# Valuation of Credit Default Swaptions and Credit Default Index Swaptions

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# Outline

# References on Valuation of Credit Default Swaptions

- D. Brigo and M. Morini: CDS market formulas and models. Working paper, Banca IMI, 2005.
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- M. Rutkowski and A. Armstrong: Valuation of credit default swaptions and credit default index swaptions. Working paper, UNSW, 2007.

# References on Modelling of CDS Spreads

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# References on Hedging of Credit Default Swaptions

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- T. Bielecki, M. Jeanblanc and M. Rutkowski: Pricing and trading credit default swaps in a hazard process model. *Annals of Applied Probability* 18 (2008), 2495-2529.
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# Credit Default Swaptions

# Hazard Process Set-up

Terminology and notation:

1. The default time is a strictly positive random variable $\tau$ defined on the underlying probability space $(\Omega, \mathcal{G}, \mathbb{P})$.
2. We define the default indicator process $H_t = \mathbb{1}_{\{\tau \leq t\}}$ and we denote by $\mathbb{H}$ its natural filtration.
3. We assume that we are given, in addition, some auxiliary filtration $\mathbb{F}$ and we write $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$, meaning that $\mathcal{G}_t = \sigma(\mathcal{H}_t, \mathcal{F}_t)$ for every $t \in \mathbb{R}_+$.
4. The filtration $\mathbb{F}$ is termed the reference filtration.
5. The filtration $\mathbb{G}$ is called the full filtration.

# Martingale Measure

The underlying market model is arbitrage-free, in the following sense:

1. Let the savings account $B$ be given by

$$B_t = \exp\left(\int_0^t r_u\, du\right), \quad \forall\, t \in \mathbb{R}_+,$$

   where the short-term rate $r$ follows an $\mathbb{F}$-adapted process.
2. A spot martingale measure $\mathbb{Q}$ is associated with the choice of the savings account $B$ as a numéraire.
3. The underlying market model is arbitrage-free, meaning that it admits a spot martingale measure $\mathbb{Q}$ equivalent to $\mathbb{P}$. Uniqueness of a martingale measure is not postulated.

# Hazard Process

Let us summarize the main features of the hazard process approach:

1. Let us denote by
$$G_t = \mathbb{Q}(\tau > t \,|\, \mathcal{F}_t)$$
the survival process of $\tau$ with respect to the reference filtration $\mathbb{F}$. We postulate that $G_0 = 1$ and $G_t > 0$ for every $t \in [0, T]$.
2. We define the hazard process $\Gamma = -\ln G$ of $\tau$ with respect to the filtration $\mathbb{F}$.
3. For any $\mathbb{Q}$-integrable and $\mathcal{F}_T$-measurable random variable $Y$, the following classic formula is valid
$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{T<\tau\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}} \, G_t^{-1} \mathbb{E}_{\mathbb{Q}}(G_T Y \,|\, \mathcal{F}_t).$$

# Default Intensity

1. Assume that the supermartingale $G$ is continuous.
2. We denote by $G = \mu - \nu$ its Doob-Meyer decomposition.
3. Let the increasing process $\nu$ be absolutely continuous, that is, $d\nu_t = \upsilon_t \, dt$ for some $\mathbb{F}$-adapted and non-negative process $\upsilon$.
4. Then the process $\lambda_t = G_t^{-1} \upsilon_t$ is called the $\mathbb{F}$-intensity of default time.

## Lemma

*The process M, given by the formula*

$$M_t = H_t - \int_0^{t \wedge \tau} \lambda_u \, du = H_t - \int_0^t (1 - H_u) \lambda_u \, du,$$

*is a* $(\mathbb{Q}, \mathbb{G})$*-martingale.*

# Defaultable Claim

A generic defaultable claim $(X, A, Z, \tau)$ consists of:

1. A promised contingent claim $X$ representing the payoff received by the holder of the claim at time $T$, if no default has occurred prior to or at maturity date $T$.
2. A process $A$ representing the dividends stream prior to default.
3. A recovery process $Z$ representing the recovery payoff at time of default, if default occurs prior to or at maturity date $T$.
4. A random time $\tau$ representing the default time.

**Definition**

The dividend process $D$ of a defaultable claim $(X, A, Z, \tau)$ maturing at $T$ equals, for every $t \in [0, T]$,

$$D_t = X \mathbb{1}_{\{\tau > T\}} \mathbb{1}_{[T,\infty[}(t) + \int_{]0,t]} (1 - H_u)\, dA_u + \int_{]0,t]} Z_u\, dH_u .$$

# Ex-dividend Price

Recall that:

- The process $B$ represents the savings account.
- A probability measure $\mathbb{Q}$ is a spot martingale measure.

## Definition

The ex-dividend price $S$ associated with the dividend process $D$ equals, for every $t \in [0, T]$,

$$S_t = B_t \, \mathbb{E}_{\mathbb{Q}} \Big( \int_{]t,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t \Big) = \mathbb{1}_{\{t<\tau\}} \, \widetilde{S}_t$$

where $\mathbb{Q}$ is a spot martingale measure.

- The ex-dividend price represents the (market) value of a defaultable claim.
- The $\mathbb{F}$-adapted process $\widetilde{S}$ is termed the pre-default value.

# Valuation Formula

**Lemma**

*The value of a defaultable claim* $(X, A, Z, \tau)$ *maturing at* $T$ *equals*

$$S_t = \mathbb{1}_{\{t<\tau\}} \frac{B_t}{G_t} \mathbb{E}_{\mathbb{Q}}\left( B_T^{-1} G_T X \mathbb{1}_{\{t<T\}} + \int_t^T B_u^{-1} G_u Z_u \lambda_u \, du + \int_t^T B_u^{-1} G_u \, dA_u \,\Big|\, \mathcal{F}_t \right)$$

*where* $\mathbb{Q}$ *is a martingale measure.*

- Recall that $\mu$ is the martingale part in the Doob-Meyer decomposition of $G$.
- Let $m$ be the $(\mathbb{Q}, \mathbb{F})$-martingale given by the formula

$$m_t = \mathbb{E}_{\mathbb{Q}}\left( B_T^{-1} G_T X + \int_0^T B_u^{-1} G_u Z_u \lambda_u \, du + \int_0^T B_u^{-1} G_u \, dA_u \,\Big|\, \mathcal{F}_t \right).$$

# Price Dynamics

## Proposition

*The dynamics of the value process* $S$ *on* $[0, T]$ *are*

$$\begin{aligned} dS_t = -S_{t-}\, dM_t + (1 - H_t)\big((r_t S_t - \lambda_t Z_t)\, dt + dA_t\big) \\ + (1 - H_t) G_t^{-1} \big(B_t\, dm_t - S_t\, d\mu_t\big) + (1 - H_t) G_t^{-2} \big(S_t\, d\langle \mu \rangle_t - B_t\, d\langle \mu, m \rangle_t\big). \end{aligned}$$

*The dynamics of the pre-default value* $\widetilde{S}$ *on* $[0, T]$ *are*

$$\begin{aligned} d\widetilde{S}_t &= \big((\lambda_t + r_t)\widetilde{S}_t - \lambda_t Z_t\big)\, dt + dA_t + G_t^{-1} \big(B_t\, dm_t - \widetilde{S}_t\, d\mu_t\big) \\ &\quad + G_t^{-2} \big(\widetilde{S}_t\, d\langle \mu \rangle_t - B_t\, d\langle \mu, m \rangle_t\big). \end{aligned}$$

# Forward Credit Default Swap

## Definition

A forward CDS issued at time $s$, with start date $U$, maturity $T$, and recovery at default is a defaultable claim $(0, A, Z, \tau)$ where

$$dA_t = -\kappa \mathbb{1}_{]U,T]}(t)\, dL_t, \quad Z_t = \delta_t \mathbb{1}_{[U,T]}(t).$$

- An $\mathcal{F}_s$-measurable rate $\kappa$ is the CDS rate.
- An $\mathbb{F}$-adapted process $L$ specifies the tenor structure of fee payments.
- An $\mathbb{F}$-adapted process $\delta : [U, T] \to \mathbb{R}$ represents the default protection.

## Lemma

*The value of the forward CDS equals, for every $t \in [s, U]$,*

$$S_t(\kappa) = B_t\, \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{U<\tau\le T\}} B_\tau^{-1} Z_\tau \,\Big|\, \mathcal{G}_t\Big) - \kappa\, B_t\, \mathbb{E}_{\mathbb{Q}}\Big( \int_{]t\wedge U, \tau\wedge T]} B_u^{-1}\, dL_u \,\Big|\, \mathcal{G}_t\Big).$$

# Valuation of a Forward CDS

**Lemma**

*The value of a credit default swap started at s, equals, for every $t \in [s, U]$,*

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \frac{B_t}{G_t} \, \mathbb{E}_{\mathbb{Q}} \left( - \int_U^T B_u^{-1} \delta_u \, dG_u - \kappa \int_{]U,T]} B_u^{-1} G_u \, dL_u \,\middle|\, \mathcal{F}_t \right).$$

Note that $S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \widetilde{S}_t(\kappa)$ where the $\mathbb{F}$-adapted process $\widetilde{S}(\kappa)$ is the pre-default value. Moreover

$$\widetilde{S}_t(\kappa) = \widetilde{P}(t, U, T) - \kappa \widetilde{A}(t, U, T)$$

where

- $\widetilde{P}(t, U, T)$ is the pre-default value of the protection leg,
- $\widetilde{A}(t, U, T)$ is the pre-default value of the fee leg per one unit of $\kappa$.

# Forward CDS Rate

- The forward CDS rate is defined similarly as the forward swap rate for a default-free interest rate swap.

## Definition

The forward market CDS at time $t \in [0, U]$ is the forward CDS in which the $\mathcal{F}_t$-measurable rate $\kappa$ is such that the contract is valueless at time $t$.

The corresponding pre-default forward CDS rate at time $t$ is the unique $\mathcal{F}_t$-measurable random variable $\kappa(t, U, T)$, which solves the equation

$$\widetilde{S}_t(\kappa(t, U, T)) = 0.$$

- Recall that for any $\mathcal{F}_t$-measurable rate $\kappa$ we have that

$$\widetilde{S}_t(\kappa) = \widetilde{P}(t, U, T) - \kappa \widetilde{A}(t, U, T).$$

# Forward CDS Rate

## Lemma

*For every* $t \in [0, U]$*,*

$$\kappa(t,U,T) = \frac{\widetilde{P}(t,U,T)}{\widetilde{A}(t,U,T)} = -\frac{\mathbb{E}_{\mathbb{Q}}\Big( \int_U^T B_u^{-1}\delta_u \, dG_u \,\Big|\, \mathcal{F}_t\Big)}{\mathbb{E}_{\mathbb{Q}}\Big( \int_{]U,T]} B_u^{-1} G_u \, dL_u \,\Big|\, \mathcal{F}_t\Big)} = \frac{M_t^P}{M_t^A}$$

*where the* $(\mathbb{Q}, \mathbb{F})$*-martingales* $M^P$ *and* $M^A$ *are given by*

$$M_t^P = -\,\mathbb{E}_{\mathbb{Q}}\Big( \int_U^T B_u^{-1}\delta_u \, dG_u \,\Big|\, \mathcal{F}_t\Big)$$

*and*

$$M_t^A = \mathbb{E}_{\mathbb{Q}}\Big( \int_{]U,T]} B_u^{-1} G_u \, dL_u \,\Big|\, \mathcal{F}_t\Big).$$

# Credit Default Swaption

## Definition

A credit default swaption is a call option with expiry date $R \leq U$ and zero strike written on the value of the forward CDS issued at time $0 \leq s < R$, with start date $U$, maturity $T$, and an $\mathcal{F}_s$-measurable rate $\kappa$.

The swaption's payoff $C_R$ at expiry equals $C_R = (S_R(\kappa))^+$.

## Lemma

*For a forward CDS with an $\mathcal{F}_s$-measurable rate $\kappa$ we have, for every $t \in [s, U]$,*

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \widetilde{A}(t, U, T)(\kappa(t, U, T) - \kappa).$$

It is clear that

$$C_R = \mathbb{1}_{\{R<\tau\}} \widetilde{A}(R, U, T)(\kappa(R, U, T) - \kappa)^+.$$

A credit default swaption is formally equivalent to a call option on the forward CDS rate with strike $\kappa$. This option is knocked out if default occurs prior to $R$.

# Credit Default Swaption

**Lemma**

*The price at time $t \in [s, R]$ of a credit default swaption equals*

$$C_t = \mathbb{1}_{\{t<\tau\}} \frac{B_t}{G_t} \mathbb{E}_{\mathbb{Q}} \left( \frac{G_R}{B_R} \widetilde{A}(R, U, T)(\kappa(R, U, T) - \kappa)^+ \,\Big|\, \mathcal{F}_t \right).$$

Define an equivalent probability measure $\widehat{\mathbb{Q}}$ on $(\Omega, \mathcal{F}_R)$ by setting

$$\frac{d\widehat{\mathbb{Q}}}{d\mathbb{Q}} = \frac{M_R^A}{M_0^A}, \quad \mathbb{Q}\text{-a.s.}$$

**Proposition**

*The price of the credit default swaption equals, for every $t \in [s, R]$,*

$$C_t = \mathbb{1}_{\{t<\tau\}} \widetilde{A}(t, U, T)\, \mathbb{E}_{\widehat{\mathbb{Q}}}\big((\kappa(R, U, T) - \kappa)^+ \,\big|\, \mathcal{F}_t\big) = \mathbb{1}_{\{t<\tau\}} \widetilde{C}_t.$$

*The forward CDS rate $(\kappa(t, U, T),\ t \leq R)$ is a $(\widehat{\mathbb{Q}}, \mathbb{F})$-martingale.*

# Brownian Case

- Let the filtration $\mathbb{F}$ be generated by a Brownian motion $W$ under $\mathbb{Q}$.
- Since $M^P$ and $M^A$ are strictly positive $(\mathbb{Q}, \mathbb{F})$-martingales, we have that

$$dM_t^P = M_t^P \sigma_t^P \, dW_t, \quad dM_t^A = M_t^A \sigma_t^A \, dW_t,$$

for some $\mathbb{F}$-adapted processes $\sigma^P$ and $\sigma^A$.

## Lemma

*The forward CDS rate* $(\kappa(t, U, T),\ t \in [0, R])$ *is* $(\widehat{\mathbb{Q}}, \mathbb{F})$*-martingale and*

$$d\kappa(t, U, T) = \kappa(t, U, T)\sigma_t^\kappa \, d\widehat{W}_t$$

*where* $\sigma^\kappa = \sigma^P - \sigma^A$ *and the* $(\widehat{\mathbb{Q}}, \mathbb{F})$*-Brownian motion* $\widehat{W}$ *equals*

$$\widehat{W}_t = W_t - \int_0^t \sigma_u^A \, du, \quad \forall \, t \in [0, R].$$

# Trading Strategies

- Let $\varphi = (\varphi^1, \varphi^2)$ be a trading strategy, where $\varphi^1$ and $\varphi^2$ are $\mathbb{G}$-adapted processes.
- The wealth of $\varphi$ equals, for every $t \in [s, R]$,

  $$V_t(\varphi) = \varphi^1_t S_t(\kappa) + \varphi^2_t A(t, U, T)$$

  and thus the pre-default wealth satisfies, for every $t \in [s, R]$,

  $$\widetilde{V}_t(\varphi) = \varphi^1_t \widetilde{S}_t(\kappa) + \varphi^2_t \widetilde{A}(t, U, T).$$

- It is enough to search for $\mathbb{F}$-adapted processes $\widetilde{\varphi}^i$, $i = 1, 2$ such that the equality

  $$\mathbb{1}_{\{t<\tau\}} \varphi^i_t = \widetilde{\varphi}^i_t$$

  holds for every $t \in [s, R]$.

# Hedging of Credit Default Swaptions

The next result yields a general representation for hedging strategy.

## Proposition

*Let the Brownian motion $W$ be one-dimensional. The hedging strategy $\widetilde{\varphi} = (\widetilde{\varphi}^1, \widetilde{\varphi}^2)$ for the credit default swaption equals, for $t \in [s, R]$,*

$$\widetilde{\varphi}^1_t = \frac{\widetilde{\xi}_t}{\kappa(t,U,T)\sigma^\kappa_t}, \quad \widetilde{\varphi}^2_t = \frac{\widetilde{C}_t - \widetilde{\varphi}^1_t \widetilde{S}_t(\kappa)}{\widetilde{A}(t,U,T)}$$

*where $\widetilde{\xi}$ is the process satisfying*

$$\frac{\widetilde{C}_R}{\widetilde{A}(R,U,T)} = \frac{\widetilde{C}_0}{\widetilde{A}(0,U,T)} + \int_0^R \widetilde{\xi}_t \, d\widehat{W}_t.$$

The main issue is an explicit computation of the process $\widetilde{\xi}$.

# Market Formula

## Proposition

*Assume that the volatility* $\sigma^\kappa = \sigma^P - \sigma^A$ *of the forward CDS spread is deterministic. Then the pre-default value of the credit default swaption with strike level* $\kappa$ *and expiry date* $R$ *equals, for every* $t \in [0, U]$*,*

$$\widetilde{C}_t = \widetilde{A}_t \Big( \kappa_t \, N\big(d_+(\kappa_t, U - t)\big) - \kappa \, N\big(d_-(\kappa_t, U - t)\big) \Big)$$

*where* $\kappa_t = \kappa(t, U, T)$ *and* $\widetilde{A}_t = \widetilde{A}(t, U, T)$*. Equivalently,*

$$\widetilde{C}_t = \widetilde{P}_t \, N\big(d_+(\kappa_t, t, R)\big) - \kappa \, \widetilde{A}_t \, N\big(d_-(\kappa_t, t, R)\big)$$

*where* $\widetilde{P}_t = \widetilde{P}(t, U, T)$ *and*

$$d_\pm(\kappa_t, t, R) = \frac{\ln(\kappa_t/\kappa) \pm \frac{1}{2} \int_t^R (\sigma^\kappa(u))^2 \, du}{\sqrt{\int_t^R (\sigma^\kappa(u))^2 \, du}} \, .$$

# Assumption 1

## Definition

For any $u \in \mathbb{R}_+$, we define the $\mathbb{F}$-martingale $G_t^u = \mathbb{Q}(\tau > u \,|\, \mathcal{F}_t)$ for $t \in [0, T]$.

- Let $G_t = G_t^t$. Then the process $(G_t,\ t \in [0, T])$ is an $\mathbb{F}$-supermartingale.
- We also assume that $G$ is a strictly positive process.

## Assumption

*There exists a family of* $\mathbb{F}$*-adapted processes* $(f_t^x;\ t \in [0, T],\ x \in \mathbb{R}_+)$ *such that, for any* $u \in \mathbb{R}_+$*,*

$$G_t^u = \int_u^\infty f_t^x \, dx, \quad \forall\, t \in [0, T].$$

# Default Intensity

- For any fixed $t \in [0, T]$, the random variable $f_t^{\cdot}$ represents the conditional density of $\tau$ with respect to the $\sigma$-field $\mathcal{F}_t$, that is,

$$f_t^x \, dx = \mathbb{Q}(\tau \in dx \,|\, \mathcal{F}_t).$$

- We write $f_t^t = f_t$ and we define $\widehat{\lambda}_t = G_t^{-1} f_t$.

**Lemma**

*Under Assumption 1, the process* $(M_t, \ t \in [0, T])$ *given by the formula*

$$M_t = H_t - \int_0^t (1 - H_u) \widehat{\lambda}_u \, du$$

*is a* $\mathbb{G}$*-martingale.*

- It can be deduced from the lemma that $\widehat{\lambda} = \lambda$ is the default intensity.

# Assumption 2

**Assumption**

*The filtration $\mathbb{F}$ is generated by a one-dimensional Brownian motion $W$.*

We now work under Assumptions 1-2. We have that

- For any fixed $u \in \mathbb{R}_+$, the $\mathbb{F}$-martingale $G^u$ satisfies, for $t \in [0, T]$,

  $$G_t^u = G_0^u + \int_0^t g_s^u \, dW_s$$

  for some $\mathbb{F}$-predictable, real-valued process $(g_t^u, \, t \in [0, T])$.
- For any fixed $x \in \mathbb{R}_+$, the process $(f_t^x, \, t \in [0, T])$ is an $(\mathbb{Q}, \mathbb{F})$-martingale and thus there exists an $\mathbb{F}$-predictable process $(\sigma_t^x, \, t \in [0, T])$ such that, for $t \in [0, T]$,

  $$f_t^x = f_0^x + \int_0^t \sigma_s^x \, dW_s.$$

# Survival Process

- The following relationship is valid, for any $u \in \mathbb{R}_+$ and $t \in [0, T]$,

$$g_t^u = \int_u^\infty \sigma_t^x \, dx.$$

- By applying the Itô-Wentzell-Kunita formula, we obtain the following auxiliary result, in which we denote $g_s^s = g_s$ and $f_s^s = f_s$.

**Lemma**

*The Doob-Meyer decomposition of the survival process G equals, for every $t \in [0, T]$,*

$$G_t = G_0 + \int_0^t g_s \, dW_s - \int_0^t f_s \, ds.$$

*In particular, G is a continuous process.*

# Volatility of Pre-Default Value

- Under the assumption that $B, Z$ and $A$ are deterministic, the volatility of the pre-default value process can be computed explicitly in terms of $\sigma_t^u$. Recall that, for $t \in [0, T]$,

$$f_t^x = f_0^x + \int_0^t \sigma_s^x \, dW_s, \quad g_t^u = \int_u^\infty \sigma_t^x \, dx.$$

## Corollary

*If $B, Z$ and $A$ are deterministic then we have that, for every $t \in [0, T]$,*

$$d\widetilde{S}_t = \Big( (r(t) + \lambda_t)\widetilde{S}_t - \lambda_t Z(t) \Big) \, dt + dA(t) + \zeta_t^T \, dW_t$$

*with* $\zeta_t^T = G_t^{-1} B(t) \nu_t^T$ *where*

$$\nu_t^T = B^{-1}(T) X G_t^T + \int_t^T B^{-1}(u) Z(u) \sigma_t^u \, du + \int_t^T B^{-1}(u) g_t^u \, dA(u).$$

# Volatility of Forward CDS Rate

## Lemma

*If $B, \delta$ and $L$ are deterministic then the forward CDS rate satisfies under $\widehat{\mathbb{Q}}$*

$$d\kappa(t,U,T) = \kappa(t,U,T)\big(\sigma_t^P - \sigma_t^A\big)\, d\widehat{W}_t$$

*where the process $\widehat{W}$, given by the formula*

$$\widehat{W}_t = W_t - \int_0^t \sigma_u^A \, du, \quad \forall\, t \in [0,R],$$

*is a Brownian motion under $\widehat{\mathbb{Q}}$ and*

$$\sigma_t^P = \Big(\int_U^T B^{-1}(u)\delta(u)\sigma_t^u\, du\Big)\Big(\int_U^T B^{-1}(u)\delta(u) f_t^u\, du\Big)^{-1}$$

$$\sigma_t^A = \Big(\int_U^Y B^{-1}(u) g_t^u\, du\Big)\Big(\int_U^T B^{-1}(u) G_t^u\, du\Big)^{-1}.$$

# CIR Default Intensity Model

We make the following standing assumptions:

1. The default intensity process $\lambda$ is governed by the CIR dynamics
$$d\lambda_t = \mu(\lambda_t)\,dt + \nu(\lambda_t)\,dW_t$$
where $\mu(\lambda) = a - b\lambda$ and $\nu(\lambda) = c\sqrt{\lambda}$.
2. The default time $\tau$ is given by
$$\tau = \inf\left\{\, t \in \mathbb{R}_+ : \int_0^t \lambda_u\,du \geq \Theta \right\}$$
where $\Theta$ is a random variable with the unit exponential distribution, independent of the filtration $\mathbb{F}$.

# Model Properties

- From the martingale property of $f^u$ we have, for every $t \leq u$,

$$f_t^u = \mathbb{E}_{\mathbb{Q}}(f_u \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(\lambda_u G_u \,|\, \mathcal{F}_t).$$

- The immersion property holds between $\mathbb{F}$ and $\mathbb{G}$ so that $G_t = \exp(-\Lambda_t)$, where $\Lambda_t = \int_0^t \lambda_u \, du$ is the hazard process. Therefore

$$f_t^s = \mathbb{E}_{\mathbb{Q}}(\lambda_s e^{-\Lambda_s} \,|\, \mathcal{F}_t).$$

- Let us denote

$$H_t^s = \mathbb{E}_{\mathbb{Q}}\big(e^{-(\Lambda_s - \Lambda_t)} \,\big|\, \mathcal{F}_t\big) = \frac{G_t^s}{G_t}.$$

- It is important to note that for the CIR model

$$H_t^s = e^{m(t,s) - n(t,s)\lambda_t} = \widehat{H}(\lambda_t, t, s)$$

where $\widehat{H}(\cdot, t, s)$ is a strictly decreasing function when $t < s$.

# Volatility of Forward CDS Rate

We assume that:

1. The tenor structure process $L$ is deterministic.
2. The savings account is $B$ is deterministic. We denote $\beta = B^{-1}$.
3. We also assume that $\delta$ is constant.

**Proposition**

*The volatility of the forward CDS rate satisfies* $\sigma^{\kappa} = \sigma^{P} - \sigma^{A}$ *where*

$$\sigma_t^P = \nu(\lambda_t)\,\frac{\beta(T)H_t^T n(t,T) - \beta(U)H_t^U n(t,U) + \int_U^T r(u)\beta(u)H_t^u n(t,u)\,du}{\beta(U)H_t^U - \beta(T)H_t^T - \int_U^T r(u)\beta(u)H_t^u\,du}$$

*and*

$$\sigma_t^A = \nu(\lambda_t)\,\frac{\int_{]U,T]} \beta(u)H_t^u n(t,u)\,dL(u)}{\int_{]U,T]} \beta(u)H_t^u\,dL(u)}\,.$$

# Equivalent Representations

- One can show that

$$C_R = \mathbb{1}_{\{R<\tau\}} \left( \delta \int_U^T B(R,u)\lambda_R^u \, du - \kappa \int_{]U,T]} B(R,u) H_R^u \, dL(u) \right)^+ .$$

- Straightforward computations lead to the following representation

$$C_R = \mathbb{1}_{\{R<\tau\}} \left( \delta B(R,U) H_R^U - \int_{]U,T]} B(R,u) H_R^u \, d\chi(u) \right)^+$$

where the function $\chi : \mathbb{R}_+ \to \mathbb{R}$ satisfies

$$d\chi(u) = -\delta \frac{\partial \ln B(R,u)}{\partial u} \, du + \kappa \, dL(u) + \delta \, d\mathbb{1}_{[T,\infty[}(u).$$

# Auxiliary Functions

- We define auxiliary functions $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ and $\psi : \mathbb{R} \to \mathbb{R}_+$ by setting
$$\zeta(x) = \delta B(R,U)\widehat{H}(x,R,U)$$
and
$$\psi(y) = \int_{]U,T]} B(R,u)\widehat{H}(y,R,u)\,d\chi(u).$$
- There exists a unique $\mathcal{F}_R$-measurable random variable $\lambda_R^*$ such that
$$\zeta(\lambda_R) = \delta B(R,U)\widehat{H}(\lambda_R,R,U) = \int_{]U,T]} B(R,u)\widehat{H}(\lambda_R^*,R,u)\,d\chi(u) = \psi(\lambda_R^*).$$
- It suffices to check that $\lambda_R^* = \psi^{-1}(\zeta(\lambda_R))$ is the unique solution to this equation.

# Explicit Valuation Formula

- The payoff of the credit default swaption admits the following representation

$$C_R = \mathbb{1}_{\{R<\tau\}} \int_{]U,T]} B(R,u)\big(\widehat{H}(\lambda^*_R, R, u) - \widehat{H}(\lambda_R, R, u)\big)^+ \, d\chi(u).$$

- Let $D^0(t,u)$ be the price at time $t$ of a unit defaultable zero-coupon bond with zero recovery maturing at $u \geq t$ and let $B(t,u)$ be the price at time $t$ of a (default-free) unit discount bond maturing at $u \geq t$.
- If the interest rate process $r$ is independent of the default intensity $\lambda$ then $D^0(t,u)$ is given by the following formula

$$D^0(t,u) = \mathbb{1}_{\{t<\tau\}} B(t,u) H^u_t.$$

# Explicit Valuation Formula

- Let $P(\lambda_t, U, u, K)$ stand for the price at time $t$ of a put bond option with strike $K$ and expiry $U$ written on a zero-coupon bond maturing at $u$ computed in the CIR model with the interest rate modeled by $\lambda$.

## Proposition

*Assume that $R = U$. Then the payoff of the credit default swaption equals*

$$C_U = \int_{]U,T]} \left(K(u)D^0(U,U) - D^0(U,u)\right)^+ d\chi(u)$$

*where $K(u) = B(U,u)\widehat{H}(\lambda^*_U, U, u)$ is deterministic, since $\lambda^*_U = \psi^{-1}(\delta)$.*

*The pre-default value of the credit default swaption equals*

$$\widetilde{C}_t = \int_{]U,T]} B(t,u)P(\lambda_t, U, u, \widehat{K}(u))\, d\chi(u)$$

*where $\widehat{K}(u) = K(u)/B(U,u) = \widehat{H}(\lambda^*_U, U, u)$.*

# Hedging Strategy

1. The price $P_t^u := P(\lambda_t, U, u, \widehat{K}(u))$ of the put bond option in the CIR model with the interest rate $\lambda$ is known to be

$$P_t^u = \widehat{K}(u) H_t^U \, \mathbb{P}_U(H_U^U \leq \widehat{K}(u) \,|\, \lambda_t) - H_t^u \, \mathbb{P}_u(H_u^U \leq \widehat{K}(u) \,|\, \lambda_t)$$

   where $H_t^u = \widehat{H}(\lambda_t, t, u)$ is the price at time $t$ of a zero-coupon bond maturing at $u$.

2. Let us denote $Z_t = H_t^u / H_t^U$ and let us set, for every $u \in [U, T]$,

$$\mathbb{P}_u(H_u^U \leq \widehat{K}(u) \,|\, \lambda_t) = \Psi_u(t, Z_t).$$

3. Then the pricing formula for the bond put option becomes

$$P_t^u = \widehat{K}(u) H_t^U \Psi_U(t, Z_t) - H_t^u \Psi_u(t, Z_t)$$

# Hedging of Credit Default Swaptions

Let us recall the general representation for the hedging strategy when $\mathbb{F}$ is the Brownian filtration.

## Proposition

*The hedging strategy* $\widetilde{\varphi} = (\widetilde{\varphi}^1, \widetilde{\varphi}^2)$ *for the credit default swaption equals, for* $t \in [s, U]$,

$$\widetilde{\varphi}^1_t = \frac{\widetilde{\xi}_t}{\kappa(t, U, T)\sigma^\kappa_t}, \quad \widetilde{\varphi}^2_t = \frac{\widetilde{C}_t - \widetilde{\varphi}^1_t \widetilde{S}_t(\kappa)}{\widetilde{A}(t, U, T)}$$

*where* $\widetilde{\xi}$ *is the process satisfying*

$$\frac{\widetilde{C}_U}{\widetilde{A}(U, U, T)} = \frac{\widetilde{C}_0}{\widetilde{A}(0, U, T)} + \int_0^U \widetilde{\xi}_t \, d\widehat{W}_t.$$

All terms were already computed, except for the process $\widetilde{\xi}$.

# Computation of $\widetilde{\xi}$

Recall that we are searching for the process $\widetilde{\xi}$ such that

$$d(\widetilde{C}_t/\widetilde{A}(t,U,T)) = \widetilde{\xi}_t \, d\widehat{W}_t.$$

## Proposition

*Assume that $R = U$. Then we have that, for every $t \in [0, U]$,*

$$\widetilde{\xi}_t = \frac{1}{\widetilde{A}_t}\left(\int_{]U,T]} B(t,u)\Big(\vartheta_t H_t^u\big(b_t^u - b_t^U\big) - P_t^u b_t^U\Big)\, d\chi(u) - \widetilde{C}_t \sigma_t^A\right)$$

*where*

$$\widetilde{A}_t = \widetilde{A}(t,U,T),\ H_t^u = \widehat{H}(\lambda_t, t, u),\ b_t^u = cn(t,u)\sqrt{\lambda_t},\ P_t^u = P(\lambda_t, U, u, \widehat{K}(u))$$

*and*

$$\vartheta_t = \widehat{K}(u)\,\frac{\partial \Psi_U}{\partial z}(t, Z_t) - \Psi_u(t, Z_t) - Z_t\,\frac{\partial \Psi_u}{\partial z}(t, Z_t).$$

# Hedging Strategy

For $R = U$, we obtain the following final result for hedging strategy.

**Proposition**

*Consider the CIR default intensity model with a deterministic short-term interest rate. The replicating strategy* $\widetilde{\varphi} = (\widetilde{\varphi}^1, \widetilde{\varphi}^2)$ *for the credit default swaption maturing at* $R = U$ *equals, for any* $t \in [0, U]$,

$$\widetilde{\varphi}^1_t = \frac{\widetilde{\xi}_t}{\kappa(t, U, T)\sigma^{\kappa}_t}, \quad \widetilde{\varphi}^2_t = \frac{\widetilde{C}_t - \widetilde{\varphi}^1_t \widetilde{S}_t(\kappa)}{\widetilde{A}(t, U, T)},$$

*where the processes* $\sigma^{\kappa}, \widetilde{C}$ *and* $\widetilde{\xi}$ *are given in previous results.*

Note that for $R < U$ the problem remains open, since a closed-form solution for the process $\widetilde{\xi}$ is not readily available in this case.

# Credit Default Index Swaptions

# Credit Default Index Swap

1. A *credit default index swap* (CDIS) is a standardized contract that is based upon a fixed portfolio of reference entities.
2. At its conception, the CDIS is referenced to $n$ fixed companies that are chosen by market makers.
3. The reference entities are specified to have equal weights.
4. If we assume each has a nominal value of one then, because of the equal weighting, the total notional would be $n$.
5. By contrast to a standard single-name CDS, the ‘buyer’ of the CDIS provides protection to the market makers.
6. By purchasing a CDIS from market makers the investor is not receiving protection, rather they are providing it to the market makers.

# Credit Default Index Swap

1. In exchange for the protection the investor is providing, the market makers pay the investor a periodic fixed premium, otherwise known as the *credit default index spread*.
2. The recovery rate $\delta \in [0, 1]$ is predetermined and identical for all reference entities in the index.
3. By purchasing the index the investor is agreeing to pay the market makers $1 - \delta$ for any default that occurs before maturity.
4. Following this, the nominal value of the CDIS is reduced by one; there is no replacement of the defaulted firm.
5. This process repeats after every default and the CDIS continues on until maturity.

# Default Times and Filtrations

1. Let $\tau_1, \dots, \tau_n$ represent default times of reference entities.
2. We introduce the sequence $\tau_{(1)} < \dots < \tau_{(n)}$ of ordered default times associated with $\tau_1, \dots, \tau_n$. For brevity, we write $\widehat{\tau} = \tau_{(n)}$.
3. We thus have $\mathbb{G} = \mathbb{H}^{(n)} \vee \hat{\mathbb{F}}$, where $\mathbb{H}^{(n)}$ is the filtration generated by the indicator process $H_t^{(n)} = \mathbb{1}_{\{\widehat{\tau} \leq t\}}$ of the last default and the filtration $\hat{\mathbb{F}}$ equals $\hat{\mathbb{F}} = \mathbb{F} \vee \mathbb{H}^{(1)} \vee \dots \vee \mathbb{H}^{(n-1)}$.
4. We are interested in events of the form $\{\widehat{\tau} \leq t\}$ and $\{\widehat{\tau} > t\}$ for a fixed $t$.
5. Morini and Brigo (2007) refer to these events as the *armageddon* and the *no-armageddon* events. We use instead the terms *collapse* event and the *pre-collapse* event.
6. The event $\{\widehat{\tau} \leq t\}$ corresponds to the total collapse of the reference portfolio, in the sense that all underlying credit names default either prior to or at time $t$.

# Basic Lemma

1. We set $\widehat{F}_t = \mathbb{Q}(\widehat{\tau} \leq t \,|\, \widehat{\mathcal{F}}_t)$ for every $t \in \mathbb{R}_+$.
2. Let us denote by $\widehat{G}_t = 1 - \widehat{F}_t = \mathbb{Q}(\widehat{\tau} > t \,|\, \widehat{\mathcal{F}}_t)$ the corresponding survival process with respect to the filtration $\widehat{\mathbb{F}}$ and let us temporarily assume that the inequality $\widehat{G}_t > 0$ holds for every $t \in \mathbb{R}_+$.
3. Then for any $\mathbb{Q}$-integrable and $\widehat{\mathcal{F}}_T$-measurable random variable $Y$ we have that

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{\{T<\widehat{\tau}\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\widehat{\tau}\}} \, \widehat{G}_t^{-1} \, \mathbb{E}_{\mathbb{Q}}(\widehat{G}_T Y \,|\, \widehat{\mathcal{F}}_t).$$

**Lemma**

*Assume that $Y$ is some $\mathbb{G}$-adapted stochastic process. Then there exists a unique $\widehat{\mathbb{F}}$-adapted process $\widehat{Y}$ such that, for every $t \in [0, T]$,*

$$Y_t = \mathbb{1}_{\{t<\widehat{\tau}\}} \widehat{Y}_t.$$

*The process $\widehat{Y}$ is termed the* pre-collapse value *of the process $Y$.*

# Notation and Assumptions

We write $T_0 = T < T_1 < \cdots < T_J$ to denote the *tenor structure* of the forward-start CDIS, where:

1. $T_0 = T$ is the inception date;
2. $T_J$ is the maturity date;
3. $T_j$ is the $j$th fee payment date for $j = 1, 2, \ldots, J$;
4. $a_j = T_j - T_{j-1}$ for every $j = 1, 2, \ldots, J$.

The process $B$ is an $\mathbb{F}$-adapted (or, at least, $\widehat{\mathbb{F}}$-adapted) and strictly positive process representing the price of the savings account.

The underlying probability measure $\mathbb{Q}$ is interpreted as a martingale measure associated with the choice of $B$ as the numeraire asset.

# Forward Credit Default Index Swap

## Definition

The discounted cash flows for the seller of the *forward CDIS* issued at time $s \in [0, T]$ with an $\mathcal{F}_s$-measurable spread $\kappa$ are, for every $t \in [s, T]$,

$$D_t^n = P_t^n - \kappa A_t^n,$$

where

$$P_t^n = (1 - \delta) B_t \sum_{i=1}^{n} B_{\tau_i}^{-1} \mathbb{1}_{\{T < \tau_i \leq T_J\}}$$

$$A_t^n = B_t \sum_{j=1}^{J} a_j B_{T_j}^{-1} \sum_{i=1}^{n} \big(1 - \mathbb{1}_{\{T_j \geq \tau_i\}}\big)$$

are discounted payoffs of the protection leg and the fee leg per one basis point, respectively. The *fair price* at time $t \in [s, T]$ of a forward CDIS equals

$$S_t^n(\kappa) = \mathbb{E}_{\mathbb{Q}}(D_t^n \,|\, \mathcal{G}_t) = \mathbb{E}_{\mathbb{Q}}(P_t^n \,|\, \mathcal{G}_t) - \kappa \, \mathbb{E}_{\mathbb{Q}}(A_t^n \,|\, \mathcal{G}_t).$$

# Forward Credit Default Index Swap

1. The quantities $P_t^n$ and $A_t^n$ are well defined for any $t \in [0, T]$ and they do not depend on the issuance date $s$ of the forward CDIS under consideration.
2. They satisfy
$$P_t^n = \mathbb{1}_{\{T < \widehat{\tau}\}} P_t^n, \quad A_t^n = \mathbb{1}_{\{T < \widehat{\tau}\}} A_t^n.$$
3. For brevity, we will write $J_t$ to denote the *reduced nominal* at time $t \in [s, T]$, as given by the formula
$$J_t = \sum_{i=1}^{n} \big(1 - \mathbb{1}_{\{t \geq \tau_i\}}\big).$$
4. In what follows, we only require that the inequality $\widehat{G}_t > 0$ holds for every $t \in [s, T_1]$, so that, in particular, $\widehat{G}_{T_1} = \mathbb{Q}(\widehat{\tau} > T_1 \,|\, \widehat{\mathcal{F}}_{T_1}) > 0$.

# Pre-collapse Price

**Lemma**

*The price at time $t \in [s, T]$ of the forward CDIS satisfies*

$$S_t^n(\kappa) = \mathbb{1}_{\{t<\widehat{\tau}\}} \widehat{G}_t^{-1} \, \mathbb{E}_{\mathbb{Q}}(D_t^n \,|\, \widehat{\mathcal{F}}_t) = \mathbb{1}_{\{t<\widehat{\tau}\}} \widehat{S}_t^n(\kappa),$$

*where the* pre-collapse price *of the forward CDIS satisfies* $\widehat{S}_t^n(\kappa) = \widehat{P}_t^n - \kappa \widehat{A}_t^n$, *where*

$$\widehat{P}_t^n = \widehat{G}_t^{-1} \, \mathbb{E}_{\mathbb{Q}}(P_t^n \,|\, \widehat{\mathcal{F}}_t) = (1-\delta)\widehat{G}_t^{-1} B_t \, \mathbb{E}_{\mathbb{Q}}\bigg( \sum_{i=1}^{n} B_{\tau_i}^{-1} \mathbb{1}_{\{T<\tau_i \le T_J\}} \,\bigg|\, \widehat{\mathcal{F}}_t \bigg)$$

$$\widehat{A}_t^n = \widehat{G}_t^{-1} \, \mathbb{E}_{\mathbb{Q}}(A_t^n \,|\, \widehat{\mathcal{F}}_t) = \widehat{G}_t^{-1} B_t \, \mathbb{E}_{\mathbb{Q}}\bigg( \sum_{j=1}^{J} a_j B_{T_j}^{-1} J_{T_j} \,\bigg|\, \widehat{\mathcal{F}}_t \bigg).$$

*The process $\widehat{A}_t^n$ may be thought of as the pre-collapse PV of receiving risky one basis point on the forward CDIS payment dates $T_j$ on the residual nominal value $J_{T_j}$. The process $\widehat{P}_t^n$ represents the pre-collapse PV of the protection leg.*

# Pre-Collapse Fair CDIS Spread

Since the forward CDIS is terminated at the moment of the *n*th default with no further payments, the forward CDS spread is defined only prior to $\widehat{\tau}$.

## Definition

The *pre-collapse fair forward CDIS spread* is the $\widehat{\mathcal{F}}_t$-measurable random variable $\kappa_t^n$ such that $\widehat{S}_t^n(\kappa_t^n) = 0$.

## Lemma

*Assume that* $\widehat{G}_{T_1} = \mathbb{Q}(\widehat{\tau} > T_1 \,|\, \widehat{\mathcal{F}}_{T_1}) > 0$. *Then the pre-collapse fair forward CDIS spread satisfies, for* $t \in [0, T]$,

$$\kappa_t^n = \frac{\widehat{P}_t^n}{\widehat{A}_t^n} = \frac{(1-\delta)\,\mathbb{E}_{\mathbb{Q}}\Big( \sum_{i=1}^n B_{\tau_i}^{-1} \mathbb{1}_{\{T<\tau_i\le T_J\}} \,\Big|\, \widehat{\mathcal{F}}_t \Big)}{\mathbb{E}_{\mathbb{Q}}\Big( \sum_{j=1}^J a_j B_{T_j}^{-1} J_{T_j} \,\Big|\, \widehat{\mathcal{F}}_t \Big)}.$$

*The price of the forward CDIS admits the following representation*

$$S_t^n(\kappa) = \mathbb{1}_{\{t<\widehat{\tau}\}} \widehat{A}_t^n (\kappa_t^n - \kappa).$$

# Market Convention for Valuing a CDIS

Market quote for the quantity $\widehat{A}^n_t$, which is essential in marking-to-market of a CDIS, is not directly available. The market convention for approximation of the value of $\widehat{A}^n_t$ hinges on the following postulates:

1. all firms are identical from time $t$ onwards (homogeneous portfolio); therefore, we just deal with a single-name case, so that either all firms default or none;
2. the implied risk-neutral default probabilities are computed using a flat single-name CDS curve with a constant spread equal to $\kappa^n_t$.

Then

$$\widehat{A}^n_t \approx J_t PV_t(\kappa^n_t),$$

where $PV_t(\kappa_t)$ is the risky present value of receiving one basis point at all CDIS payment dates calibrated to a flat CDS curve with spread equal to $\kappa^n_t$, where $\kappa^n_t$ is the quoted CDIS spread at time $t$.

The conventional market formula for the value of the CDIS with a fixed spread $\kappa$ reads, on the pre-collapse event $\{t < \widehat{\tau}\}$,

$$\widehat{S}_t(\kappa) = J_t PV_t(\kappa^n_t)(\kappa^n_t - \kappa).$$

# Market Payoff of a Credit Default Index Swaption

1. The conventional market formula for the payoff at maturity $U \leq T$ of the *payer credit default index swaption* with strike level $\kappa$ reads

$$C_U = \Big( \mathbb{1}_{\{U<\hat{\tau}\}} PV_U(\kappa_U^n) J_U(\kappa_U^n - \kappa_0^n) - \mathbb{1}_{\{U<\hat{\tau}\}} PV_U(\kappa) n(\kappa - \kappa_0^n) + L_U \Big)^+,$$

   where $L$ stands for the loss process for our portfolio so that, for every $t \in \mathbb{R}_+$,

$$L_t = (1-\delta) \sum_{i=1}^{n} \mathbb{1}_{\{\tau_i \leq t\}}.$$

2. The market convention is due to the fact that the swaption has physical settlement and the CDIS with spread $\kappa$ is not traded. If the swaption is exercised, its holder takes a long position in the on-the-run index and is compensated for the difference between the value of the on-the-run index and the value of the (non-traded) index with spread $\kappa$, as well as for defaults that occurred in the interval $[0, U]$.

# Put-Call Parity for Credit Default Index Swaptions

1. For the sake of brevity, let us denote, for any fixed $\kappa > 0$,

$$f(\kappa, L_U) = L_U - \mathbb{1}_{\{U<\widehat{\tau}\}} PV_U(\kappa) n(\kappa - \kappa_0^n).$$

2. Then the payoff of the payer credit default index swaption entered at time 0 and maturing at $U$ equals

$$C_U = \Big( \mathbb{1}_{\{U<\widehat{\tau}\}} PV_U\big(\kappa_U^n\big) J_U(\kappa_U^n - \kappa_0^n) + f(\kappa, L_U) \Big)^+,$$

whereas the payoff of the corresponding *receiver credit default index swaption* satisfies

$$P_U = \Big( \mathbb{1}_{\{U<\widehat{\tau}\}} PV_U\big(\kappa_U^n\big) J_U(\kappa_0^n - \kappa_U^n) - f(\kappa, L_U) \Big)^+.$$

3. This leads to the following equality, which holds at maturity date $U$

$$C_U - P_U = \mathbb{1}_{\{U<\widehat{\tau}\}} PV_U\big(\kappa_U^n\big) J_U(\kappa_U^n - \kappa_0^n) + f(\kappa, L_U).$$

# Model Payoff of a Credit Default Index Swaption

1. The *model payoff* of the payer credit default index swaption entered at time 0 with maturity date $U$ and strike level $\kappa$ equals

   $$C_U = \left( S^n_U(\kappa) + L_U \right)^+$$

   or, more explicitly

   $$C_U = \Big( \mathbb{1}_{\{U<\widehat{\tau}\}} \widehat{A}^n_U(\kappa_U - \kappa) + L_U \Big)^+ .$$

2. To formally derive obtain the model payoff from the market payoff, it suffices to postulate that

   $$PV_U(\kappa)n \approx PV_U\big(\kappa_U\big)J_U \approx \widehat{A}^n_U .$$

# Loss-Adjusted Forward CDIS

1. Since $L_U \geq 0$ and
$$L_U = \mathbb{1}_{\{U<\widehat{\tau}\}} L_U + \mathbb{1}_{\{U\geq\widehat{\tau}\}} L_U$$
the payoff $C_U$ can also be represented as follows
$$C_U = (S^n_U(\kappa) + \mathbb{1}_{\{U<\widehat{\tau}\}} L_U)^+ + \mathbb{1}_{\{U\geq\widehat{\tau}\}} L_U = (S^a_U(\kappa))^+ + C^L_U,$$
where we denote
$$S^a_U(\kappa) = S^n_U(\kappa) + \mathbb{1}_{\{U<\widehat{\tau}\}} L_U$$
and
$$C^L_U = \mathbb{1}_{\{U\geq\widehat{\tau}\}} L_U.$$
2. The quantity $S^a_U(\kappa)$ represents the payoff at time $U$ of the loss-adjusted forward CDIS.

# Loss-Adjusted Forward CDIS

1. The discounted cash flows for the seller of the *loss-adjusted forward CDIS* (that is, for the buyer of the protection) are, for every $t \in [0, U]$,

   $$D_t^a = P_t^a - \kappa A_t^n,$$

   where

   $$P_t^a = P_t^n + B_t B_U^{-1} \mathbb{1}_{\{U < \widehat{\tau}\}} L_U.$$

2. It is essential to observe that the payoff $D_U^a$ is the $U$-survival claim, in the sense that

   $$D_U^a = \mathbb{1}_{\{U < \widehat{\tau}\}} D_U^a.$$

3. Any other adjustments to the payoff $P_t^n$ or $A_t^n$ are also admissible, provided that the properties

   $$P_U^a = \mathbb{1}_{\{U < \widehat{\tau}\}} P_U^a, \quad A_U^a = \mathbb{1}_{\{U < \widehat{\tau}\}} A_U^a$$

   hold.

# Price of the Loss-Adjusted Forward CDIS

## Lemma

*The price of the loss-adjusted forward CDIS equals, for every* $t \in [0, U]$*,*

$$S_t^a(\kappa) = \mathbb{1}_{\{t<\widehat{\tau}\}} \widehat{G}_t^{-1} \, \mathbb{E}_{\mathbb{Q}}(D_t^a \,|\, \widehat{\mathcal{F}}_t) = \mathbb{1}_{\{t<\widehat{\tau}\}} \widehat{S}_t^a(\kappa),$$

*where the pre-collapse price satisfies* $\widehat{S}_t^a(\kappa) = \widehat{P}_t^a - \kappa \widehat{A}_t^n$*, where in turn*

$$\widehat{P}_t^a = \widehat{G}_t^{-1} \, \mathbb{E}_{\mathbb{Q}}(P_t^a \,|\, \widehat{\mathcal{F}}_t), \quad \widehat{A}_t^n = \widehat{G}_t^{-1} \, \mathbb{E}_{\mathbb{Q}}(A_t^n \,|\, \widehat{\mathcal{F}}_t)$$

*or, more explicitly,*

$$\widehat{P}_t^a = \widehat{G}_t^{-1} B_t \, \mathbb{E}_{\mathbb{Q}}\bigg( (1-\delta) \sum_{i=1}^{n} B_{\tau_i}^{-1} \mathbb{1}_{\{T<\tau_i \le T_J\}} + \mathbb{1}_{\{U<\widehat{\tau}\}} B_U^{-1} L_U \,\bigg|\, \widehat{\mathcal{F}}_t \bigg)$$

*and*

$$\widehat{A}_t^n = \widehat{G}_t^{-1} B_t \, \mathbb{E}_{\mathbb{Q}}\bigg( \sum_{j=1}^{J} a_j B_{T_j}^{-1} J_{T_j} \,\bigg|\, \widehat{\mathcal{F}}_t \bigg).$$

# Pre-Collapse Loss-Adjusted Fair CDIS Spread

We are in a position to define the fair loss-adjusted forward CDIS spread.

**Definition**

The *pre-collapse loss-adjusted fair forward CDIS spread* at time $t \in [0, U]$ is the $\widehat{\mathcal{F}}_t$-measurable random variable $\kappa_t^a$ such that $\widehat{S}_t^a(\kappa_t^a) = 0$.

**Lemma**

*Assume that* $\widehat{G}_{T_1} = \mathbb{Q}(\widehat{\tau} > T_1 \,|\, \widehat{\mathcal{F}}_{T_1}) > 0$. *Then the pre-collapse loss-adjusted fair forward CDIS spread satisfies, for* $t \in [0, U]$,

$$\kappa_t^a = \frac{\widehat{P}_t^a}{\widehat{A}_t^n} = \frac{\mathbb{E}_{\mathbb{Q}}\Big( (1-\delta) \sum_{i=1}^n B_{\tau_i}^{-1} \mathbb{1}_{\{T<\tau_i \le T_J\}} + \mathbb{1}_{\{U<\widehat{\tau}\}} B_U^{-1} L_U \,\Big|\, \widehat{\mathcal{F}}_t \Big)}{\mathbb{E}_{\mathbb{Q}}\Big( \sum_{j=1}^J a_j B_{T_j}^{-1} J_{T_j} \,\Big|\, \widehat{\mathcal{F}}_t \Big)}.$$

*The price of the forward CDIS has the following representation, for* $t \in [0, T]$,

$$S_t^a(\kappa) = \mathbb{1}_{\{t<\widehat{\tau}\}} \widehat{A}_t^n (\kappa_t^a - \kappa).$$

# Model Pricing of Credit Default Index Swaptions

1. It is easy to check that the model payoff can be represented as follows
$$C_U = \mathbb{1}_{\{U<\widehat{\tau}\}} \widehat{A}^n_U (\kappa^a_U - \kappa)^+ + \mathbb{1}_{\{U \geq \widehat{\tau}\}} L_U.$$
2. The price at time $t \in [0, U]$ of the credit default index swaption is thus given by the risk-neutral valuation formula
$$C_t = B_t \, \mathbb{E}_{\mathbb{Q}} \big( \mathbb{1}_{\{U<\widehat{\tau}\}} B_U^{-1} \widehat{A}^n_U (\kappa^a_U - \kappa)^+ \,\big|\, \mathcal{G}_t \big) + B_t \, \mathbb{E}_{\mathbb{Q}} \big( \mathbb{1}_{\{U \geq \widehat{\tau}\}} B_U^{-1} L_U \,\big|\, \mathcal{G}_t \big).$$
3. Using the filtration $\widehat{\mathbb{F}}$, we can obtain a more explicit representation for the first term in the formula above, as the following result shows.

# Model Pricing of Credit Default Index Swaptions

**Lemma**

*The price at time $t \in [0, U]$ of the payer credit default index swaption equals*

$$C_t = \mathbb{E}_{\mathbb{Q}}\left(\widehat{G}_U B_U^{-1} \widehat{A}_U^n (\kappa_U^a - \kappa)^+ \,\middle|\, \widehat{\mathcal{F}}_t\right) + B_t\, \mathbb{E}_{\mathbb{Q}}\left(\mathbb{1}_{\{U \geq \widehat{\tau}\}} B_U^{-1} L_U \,\middle|\, \mathcal{G}_t\right).$$

1. The random variable $Y = B_U^{-1} \widehat{A}_U^n (\kappa_U^a - \kappa)^+$ is manifestly $\widehat{\mathcal{F}}_U$-measurable and $Y = \mathbb{1}_{\{U < \widehat{\tau}\}} Y$. Hence the equality is an immediate consequence of the basic lemma.
2. On the collapse event $\{t \geq \widehat{\tau}\}$ we have $\mathbb{1}_{\{U \geq \widehat{\tau}\}} B_U^{-1} L_U = B_U^{-1} n(1-\delta)$ and thus the pricing formula reduces to

$$C_t = B_t\, \mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{U \geq \widehat{\tau}\}} B_U^{-1} L_U \,\big|\, \mathcal{G}_t\big) = n(1-\delta) \mathbb{E}_{\mathbb{Q}}\left(B_U^{-1} \,\middle|\, \mathcal{G}_t\right) = n(1-\delta) B(t, T),$$

where $B(t, T)$ is the price at $t$ of the $U$-maturity risk-free zero-coupon bond.

# Model Pricing of Credit Default Index Swaptions

1. Let us thus concentrate on the pre-collapse event $\{t < \widehat{\tau}\}$. We now have $C_t = C^a_t + C^L_t$, where

$$C^a_t = B_t \widehat{G}_t^{-1} \, \mathbb{E}_{\mathbb{Q}} \left( \widehat{G}_U B_U^{-1} \widehat{A}^n_U (\kappa^a_U - \kappa)^+ \,\middle|\, \widehat{\mathcal{F}}_t \right)$$

and

$$C^L_t = B_t \, \mathbb{E}_{\mathbb{Q}} \big( \mathbb{1}_{\{U \geq \widehat{\tau} > t\}} B_U^{-1} L_U \,\big|\, \widehat{\mathcal{F}}_t \big).$$

The last equality follows from the well known fact that on $\{t < \widehat{\tau}\}$ any $\mathcal{G}_t$-measurable event can be represented by an $\widehat{\mathcal{F}}_t$-measurable event, in the sense that for any event $A \in \mathcal{G}_t$ there exists an event $\widehat{A} \in \widehat{\mathcal{F}}_t$ such that $\mathbb{1}_{\{t<\widehat{\tau}\}} A = \mathbb{1}_{\{t<\widehat{\tau}\}} \widehat{A}$.

# Model Pricing of Credit Default Index Swaptions

1. The computation of $C_t^L$ relies on the knowledge of the risk-neutral conditional distribution of $\widehat{\tau}$ given $\widehat{\mathcal{F}}_t$ and the term structure of interest rates, since on the event $\{U \geq \widehat{\tau} > t\}$ we have $B_U^{-1} L_U = B_U^{-1} n(1-\delta)$.
2. For $C_t^a$, we define an equivalent probability measure $\widehat{\mathbb{Q}}$ on $(\Omega, \widehat{\mathcal{F}}_U)$

$$\frac{d\widehat{\mathbb{Q}}}{d\mathbb{Q}} = c\widehat{G}_U B_U^{-1} \widehat{A}_U^n, \quad \mathbb{Q}\text{-a.s.}$$

3. Note that the process $\widehat{\eta}_t = c\widehat{G}_t B_t^{-1} \widehat{A}_t^n$, $t \in [0, U]$, is a strictly positive $\widehat{\mathbb{F}}$-martingale under $\mathbb{Q}$, since

$$\widehat{\eta}_t = c\widehat{G}_t B_t^{-1} \widehat{A}_t^n = c\, \mathbb{E}_{\mathbb{Q}}\Big( \sum_{j=1}^J a_j B_{T_j}^{-1} J_{T_j} \,\Big|\, \widehat{\mathcal{F}}_t \Big)$$

and $\mathbb{Q}(\tau > T_j \,|\, \widehat{\mathcal{F}}_{T_j}) = \widehat{G}_{T_j} > 0$ for every $j$.

4. Therefore, for every $t \in [0, U]$,

$$\frac{d\widehat{\mathbb{Q}}}{d\mathbb{Q}} \Big|\widehat{\mathcal{F}}_t = \mathbb{E}_{\mathbb{Q}}(\widehat{\eta}_U \,|\, \widehat{\mathcal{F}}_t) = \widehat{\eta}_t, \quad \mathbb{Q}\text{-a.s.}$$

# Model Pricing Formula for Credit Default Index Swaptions

**Lemma**

*The price at time $t \in [0, U]$ of the payer credit default index swaption on the pre-collapse event $\{t < \widehat{\tau}\}$ equals*

$$C_t = \widehat{A}^n_t \, \mathbb{E}_{\widehat{\mathbb{Q}}}\big((\kappa^a_U - \kappa)^+ \,\big|\, \widehat{\mathfrak{F}}_t\big) + B_t \, \mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{U \geq \widehat{\tau} > t\}} B_U^{-1} L_U \,\big|\, \widehat{\mathfrak{F}}_t\big).$$

The next lemma establishes the martingale property of the process $\kappa^a$ under $\widehat{\mathbb{Q}}$.

**Lemma**

*The pre-collapse loss-adjusted fair forward CDIS spread $\kappa^a_t$, $t \in [0, U]$, is a strictly positive $\widehat{\mathbb{F}}$-martingale under $\widehat{\mathbb{Q}}$.*

# Black Formula for Credit Default Index Swaptions

1. Our next goal is to establish a suitable version of the Black formula for the credit default index swaption.
2. To this end, we postulate that the pre-collapse loss-adjusted fair forward CDIS spread satisfies

$$\kappa_t^a = \kappa_0^a + \int_0^t \sigma_u \kappa_u^a \, d\widehat{W}_u, \quad \forall \, t \in [0, U],$$

   where $\widehat{W}$ is the one-dimensional standard Brownian motion under $\widehat{\mathbb{Q}}$ with respect to $\widehat{\mathbb{F}}$ and $\sigma$ is an $\widehat{\mathbb{F}}$-predictable process.
3. The assumption that the filtration $\widehat{\mathbb{F}}$ is the Brownian filtration would be too restrictive, since $\hat{\mathbb{F}} = \mathbb{F} \vee \mathbb{H}^{(1)} \vee \cdots \vee \mathbb{H}^{(n-1)}$ and thus $\hat{\mathbb{F}}$ will typically need to support also discontinuous martingales.

# Market Pricing Formula for Credit Default Index Swaptions

**Proposition**

*Assume that the volatility $\sigma$ of the pre-collapse loss-adjusted fair forward CDIS spread is a positive function. Then the pre-default price of the payer credit default index swaption equals, for every $t \in [0, U]$ on the pre-collapse event $\{t < \widehat{\tau}\}$,*

$$C_t = \widehat{A}^n_t \Big( \kappa^a_t N\big(d_+(\kappa^a_t, t, U)\big) - \kappa N\big(d_-(\kappa^a_t, t, U)\big)\Big) + C^L_t$$

*or, equivalently,*

$$C_t = \widehat{P}^a_t \, N\big(d_+(\kappa^a_t, t, U)\big) - \kappa \widehat{A}^n_t \, N\big(d_-(\kappa^a_t, t, U)\big) + C^L_t,$$

*where*

$$d_\pm(\kappa^a_t, t, U) = \frac{\ln(\kappa^a_t/\kappa) \pm \frac{1}{2} \int_t^U \sigma^2(u)\, du}{\big(\int_t^U \sigma^2(u)\, du\big)^{1/2}}.$$

# Approximation

## Proposition

*The price of a payer credit default index swaption can be approximated as follows*

$$C_t \approx \mathbb{1}_{\{t<\widehat{\tau}\}} \widehat{A}_t^n \Big( \kappa_t^n N\big(d_+(\kappa_t^n, t, U)\big) - (\kappa - \bar{L}_t) N\big(d_-(\kappa_t^n, t, U)\big) \Big),$$

*where for every $t \in [0, U]$*

$$d_\pm(\kappa_t^n, t, U) = \frac{\ln(\kappa_t^n/(\kappa - \bar{L}_t)) \pm \frac{1}{2} \int_t^U \sigma^2(u)\, du}{\big( \int_t^U \sigma^2(u)\, du \big)^{1/2}}$$

*and*

$$\bar{L}_t = \mathbb{E}_{\widehat{\mathbb{Q}}}\big( (A_U^n)^{-1} L_U \,|\, \widehat{\mathfrak{F}}_t \big).$$

# Comments

1. Under usual circumstances, the probability of all defaults occurring prior to $U$ is expected to be very low.
2. However, as argued by Morini and Brigo (2007), this assumption is not always justified, in particular, it is not suitable for periods when the market conditions deteriorate.
3. It is also worth mentioning that since we deal here with the risk-neutral probability measure, the probabilities of default events are known to drastically exceed statistically observed default probabilities, that is, probabilities of default events under the physical probability measure.

# Market Models for CDS Spreads

# Notation

1. Let $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{Q})$ be a filtered probability space, where $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ is a filtration such that $\mathcal{F}_0$ is trivial.
2. We assume that the random time $\tau$ defined on this space is such that the $\mathbb{F}$-*survival process* $G_t = \mathbb{Q}(\tau > t \,|\, \mathcal{F}_t)$ is positive.
3. The probability measure $\mathbb{Q}$ is interpreted as the risk-neutral measure.
4. Let $0 < T_0 < T_1 < \cdots < T_n$ be a fixed *tenor structure* and let us write $a_i = T_i - T_{i-1}$.
5. We denote $\widetilde{a}_i = a_i/(1 - \delta_i)$ where $\delta_i$ is the recovery rate if default occurs between $T_{i-1}$ and $T_i$.
6. We denote by $\beta(t, T)$ the default-free discount factor over the time period $[t, T]$.

# Bottom-up Approach under Deterministic Interest Rates

1. Assume first that the interest rate is deterministic.
2. The *pre-default forward CDS spread* $\kappa^i$ corresponding to the single-period forward CDS starting at time $T_{i-1}$ and maturing at $T_i$ equals

$$1 + \widetilde{a}_i \kappa^i_t = \frac{\mathbb{E}_{\mathbb{Q}}\left(\beta(t,T_i)\mathbb{1}_{\{\tau > T_{i-1}\}} \,\middle|\, \mathcal{F}_t\right)}{\mathbb{E}_{\mathbb{Q}}\left(\beta(t,T_i)\mathbb{1}_{\{\tau > T_i\}} \,\middle|\, \mathcal{F}_t\right)}, \quad \forall\, t \in [0, T_{i-1}].$$

3. Since the interest rate is deterministic, we obtain, for $i = 1, \dots, n$,

$$1 + \widetilde{a}_i \kappa^i_t = \frac{\mathbb{Q}(\tau > T_{i-1} \,|\, \mathcal{F}_t)}{\mathbb{Q}(\tau > T_i \,|\, \mathcal{F}_t)}, \quad \forall\, t \in [0, T_{i-1}],$$

and thus

$$\frac{\mathbb{Q}(\tau > T_i \,|\, \mathcal{F}_t)}{\mathbb{Q}(\tau > T_0 \,|\, \mathcal{F}_t)} = \prod_{j=1}^{i} \frac{1}{1 + \widetilde{a}_j \kappa^j_t}, \quad \forall\, t \in [0, T_0].$$

# Auxiliary Probability Measure $\mathbb{P}$

We define the probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $(\Omega, \mathcal{F}_T)$ by setting, for every $t \in [0, T]$,

$$\eta_t = \frac{d\mathbb{P}}{d\mathbb{Q}}\Big|_{\mathcal{F}_t} = \frac{\mathbb{Q}(\tau > T_n \,|\, \mathcal{F}_t)}{\mathbb{Q}(\tau > T_n \,|\, \mathcal{F}_0)}.$$

## Lemma

*For every $i = 1, \dots, n$, the process $Z^{\kappa,i}$ given by*

$$Z_t^{\kappa,i} = \prod_{j=i+1}^{n} \left(1 + \widetilde{a}_j \kappa_t^j\right), \quad \forall\, t \in [0, T_i],$$

*is a positive $(\mathbb{P}, \mathbb{F})$-martingale.*

# CDS Martingale Measures

1. For any $i = 1, \ldots, n$ we define the probability measure $\mathbb{P}^i$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ by setting (note that $Z_t^{\kappa,n} = 1$ and thus $\mathbb{P}^n = \mathbb{P}$)

$$\frac{d\mathbb{P}^i}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = c_i Z_t^{\kappa,i} = \frac{\mathbb{Q}(\tau > T_i)}{\mathbb{Q}(\tau > T_n)} \prod_{j=i+1}^{n} \left(1 + \widetilde{a}_j \kappa_t^j\right).$$

2. Assume that the PRP holds under $\mathbb{P} = \mathbb{P}^n$ with the $\mathbb{R}^k$-valued spanning $(\mathbb{P}, \mathbb{F})$-martingale $M$. Then the PRP is also valid with respect to $\mathbb{F}$ under any probability measure $\mathbb{P}^i$ for $i = 1, \ldots, n$.
3. The positive process $\kappa^i$ is a $(\mathbb{P}^i, \mathbb{F})$-martingale and thus it satisfies, for $i = 1, \ldots, n$,

$$\kappa_t^i = \kappa_0^i + \int_{(0,t]} \kappa_s^i \sigma_s^i \cdot d\Psi^i(M)_s$$

for some $\mathbb{R}^k$-valued, $\mathbb{F}$-predictable process $\sigma^i$, where $\Psi^i(M)$ is the $\mathbb{P}^i$-Girsanov transform of $M$

$$\Psi^i(M)_t = M_t^i - \int_{(0,t]} (Z_s^i)^{-1} \, d[Z^i, M]_s.$$

# Dynamics of Forward CDS Spreads

## Proposition

*Let the processes* $\kappa^i$, $i = 1, \ldots, n$, *be defined by*

$$1 + \widetilde{a}_i \kappa^i_t = \frac{\mathbb{E}_{\mathbb{Q}}\left(\beta(t, T_i)\mathbb{1}_{\{\tau > T_{i-1}\}} \,\middle|\, \mathcal{F}_t\right)}{\mathbb{E}_{\mathbb{Q}}\left(\beta(t, T_i)\mathbb{1}_{\{\tau > T_i\}} \,\middle|\, \mathcal{F}_t\right)}, \quad \forall\, t \in [0, T_{i-1}].$$

*Assume that the PRP holds with respect to* $\mathbb{F}$ *under* $\mathbb{P}$ *with the spanning* $(\mathbb{P}, \mathbb{F})$*-martingale* $M = (M^1, \ldots, M^k)$. *Then there exist* $\mathbb{R}^k$*-valued,* $\mathbb{F}$*-predictable processes* $\sigma^i$ *such that the joint dynamics of processes* $\kappa^i$, $i = 1, \ldots, n$ *under* $\mathbb{P}$ *are given by*

$$d\kappa^i_t = \sum_{l=1}^{k} \kappa^i_t \sigma^{i,l}_t \, dM^l_t - \sum_{j=i+1}^{n} \frac{\widetilde{a}_j \kappa^i_t \kappa^j_t}{1 + \widetilde{a}_j \kappa^j_t} \sum_{l,m=1}^{k} \sigma^{i,l}_t \sigma^{j,m}_t \, d[M^{l,c}, M^{m,c}]_t$$

$$- \frac{1}{Z^i_{t-}} \Delta Z^i_t \sum_{l=1}^{k} \kappa^i_t \sigma^{i,l}_t \Delta M^l_t.$$

# Top-down Approach: First Step

## Proposition

*Assume that:*
*(i) the positive processes* $\kappa^i$, $i = 1, \dots, n$, *are such that the processes* $Z^{\kappa,i}$, $i = 1, \dots, n$ *are* $(\mathbb{P}, \mathbb{F})$*-martingales, where*

$$Z_t^{\kappa,i} = \prod_{j=i+1}^{n} \left(1 + \widetilde{a}_j \kappa_t^j\right).$$

*(ii)* $M = (M^1, \dots, M^k)$ *is a spanning* $(\mathbb{P}, \mathbb{F})$*-martingale.*
*(iii)* $\sigma^i$, $i = 1, \dots, n$ *are* $\mathbb{R}^k$*-valued,* $\mathbb{F}$*-predictable processes.*
*Then:*
*(i) for every* $i = 1, \dots, n$, *the process* $\kappa^i$ *is a* $(\mathbb{P}^i, \mathbb{F})$*-martingale where*

$$\frac{d\mathbb{P}^i}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} = c_i \prod_{j=i+1}^{n} \left(1 + \widetilde{a}_j \kappa_t^j\right),$$

*(ii) the joint dynamics of processes* $\kappa^i$, $i = 1, \dots, n$ *under* $\mathbb{P}$ *are given by the previous proposition.*

# Top-down Approach: Second Step

1. We will now construct a default time $\tau$ consistent with the dynamics of forward CDS spreads. Let us set

$$M^{i-1}_{T_{i-1}} = \prod_{j=1}^{i-1} \frac{1}{1 + \widetilde{a}_j \kappa^j_{T_{i-1}}}, \qquad M^{i}_{T_i} = \prod_{j=1}^{i} \frac{1}{1 + \widetilde{a}_j \kappa^j_{T_i}}.$$

2. Since the process $\widetilde{a}_i \kappa^i$ is positive, we obtain, for every $i = 0, \dots, n$,

$$G_{T_i} := M^i_{T_i} = \frac{M^{i-1}_{T_{i-1}}}{1 + \widetilde{a}_i \kappa^i_{T_i}} \leq M^{i-1}_{T_{i-1}} =: G^{i-1}_{T_{i-1}}.$$

3. The process $G_{T_i} = M^i_{T_i}$ is thus decreasing for $i = 0, \dots, n$.
4. We make use of the canonical construction of default time $\tau$ taking values in $\{T_0, \dots, T_n\}$.
5. We obtain, for every $i = 0, \dots, n$,

$$\mathbb{P}(\tau > T_i \,|\, \mathcal{F}_{T_i}) = G_{T_i} = \prod_{j=1}^{i} \frac{1}{1 + \widetilde{a}_j \kappa^j_{T_i}}.$$

# Bottom-up Approach under Independence

Assume that we are given a model for Libors $(L^1, \dots, L^n)$ where $L^i = L(t, T_{i-1})$ and CDS spreads $(\kappa^1, \dots, \kappa^n)$ in which:

1. The default intensity $\gamma$ generates the filtration $\mathbb{F}^\gamma$.
2. The interest rate process $r$ generates the filtration $\mathbb{F}^r$.
3. The probability measure $\mathbb{Q}$ is the spot martingale measure.
4. The $\mathbb{H}$-hypothesis holds, that is, $\mathbb{F} \overset{\mathbb{Q}}{\hookrightarrow} \mathbb{G}$, where $\mathbb{F} = \mathbb{F}^r \vee \mathbb{F}^\gamma$.
5. The PRP holds with the $(\mathbb{Q}, \mathbb{F})$-spanning martingale $M$.

**Lemma**

*It is possible to determine the joint dynamics of Libors and CDS spreads* $(L^1, \dots, L^n, \kappa^1, \dots, \kappa^n)$ *under any martingale measure* $\mathbb{P}^i$.

# Top-down Approach under Independence

To construct a model we assume that:

1. A martingale $M = (M^1, \dots, M^k)$ has the PRP with respect to $(\mathbb{P}, \mathbb{F})$.
2. The family of process $Z^i$ given by

$$Z_t^{L,\kappa,i} := \prod_{j=i+1}^{n} (1 + a_j L_t^j)(1 + \widetilde{a}_j \kappa_t^j)$$

   are martingales on the filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$.
3. Hence there exists a family of probability measures $\mathbb{P}^i$, $i = 1, \dots, n$ on $(\Omega, \mathcal{F}_T)$ with the densities

$$\frac{d\mathbb{P}^i}{d\mathbb{P}} = c_i Z^{L,\kappa,i}.$$

# Dynamics of LIBORs and CDS Spreads

## Proposition

*The dynamics of* $L^i$ *and* $\kappa^i$ *under* $\mathbb{P}^n$ *with respect to the spanning* $(\mathbb{P},\mathbb{F})$*-martingale* $M$ *are given by*

$$
\begin{aligned}
dL^i_t = {} & \sum_{l=1}^{k} \xi^{i,l}_t \, dM^l_t - \sum_{j=i+1}^{n} \frac{a_j}{1+a_j L^j_t} \sum_{l,m=1}^{k} \xi^{i,l}_t \xi^{j,m}_t \, d[M^{l,c}, M^{m,c}]_t \\
& - \sum_{j=i+1}^{n} \frac{\widetilde{a}_j}{1+\widetilde{a}_j \kappa^j_t} \sum_{l,m=1}^{k} \xi^{i,l}_t \sigma^{j,m}_t \, d[M^{l,c}, M^{m,c}]_t - \frac{1}{Z^i_t} \Delta Z^i_t \sum_{l=1}^{k} \xi^{i,l}_t \Delta M^l_t
\end{aligned}
$$

*and*

$$
\begin{aligned}
d\kappa^i_t = {} & \sum_{l=1}^{k} \sigma^{i,l}_t \, dM^l_t - \sum_{j=i+1}^{n} \frac{a_j}{1+a_j L^j_t} \sum_{l,m=1}^{k} \sigma^{i,l}_t \xi^{j,m}_t \, d[M^{l,c}, M^{m,c}]_t \\
& - \sum_{j=i+1}^{n} \frac{\widetilde{a}_j}{1+\widetilde{a}_j \kappa^j_t} \sum_{l,m=1}^{k} \sigma^{i,l}_t \sigma^{j,m}_t \, d[M^{l,c}, M^{m,c}]_t - \frac{1}{Z^i_t} \Delta Z^i_t \sum_{l=1}^{k} \sigma^{i,l}_t \Delta M^l_t.
\end{aligned}
$$

# Bottom-up Approach: One- and Two-Period Spreads

1. Let $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{Q})$ be a filtered probability space, where $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ is a filtration such that $\mathcal{F}_0$ is trivial.
2. We assume that the random time $\tau$ defined on this space is such that the $\mathbb{F}$-*survival process* $G_t = \mathbb{Q}(\tau > t \,|\, \mathcal{F}_t)$ is positive.
3. The probability measure $\mathbb{Q}$ is interpreted as the risk-neutral measure.
4. Let $0 < T_0 < T_1 < \cdots < T_n$ be a fixed *tenor structure* and let us write $a_i = T_i - T_{i-1}$ and $\widetilde{a}_i = a_i/(1 - \delta_i)$
5. We no longer assume that the interest rate is deterministic.
6. We denote by $\beta(t, T)$ the default-free discount factor over the time period $[t, T]$.

# One-Period CDS Spreads

The *one-period forward CDS spread* $\kappa^i = \kappa^{i-1,i}$ satisfies, for $t \in [0, T_{i-1}]$,

$$1 + \widetilde{a}_i \kappa^i_t = \frac{\mathbb{E}_{\mathbb{Q}} \left( \beta(t, T_i) \mathbb{1}_{\{\tau > T_{i-1}\}} \,\middle|\, \mathcal{F}_t \right)}{\mathbb{E}_{\mathbb{Q}} \left( \beta(t, T_i) \mathbb{1}_{\{\tau > T_i\}} \,\middle|\, \mathcal{F}_t \right)}.$$

Let $A^{i-1,i}$ be the *one-period CDS annuity*

$$A^{i-1,i}_t = \widetilde{a}_i \, \mathbb{E}_{\mathbb{Q}} \left( \beta(t, T_i) \mathbb{1}_{\{\tau > T_i\}} \,\middle|\, \mathcal{F}_t \right)$$

and let

$$P^{i-1,i}_t = \mathbb{E}_{\mathbb{Q}} \left( \beta(t, T_i) \mathbb{1}_{\{\tau > T_{i-1}\}} \,\middle|\, \mathcal{F}_t \right) - \mathbb{E}_{\mathbb{Q}} \left( \beta(t, T_i) \mathbb{1}_{\{\tau > T_i\}} \,\middle|\, \mathcal{F}_t \right).$$

Then

$$\kappa^i_t = \frac{P^{i-1,i}_t}{A^{i-1,i}_t}, \quad \forall\, t \in [0, T_{i-1}].$$

# One-Period CDS Spreads

Let $A^{i-2,i}$ stand for the *two-period CDS annuity*

$$A_t^{i-2,i} = \widetilde{a}_{i-1}\,\mathbb{E}_{\mathbb{Q}}\left(\beta(t,T_{i-1})\mathbb{1}_{\{\tau>T_{i-1}\}}\,\middle|\,\mathcal{F}_t\right) + \widetilde{a}_i\,\mathbb{E}_{\mathbb{Q}}\left(\beta(t,T_i)\mathbb{1}_{\{\tau>T_i\}}\,\middle|\,\mathcal{F}_t\right)$$

and let

$$P_t^{i-2,i} = \sum_{j=i-1}^{i}\left(\mathbb{E}_{\mathbb{Q}}\left(\beta(t,T_j)\mathbb{1}_{\{\tau>T_{j-1}\}}\,\middle|\,\mathcal{F}_t\right) - \mathbb{E}_{\mathbb{Q}}\left(\beta(t,T_j)\mathbb{1}_{\{\tau>T_j\}}\,\middle|\,\mathcal{F}_t\right)\right).$$

The *two-period CDS spread* $\widetilde{\kappa}^i = \kappa^{i-2,i}$ is given by the following expression

$$\widetilde{\kappa}_t^i = \kappa_t^{i-2,i} = \frac{P_t^{i-2,i}}{A_t^{i-2,i}} = \frac{P_t^{i-2,i-1} + P_t^{i-1,i}}{A_t^{i-2,i-1} + A_t^{i-1,i}}, \quad \forall\, t \in [0, T_{i-1}].$$

# One-Period CDS Measures

1. Our aim is to derive the semimartingale decomposition of $\kappa^i, i = 1, \dots, n$ and $\widetilde{\kappa}^i$, $i = 2, \dots, n$ under a common probability measure.
2. We start by noting that the process $A^{n-1,n}$ is a positive $(\mathbb{Q}, \mathbb{F})$-martingale and thus it defines the probability measure $\mathbb{P}^n$ on $(\Omega, \mathcal{F}_T)$.
3. The following processes are easily seen to be $(\mathbb{P}^n, \mathbb{F})$-martingales

$$\frac{A_t^{i-1,i}}{A_t^{n-1,n}} = \prod_{j=i+1}^{n} \frac{\widetilde{a}_j(\widetilde{\kappa}_t^j - \kappa_t^j)}{\widetilde{a}_{j-1}(\kappa_t^{j-1} - \widetilde{\kappa}_t^j)} = \frac{\widetilde{a}_n}{\widetilde{a}_i} \prod_{j=i+1}^{n} \frac{\widetilde{\kappa}_t^j - \kappa_t^j}{\kappa_t^{j-1} - \widetilde{\kappa}_t^j}.$$

4. Given this family of positive $(\mathbb{P}^n, \mathbb{F})$-martingales, we define a family of probability measures $\mathbb{P}^i$ for $i = 1, \dots, n$ such that $\kappa^i$ is a martingale under $\mathbb{P}^i$.

# Two-Period CDS Measures

1. For every $i = 2, \dots, n$, the following process is a $(\mathbb{P}^i, \mathbb{F})$-martingale

$$\begin{aligned} \frac{A_t^{i-2,i}}{A_t^{i-1,i}} &= \frac{\widetilde{a}_{i-1} \mathbb{E}_{\mathbb{Q}} \left( \beta(t, T_{i-1}) \mathbb{1}_{\{\tau > T_{i-1}\}} \,\middle|\, \mathcal{F}_t \right) + \widetilde{a}_i \mathbb{E}_{\mathbb{Q}} \left( \beta(t, T_i) \mathbb{1}_{\{\tau > T_i\}} \,\middle|\, \mathcal{F}_t \right)}{\mathbb{E}_{\mathbb{Q}} \left( \beta(t, T_i) \mathbb{1}_{\{\tau > T_i\}} \,\middle|\, \mathcal{F}_t \right)} \\ &= \widetilde{a}_{i-1} \left( \frac{A_t^{i-2,i-1}}{A_t^{i-1,i}} + 1 \right) \\ &= \widetilde{a}_i \left( \frac{\widetilde{\kappa}_t^i - \kappa_t^i}{\kappa_t^{i-1} - \widetilde{\kappa}_t^i} + 1 \right). \end{aligned}$$

2. Therefore, we can define a family of the associated probability measures $\widetilde{\mathbb{P}}^i$ on $(\Omega, \mathcal{F}_T)$, for every $i = 2, \dots, n$.
3. It is obvious that $\widetilde{\kappa}^i$ is a martingale under $\widetilde{\mathbb{P}}^i$ for every $i = 2, \dots, n$.

# One and Two-Period CDS Measures

We will summarise the above in the following diagram

$$\begin{tikzcd}
\mathbb{Q} \arrow[r, "\frac{d\mathbb{P}^n}{d\mathbb{Q}}"] & \mathbb{P}^n \arrow[r, "\frac{d\mathbb{P}^{n-1}}{d\mathbb{P}^n}"] \arrow[d, "\frac{d\widetilde{\mathbb{P}}^n}{d\mathbb{P}^n}"'] & \mathbb{P}^{n-1} \arrow[r, "\frac{d\mathbb{P}^{n-2}}{d\mathbb{P}^{n-1}}"] \arrow[d, "\frac{d\widetilde{\mathbb{P}}^{n-1}}{d\mathbb{P}^{n-1}}"'] & \dots \arrow[r] \arrow[d] & \mathbb{P}^2 \arrow[r] \arrow[d, "\frac{d\widetilde{\mathbb{P}}^2}{d\mathbb{P}^2}"'] & \mathbb{P}^1 \\
 & \widetilde{\mathbb{P}}^n & \widetilde{\mathbb{P}}^{n-1} & \dots & \widetilde{\mathbb{P}}^2 &
\end{tikzcd}$$

where

$$\frac{d\mathbb{P}^n}{d\mathbb{Q}} = A_t^{n-1,n}$$

$$\frac{d\mathbb{P}^i}{d\mathbb{P}^{i+1}} = \frac{A_t^{i-1,i}}{A_t^{i,i+1}} = \frac{\widetilde{a}_{i+1}}{\widetilde{a}_i} \left( \frac{\widetilde{\kappa}_t^{i+1} - \kappa_t^{i+1}}{\kappa_t^i - \widetilde{\kappa}_t^{i+1}} \right)$$

$$\frac{d\widetilde{\mathbb{P}}^i}{d\mathbb{P}^i} = \frac{A_t^{i-2,i}}{A_t^{i-1,i}} = \widetilde{a}_i \left( \frac{\widetilde{\kappa}_t^i - \kappa_t^i}{\kappa_t^{i-1} - \widetilde{\kappa}_t^i} + 1 \right).$$

# Bottom-up Approach: Joint Dynamics

1. We are in a position to calculate the semimartingale decomposition of $(\kappa^1, \ldots, \kappa^n, \widetilde{\kappa}^2, \ldots, \widetilde{\kappa}^n)$ under $\mathbb{P}^n$.
2. It suffices to use the following Radon-Nikodým densities

$$\frac{d\mathbb{P}^i}{d\mathbb{P}^n} = \frac{A_t^{i-1,i}}{A_t^{n-1,n}} = \frac{\widetilde{a}_n}{\widetilde{a}_i} \prod_{j=i+1}^{n} \frac{\widetilde{\kappa}_t^j - \kappa_t^j}{\kappa_t^{j-1} - \widetilde{\kappa}_t^j}$$

$$\begin{aligned} \frac{d\widetilde{\mathbb{P}}^i}{d\mathbb{P}^n} = \frac{A_t^{i-2,i}}{A_t^{n-1,n}} &= \widetilde{a}_n \left( \frac{\widetilde{\kappa}_t^i - \kappa_t^i}{\kappa_t^{i-1} - \widetilde{\kappa}_t^i} + 1 \right) \prod_{j=i+1}^{n} \frac{\widetilde{\kappa}_t^j - \kappa_t^j}{\kappa_t^{j-1} - \widetilde{\kappa}_t^j} \\ &= \widetilde{a}_n \left( \prod_{j=i}^{n} \frac{\widetilde{\kappa}_t^j - \kappa_t^j}{\kappa_t^{j-1} - \widetilde{\kappa}_t^j} + \prod_{j=i+1}^{n} \frac{\widetilde{\kappa}_t^j - \kappa_t^j}{\kappa_t^{j-1} - \widetilde{\kappa}_t^j} \right) \\ &= \widetilde{a}_{i-1} \frac{d\mathbb{P}^{i-1}}{d\mathbb{P}^n} + \widetilde{a}_i \frac{d\mathbb{P}^i}{d\mathbb{P}^n}. \end{aligned}$$

3. Explicit formulae for the joint dynamics of one and two-period spreads are available.

# Top-down Approach: Postulates

1. The processes $\kappa^1, \dots, \kappa^n$ and $\widetilde{\kappa}^2, \dots, \widetilde{\kappa}^n$ are $\mathbb{F}$-adapted.
2. For every $i = 1, \dots, n$, the process $Z^{\kappa,i}$

$$Z_t^{\kappa,i} = \frac{c_n}{c_i} \prod_{j=i+1}^{n} \frac{\widetilde{\kappa}_t^j - \kappa_t^j}{\kappa_t^{j-1} - \widetilde{\kappa}_t^j}$$

   is a positive $(\mathbb{P}, \mathbb{F})$-martingale where $c_1, \dots, c_n$ are constants.
3. For every $i = 2, \dots, n$, the process $Z^{\widetilde{\kappa},i}$ given by the formula

$$Z^{\widetilde{\kappa},i} = \widetilde{c}_i (Z^{\kappa,i} + Z^{\kappa,i-1}) = \widetilde{c}_i \frac{\kappa^{i-1} - \kappa^i}{\kappa^{i-1} - \widetilde{\kappa}^i} Z^{\kappa,i}$$

   is a positive $(\mathbb{P}, \mathbb{F})$-martingale where $\widetilde{c}_2, \dots, \widetilde{c}_n$ are constants.
4. The process $M = (M^1, \dots, M^k)$ is the $(\mathbb{P}, \mathbb{F})$-spanning martingale.
5. Probability measures $\mathbb{P}^i$ and $\widetilde{\mathbb{P}}^i$ have the density processes $Z^{\kappa,i}$ and $Z^{\widetilde{\kappa},i}$. In particular, the equality $\mathbb{P}^n = \mathbb{P}$ holds, since $Z^{\kappa,n} = 1$.
6. Processes $\kappa^i$ and $\widetilde{\kappa}^i$ are martingales under $\mathbb{P}^i$ and $\widetilde{\mathbb{P}}^i$, respectively.

# Top-down Approach: Lemma

## Lemma

*Let* $M = (M^1, \dots, M^k)$ *be the* $(\mathbb{P}, \mathbb{F})$*-spanning martingale. For any* $i = 1, \dots, n$*, the process* $X^i$ *admits the integral representation*

$$\kappa_t^i = \int_{(0,t]} \sigma_s^i \cdot d\Psi^i(M)_s$$

*and*

$$\widetilde{\kappa}_t^i = \int_{(0,t]} \zeta_s^i \cdot d\widetilde{\Psi}^i(M)_s$$

*where* $\sigma^i = (\sigma^{i,1}, \dots, \sigma^{i,k})$ *and* $\zeta^i = (\zeta^{i,1}, \dots, \zeta^{i,k})$ *are* $\mathbb{R}^k$*-valued,* $\mathbb{F}$*-predictable processes that can be chosen arbitrarily. The* $(\mathbb{P}^i, \mathbb{F})$*-martingale* $\Psi^i(M^l)$ *is given by*

$$\Psi^i(M^l)_t = M_t^l - \left[(\ln Z^{\kappa,i})^c, M^{l,c}\right]_t - \sum_{0<s\leq t} \frac{1}{Z_s^{\kappa,i}} \Delta Z_s^{\kappa,i} \Delta M_s^l.$$

*An analogous formula holds for the Girsanov transform* $\widetilde{\Psi}^i(M^l)$*.*

# Top-down Approach: Joint Dynamics

## Proposition

*The semimartingale decomposition of the* $(\mathbb{P}^i, \mathbb{F})$*-spanning martingale* $\Psi^i(M)$ *under the probability measure* $\mathbb{P}^n = \mathbb{P}$ *is given by, for* $i = 1, \ldots, n$*,*

$$\Psi^i(M)_t = M_t - \sum_{j=i+1}^{n} \int_{(0,t]} \frac{(\kappa_s^{j-1} - \kappa_s^j)\, \zeta_s^j \cdot d[M^c]_s}{(\widetilde{\kappa}_s^j - \kappa_s^j)(\kappa_s^{j-1} - \widetilde{\kappa}_s^j)} - \sum_{j=i+1}^{n} \int_{(0,t]} \frac{\sigma_s^j \cdot d[M^c]_s}{\widetilde{\kappa}_s^j - \kappa_s^j}$$

$$- \sum_{j=i+1}^{n} \int_{(0,t]} \frac{\sigma_s^{j-1} \cdot d[M^c]_s}{\kappa_s^{j-1} - \widetilde{\kappa}_s^j} - \sum_{0<s\leq t} \frac{1}{Z_s^{\kappa,i}} \Delta Z_s^{\kappa,i} \Delta M_s.$$

*An analogous formula holds for* $\widetilde{\Psi}^i(M)$*. Hence the joint dynamics of the process* $(\kappa^1, \ldots, \kappa^n, \widetilde{\kappa}^2, \ldots, \widetilde{\kappa}^n)$ *under* $\mathbb{P} = \mathbb{P}^n$ *are explicitly known.*

# Towards Generic Swap Models

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a filtered probability space. Suppose that we are given a family of swaps $\mathcal{S} = \{\kappa^1, \dots, \kappa^l\}$ and a family of processes $\{Z^1, \dots, Z^l\}$ satisfying the following conditions for every $j = 1, \dots, l$:

1. the process $\kappa^j$ is a positive special semimartingale,
2. the process $\kappa^j Z^j$ is a $(\mathbb{P}, \mathbb{F})$-martingale,
3. the process $Z^j$ is a positive $(\mathbb{P}, \mathbb{F})$-martingale with $Z^j_0 = 1$,
4. the process $Z^j$ is uniquely expressed as a function of some subset of swaps in $\mathcal{S}$, specifically, $Z^j = f_j(\kappa^{n_1}, \dots, \kappa^{n_k})$ where $f_j : \mathbb{R}^k \to \mathbb{R}$ is a $C^2$ function in variables belonging to $\{\kappa^{n_1}, \dots, \kappa^{n_k}\} \subset \mathcal{S}$.

# Volatility-Based Modelling

1. For the purpose of modelling, we select a $(\mathbb{P}, \mathbb{F})$-martingale $M$ and we define $\kappa^j$ under $\mathbb{P}^j$ as follows
$$\kappa^j_t = \int_0^t \kappa^j_s \sigma^j_s \cdot d\Psi^j(M)_s.$$
2. Therefore, specifying $\kappa^j$ is equivalent to specifying the "volatility" $\sigma^j$.
3. The martingale part of $\kappa^j$ can be expressed as
$$(\kappa^j)^m_t = \int_0^t \kappa^j_s \sigma^j_s \cdot d\Psi^j(M)_s - \int_{(0,t]} Z^j_s \kappa^j_s \sigma^j_s \cdot d\left[\frac{1}{Z^j}, \Psi^j(M)\right]_s = \int_0^t \kappa^j_s \sigma^j_s \cdot dM^j_s$$
where $M^j$ is a $(\mathbb{P}, \mathbb{F})$-martingale.
4. The Radon-Nikodým density process $Z^j$ has the following decomposition
$$Z^j_t = \sum_{i=1}^k \int_{[0,t)} \frac{\partial f_j}{\partial x_i}(\kappa^{n_1}_s, \dots, \kappa^{n_k}_s) \kappa^{n_i}_s \sigma^{n_i}_s \cdot dM^{n_i}_s.$$
5. Hence the choice of "volatilities" completely specifies the model.