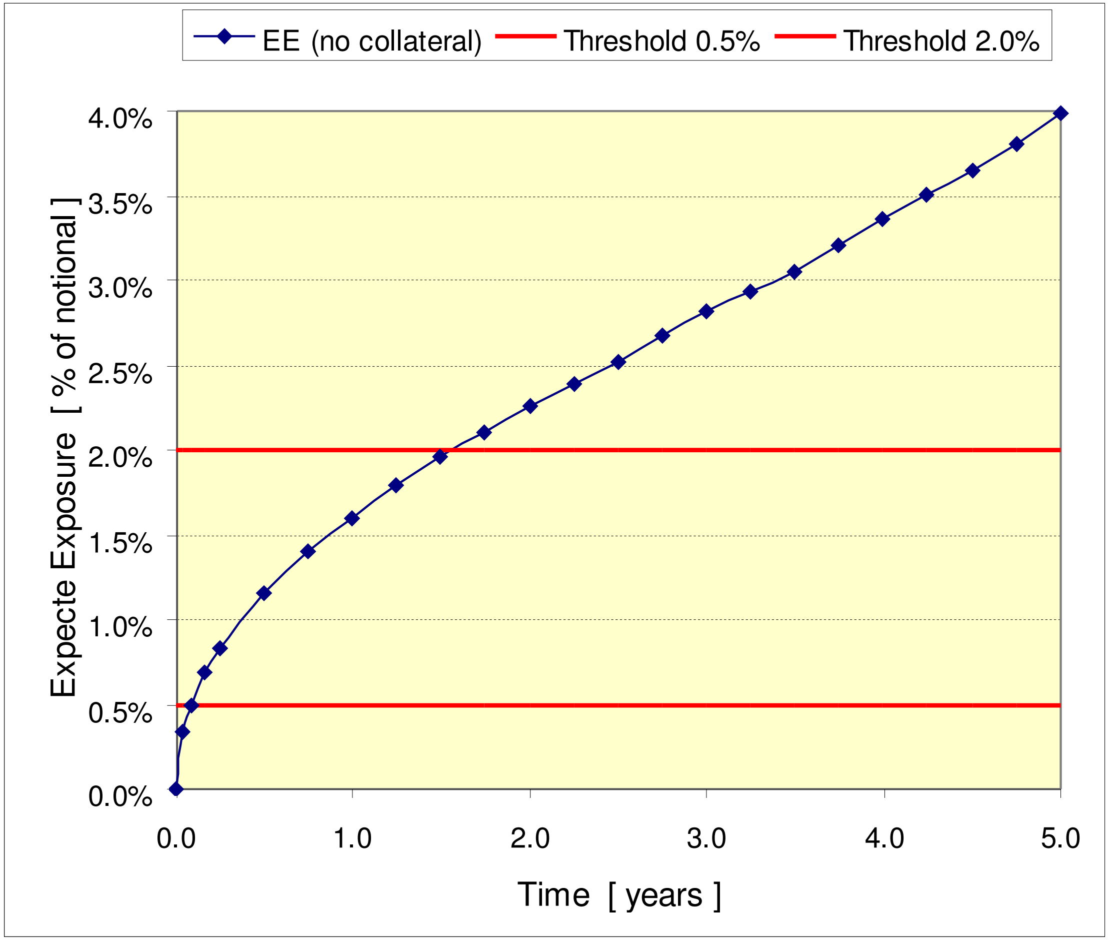
$$\begin{aligned}\mathrm{EE}_C^{(j+)}(t) = \ & H_{\mathrm{cpt}}^{(\mathrm{e})} + da^{(j)}(t) \quad F\left(d_{\mathrm{cpt}}^{(2)}\right) - F\left(d_{\mathrm{cpt}}^{(1)}\right) \\ & + b^{(j)}(t)\ \ f\left(d_{\mathrm{cpt}}^{(2)}\right) - f\left(d_{\mathrm{cpt}}^{(1)}\right) \ \ + V^{(j)}(t)\ \ F\left(d_{\mathrm{cpt}}^{(1)}\right) - F\left(d_{\mathrm{bnk}}^{(1)}\right) \\ & + \ \ H_{\mathrm{bnk}}^{(\mathrm{e})} + da^{(j)}(t)\ \ F\left(d_{\mathrm{bnk}}^{(1)}\right) + b^{(j)}(t) f\left(d_{\mathrm{bnk}}^{(1)}\right)\end{aligned}$$

and

$$\mathrm{EE}_C^{(j-)}(t) = \ \ H_{\mathrm{bnk}}^{(\mathrm{e})} + da^{(j)}(t)\ \ F\left(d_{\mathrm{bnk}}^{(2)}\right) + b^{(j)}(t) f\left(d_{\mathrm{bnk}}^{(2)}\right)$$

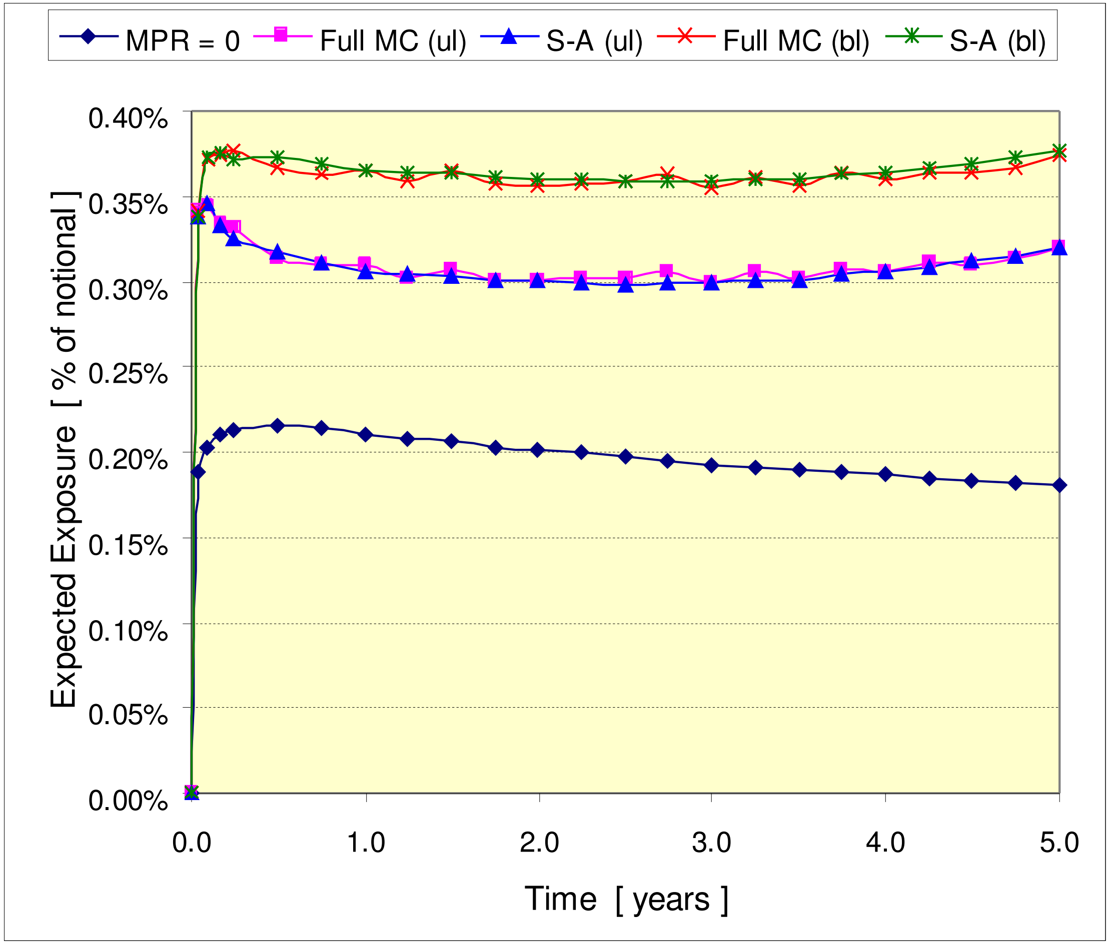
Example 1: 5-Year IR Swap Starting in 5 Years

- ***Uncollateralized EE*** and the ***two thresholds*** we will consider



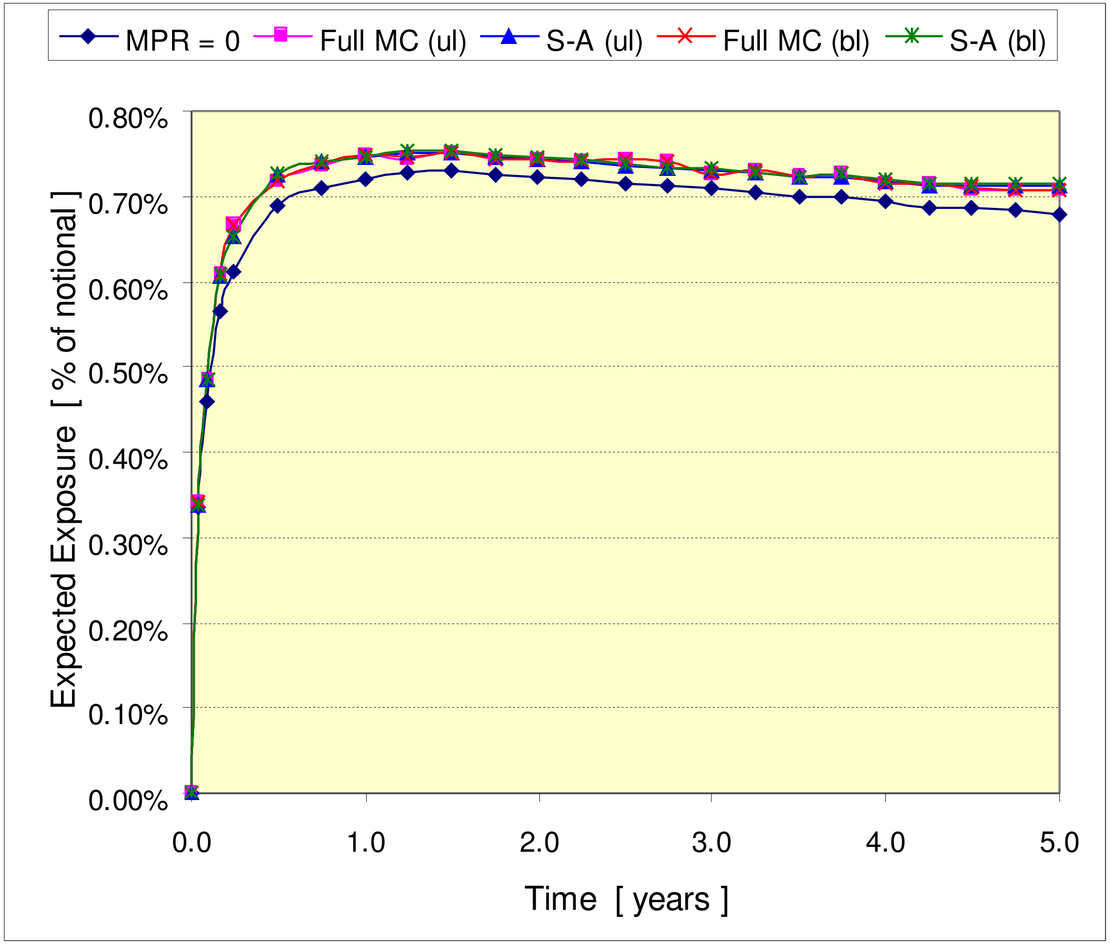
Forward Starting Swap and Small Threshold

- ***Collateralized EE*** when threshold is **0.5%**



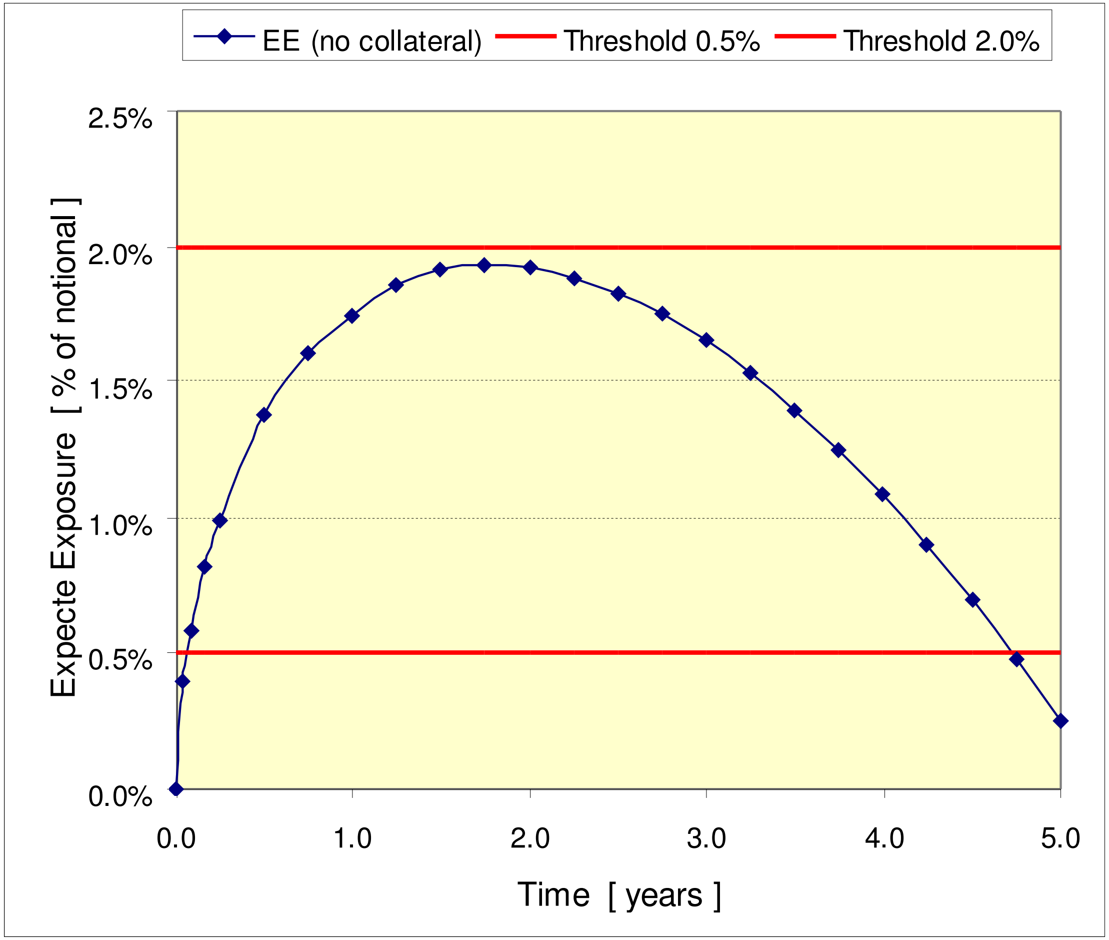
Forward Starting Swap and Large Threshold

- ***Collateralized EE*** when threshold is **2.0%**



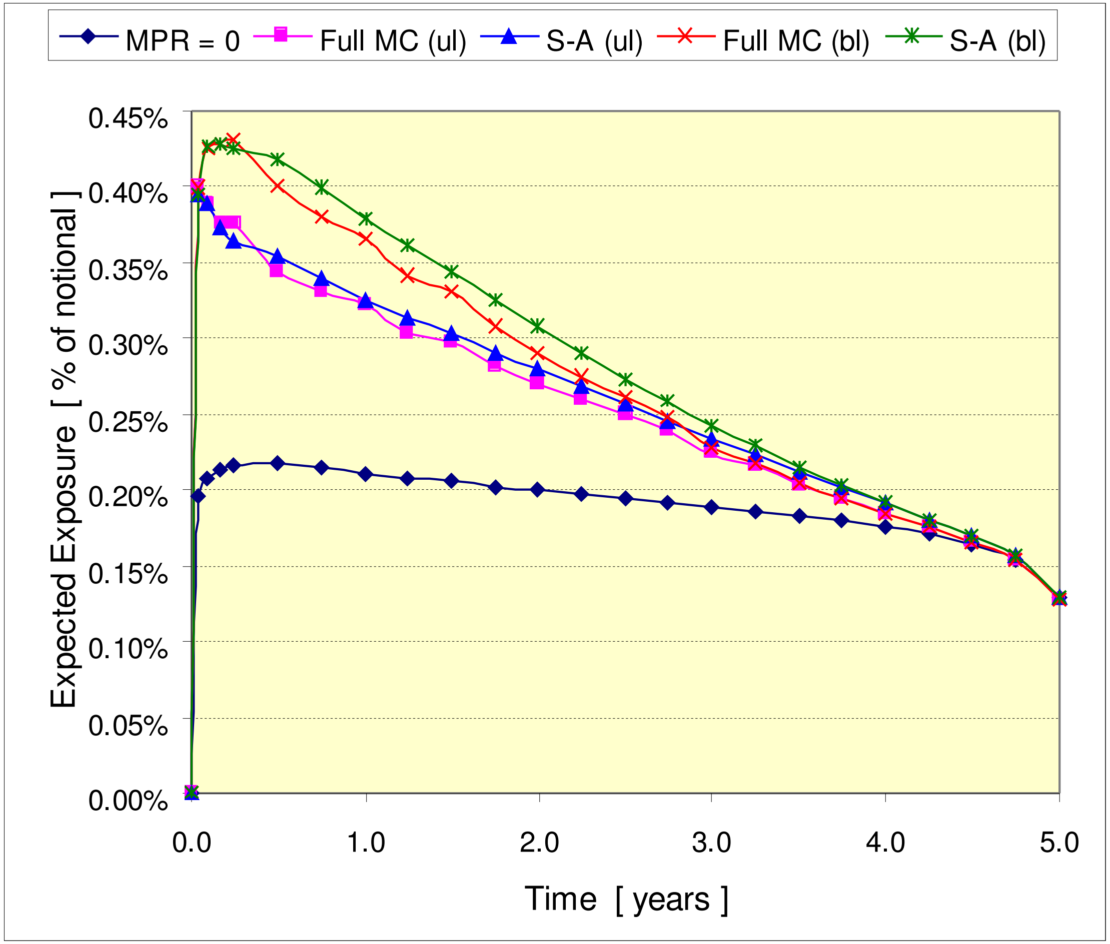
Example 2: 5-Year IR Swap Starting Now

- ***Uncollateralized EE*** and the ***two thresholds*** we will consider



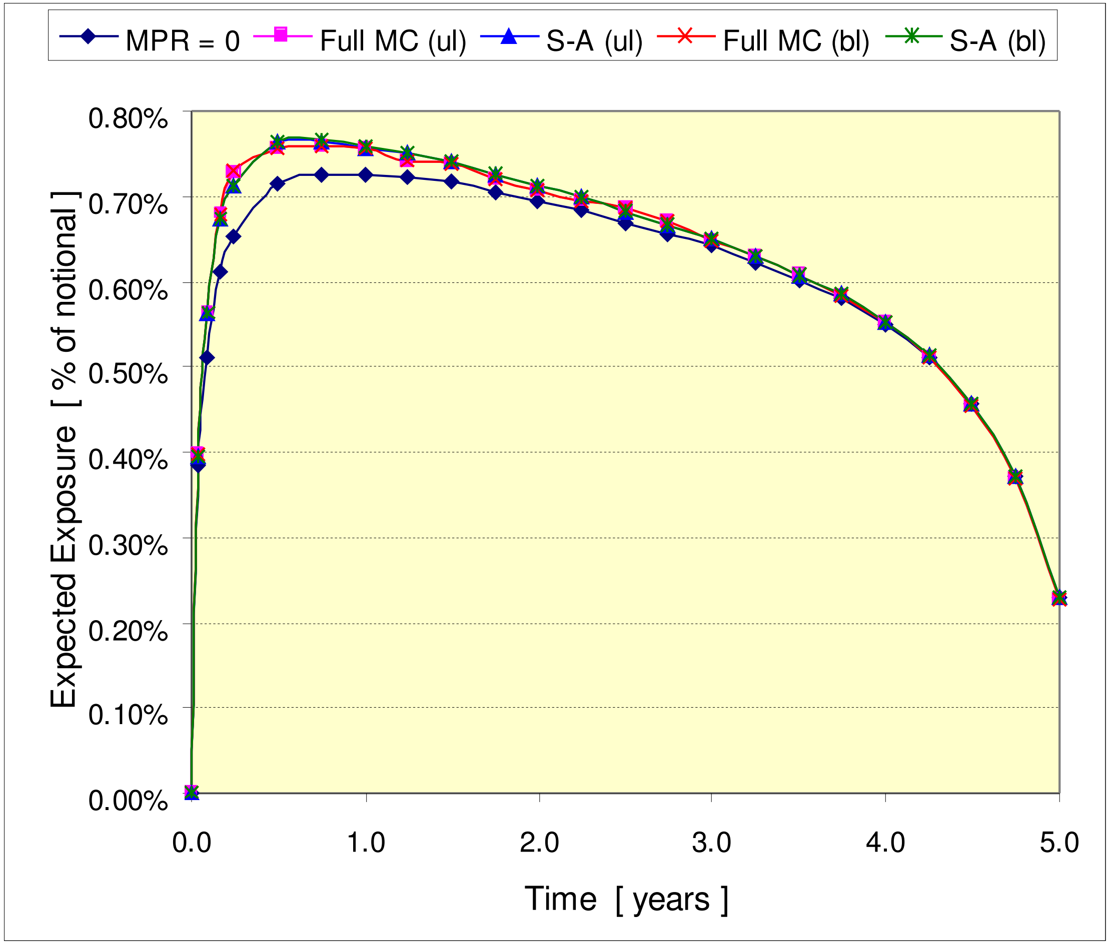
Swap Starting Now and Small Threshold

- ***Collateralized EE*** when threshold is **0.5%**



Swap Starting Now and Large Threshold

- ***Collateralized EE*** when threshold is **2.0%**



Conclusion

- ***Margin agreements*** are important risk mitigation tools that need to be modeled accurately
- ***Collateral*** available at a primary time point depends on the portfolio value at the corresponding look-back time point
- ***Full Monte Carlo*** is the most flexible approach, but it requires simulating trade values at both primary and look-back time points
  - Simulation time is doubled in comparison to non-margined counterparties
- We have developed a ***semi-analytical*** method of calculating collateralized EE that avoids doubling the simulation time
  - Portfolio value is simulated only at primary time points
  - For each portfolio value scenario at a primary time point, conditional collateralized EE is calculated in closed form
  - Unconditional collateralized EE at a primary time point is obtained by averaging the conditional collateralized EE over all scenarios