CDS Calibration with Tractable Structural Models with an application to Equity Return Swap valuation under Counterparty Risk

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Talk outline

- Basic ideas of Structural Models: comparison with Intensity Models;
- Analytically Tractable First Passage model (AT1P);
- Credit Default Swap calibration with the AT1P model: the Lehman example;
- Extension: Scenario Barrier Time-Varying Volatility model (SBTV);
- Using Structural Models to price counterparty risk and hybrid equity/credit products: the Equity Return Swap example;
- Structural Models: multiname products;
- Conclusions;
- References.

Structural Models: Basic Ideas

The stylized structure of the firm economy is modeled through:

- $V(t)$: **stochastic value for the value of the firm**;
- $t \mapsto H(t)$: **default barrier representing the debt** of the firm and safety covenants;
- τ: **the default time is the first time instant where the value of the firm V touches the safety barrier H**.

The basic idea is that if the firm value goes below the safety level, then the firm is deemed to be no longer able to pay its debt and is forced into bankruptcy.

Default is induced by observable market information (the value of the firm V).

Important difference with Intensity Models: In basic Structural Models there is nothing external to the basic market information in the default process. Default is induced by a completely observable variable, the value of the firm.

For these models we have $\mathcal{G}_t = \mathcal{F}_t$, i.e. history on past and present default plus the basic market information coincides with the history of the basic market information itself.

Intensity (Reduced Form) Models: Basic Ideas

- The default time τ obeys roughly the following:

$$\text{Prob}(\tau \in [t, t+dt) | \tau > t, \text{ market info up to } t) = \lambda(t)dt$$

- λ is called **intensity** or **hazard rate**. It is also an instantaneous credit spread.
- If λ is deterministic, τ is the first jump time of a Poisson Process.
- The jump is **NOT** triggered by basic market observables: It is an **exogenous component**.
- The survival probability can be computed by

$$\text{Prob}(\tau > t) = e^{-\int_0^t \lambda(u)du}$$

Intensity and Structural Models: Different Uses

At this stage it is not possible to say that intensity models are better than structural models or vice versa. The two kind of models are both useful and both needed, and are to be used for different products.

Intensity Models:

- Intensity models offer parallels with interest rate models, and are thus more suited to model credit spreads;
- They are typically easier to calibrate to corporate bond or Credit Default Swap market information;
- More suited to refined relative value pricing (CDS options etc);
- In cases with stochastic intensity the extension to multiname situations can be difficult (First to default, CDO's, etc);
- Calibration of the "default correlation" component (jump-terms Copula function) among different names is not clear and often based on debatable links with the equity market.

Intensity and Structural Models: Different Uses

Structural Models:

- Structural models are easier to use in situations where we need to model also equity variables taking into account correlation;
- In their basic formulation they are more suited to "fundamental pricing" that to refined relative value pricing;
- Cases include equity return swaps with counterparty risk, total rate of return swaps, and counterparty risk in any equity product;
- They are more difficult to calibrate with precision to Credit Default Swaps or Corporate Bonds data;
- They are more naturally extended to multi-name situation (no "out of the blue" copula) than stochastic intensity models;
- Different names default correlation has a much more grounded link with equity correlation than in intensity models; in principle it would suffice to estimate historical correlation of equity returns.

Structural Models: Merton's Model

The first Structural Model is due to Merton (1974). **The value of the firm V is assumed to be a tradable asset and to follow a standard Geometric Brownian Motion**. Under the risk neutral measure:

$$dV(t) = (r - k)V(t)dt + \sigma V(t)dW(t)$$

(r is the risk free rate, k is the payout ratio and σ is the volatility, all constant).

This dynamics is lognormal; Crouhy et al (2000) notice that "this assumption [lognormal V] is quite robust and, according to KMVs own empirical studies, actual data conform quite well to this hypothesis".

V is seen as composed by the equity part S and the debt part D, so that at each point in time the following equivalence holds:

$$\text{Firm Value} = \text{Debt Value} + \text{Equity}, \quad V(t) = D(t) + S(t)$$

Structural Models: Merton's Model

Simple assumption for debt: **zero coupon debt** with maturity $\bar{T}$ and face value L.

Default is linked to capability of the firm to pay back all the debt issued.

If at maturity $\bar{T}$ the firm value V is greater than L, then all the debt is paid back and the firm survives; if V is smaller than L then the company is not able to pay the bondholders and then there is the default. Analytically

$$\tau = \bar{T}\mathbf{1}_{\{V_{\bar{T}}<L\}} + \infty\mathbf{1}_{\{V_{\bar{T}}\geq L\}}$$

Default can happen only at the debt maturity $\bar{T}$. This is a quite restrictive assumption and we will see later in the discussion how it can be relaxed.

The value of the debt at maturity is hence $D_{\bar{T}} = \min(V_{\bar{T}}; L)$, from which, at $t < \bar{T}$

$$D_t = \mathbb{E}_t\left[D(t;\bar{T})\min(V_{\bar{T}};L)\right] = \mathbb{E}_t\left[D(t;\bar{T})\left[V_{\bar{T}} - (V_{\bar{T}} - L)^+\right]\right] =$$

$$= \mathbb{E}_t\left[D(t;\bar{T})\left[L - (L - V_{\bar{T}})^+\right]\right] = P(t;\bar{T})L - \mathsf{Put}(t;\bar{T};V_t;\sigma^2;L)$$

Structural Models: Black-Cox Model

Drawbacks of Merton's Model: default can happen only at the debt maturity $\bar{T}$.

Unsatisfactory: there could be scenarios in which default happens before $\bar{T}$, related to problems of optimal capital structure and stockholders decisions to reorganize the firm.

Black and Cox (1976) assume a barrier representing safety covenants for the firm. Default is triggered by the firm value V hitting this barrier from above. At default the firm reimburses the debt-holders. Let $H(t;\bar{T})$ be the barrier with time dependence on t and final zero coupon debt maturity $\bar{T}$.

Black and Cox assume a constant parameters Geometric Brownian Motion

$$dV(t) = (r-k)V(t)dt + \sigma_V V(t)dW(t)$$

and an exponential barrier (we omit the $\bar{T}$ dependence in H)

$$H(t) = \begin{cases} L & \text{if } t = \bar{T} \\ Ke^{-\gamma(\bar{T}-t)} & \text{if } t < \bar{T} \end{cases}$$

Structural Models: Black-Cox Model

$$H(t) = \begin{cases} L & \text{if } t = \bar{T} \\ Ke^{-\gamma(\bar{T}-t)} & \text{if } t < \bar{T} \end{cases}$$

γ and K are positive parameters. Black-Cox assume also $Ke^{-\gamma(\bar{T}-t)} < Le^{-r(\bar{T}-t)}$.

In the Black-Cox model, the survival probability is given by (see Bielecki and Rutkowski (2002) for derivation)

$$\text{Prob}(\tau > t) = \Phi\left(\frac{\ln\left(\frac{V_0}{H(0)}\right) + \tilde{\nu}t}{\sigma_V\sqrt{t}}\right) - \left(\frac{H(0)}{V_0}\right)^{2\tilde{a}} \Phi\left(\frac{\ln\left(\frac{H(0)}{V_0}\right) + \tilde{\nu}t}{\sigma_V\sqrt{t}}\right)$$

where $\tilde{\nu} = r - k - \gamma - \frac{1}{2}\sigma_V^2$ and $\tilde{a} = \frac{\tilde{\nu}}{\sigma_V^2}$.

Fundamental Credit Derivative: Credit Default Swap

A Credit Default Swap (CDS) is a basic contract ensuring protection against default.

Two parties: The **Protection Buyer** "A" and the **Protection Seller** "B". The contract is written on the default of an underlying company "C" (**Reference Entity**); τ is the the default time of "C".

Protection	$\rightarrow$	protection LGD at default τ if $T_a < \tau \leq T_b$	$\rightarrow$	Protection
Seller "B"	$\leftarrow$	rate R at $T_{a+1}, \dots, T_b$ or until default τ	$\leftarrow$	Buyer "A"

LGD $= 1 -$ REC, where LGD is the **Loss Given Default** and REC is the **Recovery Rate**. The CDS discounted payoff (from a protection buyer viewpoint) is:

$$\Pi_{\text{CDS}}(t; T_a, T_b, \text{Rec}, R) = +\text{Lgd}\left(\mathbf{1}_{\{T_a < \tau \leq T_b\}} D(t;\tau)\right)$$

$$-R\left[\sum_{i=a+1}^{b}\left((T_i - T_{i-1})\mathbf{1}_{\{\tau > T_i\}} D(t;T_i) + (\tau - T_{i-1})\mathbf{1}_{\{T_{i-1} < \tau \leq T_i\}} D(t;\tau)\right)\right] \quad (1)$$

Fundamental Credit Derivative: Credit Default Swap

Ignoring accrued amounts, we can approximate the discounted payoff (1) as:

$$\Pi_{\mathsf{CDS}}(t;T_a,T_b) = \mathrm{L{\scriptstyle GD}} \sum_{i=a+1}^{b} \mathbf{1}_{\{T_{i-1}<\tau\le T_i\}} D(t;T_i) - R \sum_{i=a+1}^{b} (T_i - T_{i-1}) \mathbf{1}_{\{\tau\ge T_i\}} D(t;T_i) \tag{2}$$

According to **Risk Neutral valuation**, the CDS value can be computed as

$$\mathsf{CDS}(t;T_a,T_b,\mathrm{R{\scriptstyle EC}},R) = \mathbb{E}\left[\Pi_{\mathsf{CDS}}(t;T_a,T_b,\mathrm{R{\scriptstyle EC}},R)\right]$$

Since we know that $\text{Prob}(\tau > t) \equiv \mathbb{Q}(\tau > t) = \mathbb{E}\left[\mathbf{1}_{\{\tau>t\}}\right]$ and considering deterministic interest rates, we get:

$$\begin{aligned} \mathsf{CDS}(t;T_a,T_b,\mathrm{R{\scriptstyle EC}},R) &= \mathrm{L{\scriptstyle GD}} \sum_{i=a+1}^{b} \left[\mathbb{Q}(\tau > T_{i-1}) - \mathbb{Q}(\tau > T_i)\right] P(t;T_i) \\ &- R \sum_{i=a+1}^{b} (T_i - T_{i-1}) \mathbb{Q}(\tau \ge T_i) P(t;T_i) \end{aligned} \tag{3}$$

Fundamental Credit Derivative: Credit Default Swap

Market quotes a CDS rate that makes the contract fair, i.e.:

$$\text{CDS}(t; T_a, T_b, \text{Rec}, R) = 0$$

Hence if we have a **series of market quotes** $\hat{R}_1, \ldots, \hat{R}_n$ for CDS with **different tenors** $\hat{T}_1, \ldots, \hat{T}_n$, using (3) we can find a **term structure of survival probability for the Reference Entity**.

For example, in the case of **Intensity Models**, we can choose a piecewise linear (or constant) intensity $\lambda(t)$ and calibrate to CDS premia. We know, in fact, that $\mathbb{Q}(\tau > t) = \exp\left(-\int_0^t \lambda(u)du\right)$; so we can choose a value $\hat{\lambda}_i$ for each tenor $\hat{T}_i$ and make a calibration to fit all market quotes.

$\hat{T}_i$	$\hat{R}_i$ (bps)	$\hat{\lambda}_i$	Surv$_i$
16 Sep 2009			100.0%
1y	28.0	0.466%	99.5%
3y	36.5	0.902%	98.2%
5y	44.5	1.022%	96.3%
7y	48.5	0.982%	94.4%
10y	52.5	1.170%	91.4%

Black Cox: CDS Calibration?

Can one make the Black-Cox model reproduce liquid CDS data (CDS Calibration)?

$$
\begin{array}{c} \hat{R}_{1y} \\ \hat{R}_{2y} \\ \vdots \\ \hat{R}_{10y} \end{array} \quad \longleftrightarrow \quad
\begin{array}{c}
dV = (r-k)dt + \sigma_V V(t) dW(t) \\
H(t) = \begin{cases} L & \text{if } t = \bar{T} \\ K e^{-\gamma(\bar{T}-t)} & \text{if } t < \bar{T} \end{cases} \\
\text{model parameters: } \sigma_V,\ L,\ K \text{ and } \gamma
\end{array}
$$

Typically one has from 5 to 10 CDS market quotes and just 4 parameters in the Black-Cox model to calibrate them. Further, even if we had only 4 CDS quotes, the 4 parameters σ_V, L, K and γ are not much flexible.

Can we extend the model to make it more flexible and capable of exactly retrieving any number of quoted CDS?

Structural Models: CDS Calibration?

Our strategy:

$$\begin{matrix} \hat{R}_{1y} \\ \hat{R}_{2y} \\ \vdots \\ \hat{R}_{10y} \end{matrix} \quad \longleftrightarrow \quad \begin{matrix} dV = (r-k)dt + \boxed{\sigma_V(t)} V(t)dW(t) \\ H(t) = \dots \\ t \mapsto \sigma_V(t),\ t \mapsto H(t) \end{matrix}$$

Now we would have infinite parameters (all the values of $\sigma_V(t)$, for example) to account for all CDS market quotes.

The problem is: can we insert a time-dependent V dynamics **and** preserve barrier-like analytical formulas for survival probabilities $\mathbb{Q}(\tau > t)$ (and thus CDS, etc.)?

Structural Models: Barrier Options. CDS Calibration?

The diffculties in formulating such a model (like our AT1P below) and the reason why nobody tried it before is that in general Barrier option problems are difficult or impossible in presence of time-dependent volatilities or general curved barriers.

However, some recent work shows that it is possible to find analytical barrier option prices when the barrier has a particular curved shape depending partly on the time dependent volatility (Lo et al. (2003), Rapisarda (2003)).

Our AT1P model builds on these results: indeed, our curved barrier $\hat{H}(t)$ will depend on $\sigma_V(t)$.

Analytically Tractable 1st Passage (AT1P) Model

AT1P model: Let the risk neutral firm V dynamics and the default barrier $\hat{H}(t)$ be

$$dV(t) = V(t)\boxed{(r(t) - k(t))}dt + V(t)\boxed{\sigma_V(t)}dW(t)$$

$$\hat{H}(t) = H \exp\left(\int_0^t \left(\boxed{(r(s) - k(s))} - B\boxed{\sigma_V(s)^2}\right) ds\right) = \frac{H}{V_0}\mathbb{E}[V_t]e^{\left(-B\int_0^t \sigma_s^2 ds\right)}$$

and let the default time τ be **the 1st time** V hits $\hat{H}$ **from above**, starting from $V_0 > H$. Here $H > 0$ and B are free parameters we may use to shape the barrier.

Then the survival probability is given analytically by

$$\mathbb{Q}(\tau > t) = \Phi\left(\frac{\ln\left(\frac{V_0}{H}\right) + \frac{2B-1}{2}\int_0^t \sigma(s)^2 ds}{\sqrt{\int_0^t \sigma(s)^2 ds}}\right) - \left(\frac{H}{V_0}\right)^{2B-1}\Phi\left(\frac{\ln\left(\frac{H}{V_0}\right) + \frac{2B-1}{2}\int_0^t \sigma(s)^2 ds}{\sqrt{\int_0^t \sigma(s)^2 ds}}\right) \quad (4)$$

Analytically Tractable 1st Passage (AT1P) Model

Proof of the previous result can be found, for example, in Brigo and Tarenghi (2004), using $B = (1 + 2\beta)/2$.

The default barrier $\hat{H}(t)$ varies in time, following company and market conditions.

$$\begin{aligned}\hat{H}(t) &= H \exp\left[\int_0^t \left(r(s) - k(s) - B\sigma_s^2\right) ds\right] \\ &= \underbrace{\boxed{\frac{H}{V_0}\mathbb{E}[V_t]}}_{\substack{\text{Backbone of the barrier,} \\ \text{proportional to expected asset value}}} \times \underbrace{\exp\left(-B\int_0^t \sigma_s^2 ds\right)}_{\substack{\text{Cutting some slack} \\ \text{in high volatility conditions}}}\end{aligned}$$

Also, observe that H and V always appear in the formulas in ratios like $\frac{V}{H}$: This **homogeneity property** allows us to rescale the initial value of the firm $V_0 = 1$, and express the barrier parameter H as a fraction of it. In this case we do not need to know the real value of the firm.

The AT1P Structural Model: CDS calibration

In the AT1P model we have $\mathbb{Q}(\tau > t) =$ formula in $(\sigma(t), H, B)$. We could choose, for example, a piecewise constant shape for the volatility, assigning a pillar value $\hat{\sigma}_i$ to each tenor of CDS market quotes. In this case we would have:

$$\hat{R}_{1y}, \hat{R}_{2y}, \ldots, \hat{R}_{10y} \quad \longleftrightarrow \quad \hat{\sigma}_{1y}, \hat{\sigma}_{2y}, \ldots, \hat{\sigma}_{10y}, H, B$$

where $\sigma(t) = \hat{\sigma}_i$ for $\hat{T}_{i-1} \leq t < \hat{T}_i$ $(T_0 = 0)$.

This means that we have $n + 2$ free parameters for n market quotes.

Our choice is to fix B (as previously seen, B can change the shape of the barrier, giving some freedom in calibration).

Still, we have too many degrees of freedom; we can proceed in two directions:

- Use an exogenous estimate for H and calibrate all the $\hat{\sigma}_i$'s;
- Use some estimate of the firm value volatility and calibrate H.

The AT1P Structural Model: CDS calibration

Our choice is to use some estimate for H; we set $H/V_0 = 0.4$ (analogy with CDS recovery).

We then **calibrate all the market quotes of CDS**: for each maturity $\hat{T}_i$, we look for a volatility $\hat{\sigma}_i$ coherent with the market quote $\hat{R}_i$.

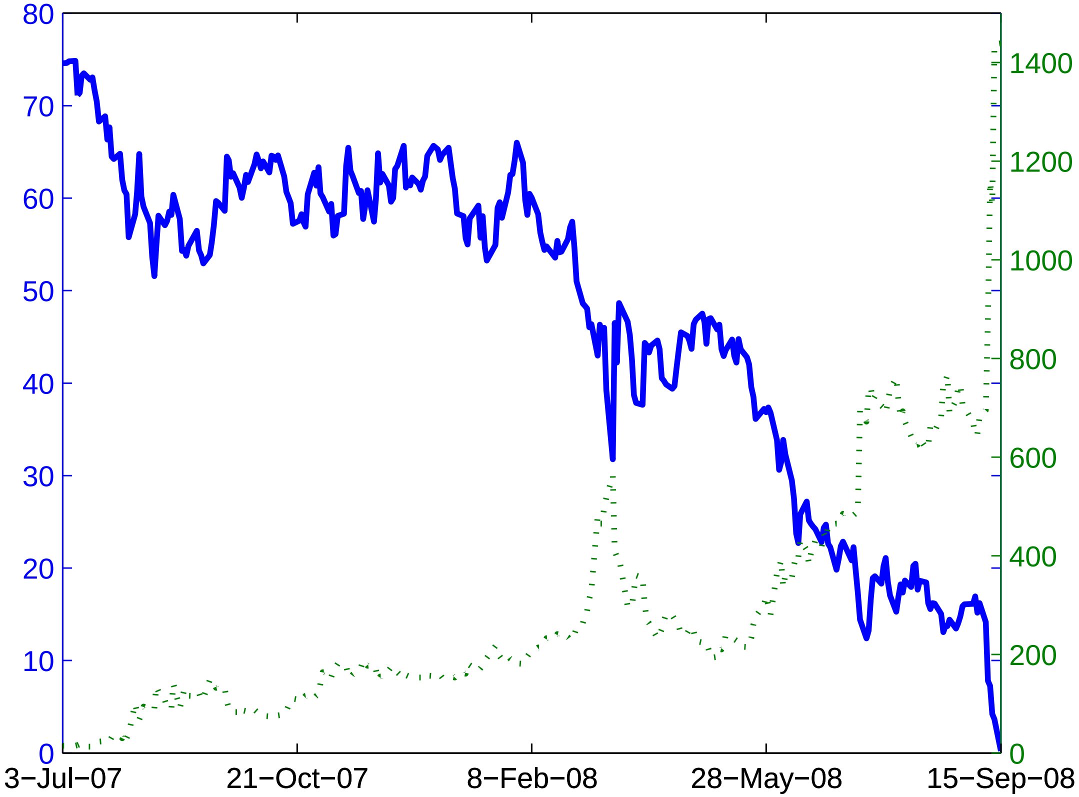
This procedure is justified: we are not interested in estimating the real process of the firm value, neither the real capital structure of the firm, but only in **reproducing risk neutral default probabilities with a model that makes sense also economically**. We are interested in the economic interpretation and not in sharply estimating the capital structure of the firm.

We appreciate the structural model interpretation as a tool for assessing the realism of the outputs of calibrations, and as an instrument to check economic consequences and possible diagnostics.

A Case Study with AT1P: Lehman Brothers default history

- **August 23, 2007**: Lehman announces that it is going to shut one of its home lending units (*BNC Mortgage*) and lay off 1,200 employees. The bank says it would take a $52 million charge to third-quarter earnings.
- **March 18, 2008**: Lehman announces better than expected first-quarter results (but profits have more than halved).
- **June 9, 2008**: Lehman confirms the booking of a $2.8 billion loss and announces plans to raise $6 billion in fresh capital by selling stock. Lehman shares lose more than 9% in afternoon trade.
- **June 12, 2008**: Lehman shakes up its management; its chief operating officer and president, and its chief financial officer had been removed from their posts.
- **August 28, 2008**: Lehman prepares to lay off 1,500 people. The Lehman executives have been knocking on doors all over the world seeking a capital infusion.
- **September 9, 2008**: Lehman shares fall 45%.
- **September 14, 2008**: Lehman files for bankruptcy protection and hurtles toward liquidation after it failed to find a buyer.

A Case Study with AT1P: Lehman Brothers CDS Calibration



Stock Price (solid line); CDS spread 1y (dashed line)

A Case Study with AT1P: Lehman Brothers CDS Calibration

Date: **July 10, 2007**

$\text{Rec} = 40\%$

$H = 0.4$ (AT1P)

$\hat{T}_i$	$\hat{R}_i$ (bps)
1y	16
3y	29
5y	45
7y	50
10y	58

$\hat{T}_i$	$\hat{\lambda}_i$ (bps)	Surv (Int)	$\hat{\sigma}_i$	Surv (AT1P)
10 Jul 2007		100.0%		100.0%
1y	0.267%	99.7%	29.2%	99.7%
3y	0.949%	98.5%	14.0%	98.5%
5y	1.499%	96.1%	14.5%	96.1%
7y	0.676%	94.1%	12.0%	94.1%
10y	2.191%	90.1%	12.7%	90.2%

A Case Study with AT1P: Lehman Brothers CDS Calibration

Date: **March 18, 2008**

$\text{Rec} = 40\%$

$H = 0.4$ (AT1P)

$\hat{T}_i$	$\hat{R}_i$ (bps)
1y	409
3y	369
5y	309
7y	290
10y	264

$\hat{T}_i$	$\hat{\lambda}_i$ (bps)	Surv (Int)	$\hat{\sigma}_i$	Surv (AT1P)
18 Mar 2008		100.0%		100.0%
1y	6.759%	93.5%	45.3%	93.3%
3y	4.638%	83.4%	24.7%	83.2%
5y	1.872%	78.1%	18.2%	77.9%
7y	5.769%	72.4%	19.6%	72.3%
10y	-0.364%	66.8%	17.4%	66.4%

A Case Study with AT1P: Lehman Brothers CDS Calibration

Date: **June 12, 2008**

$\text{Rec} = 40\%$

$H = 0.4$ (AT1P)

$\hat{T}_i$	$\hat{R}_i$ (bps)
1y	397
3y	315
5y	277
7y	258
10y	240

$\hat{T}_i$	$\hat{\lambda}_i$ (bps)	Surv (Int)	$\hat{\sigma}_i$	Surv (AT1P)
12 Jun 2008		100.0%		100.0%
1y	6.563%	93.6%	45.0%	93.5%
3y	2.174%	85.8%	21.9%	85.6%
5y	4.742%	80.1%	18.6%	79.9%
7y	1.579%	75.2%	18.1%	75.0%
10y	4.369%	68.8%	17.5%	68.7%

A Case Study with AT1P: Lehman Brothers CDS Calibration

Date: **September 12, 2008**

$\text{Rec} = 40\%$

$H = 0.4$ (AT1P)

$\hat{T}_i$	$\hat{R}_i$ (bps)
1y	1437
3y	902
5y	710
7y	636
10y	588

$\hat{T}_i$	$\hat{\lambda}_i$ (bps)	Surv (Int)	$\hat{\sigma}_i$	Surv (AT1P)
12 Sep 2008		100.0%		100.0%
1y	23.260%	79.2%	62.2%	78.4%
3y	-6.432%	67.0%	30.8%	65.5%
5y	18.686%	59.2%	24.3%	59.1%
7y	-8.049%	53.3%	26.9%	52.5%
10y	23.294%	42.4%	29.5%	43.4%

AT1P Structural Model: Comments

We have seen that AT1P can calibrate exactly CDS market quotes. Also, according to our experience, sometimes it shows a stronger calibration power than common intensity models (under stress conditions, intensity pillar values may become negative).

However, looking at calibration results with more attention, we find:

- Scarce relevance of barriers in calibration.
- High discrepancy between first volatility bucket and the following values.

When the **default boundary is deterministic**, diffusion models tend to calibrate a considerably probability of default by one-year (shortest horizon credit spread) only **supposing particularly high one-year volatility**.

Accounting for Market Uncertainty

Hence the problem is also related to the fundamental assumption that the **default threshold is a deterministic, known function of time**, based on reliable accounting data.

This is a very **strong assumption and usually it is not true**: **balance sheet information is not certain**, possibly because the company is hiding information, or because a real valuation of the firm assets is not easy (for example in case of derivative instruments).

Public investors, then, may have only a **partial and coarse information** about the true value of the firm assets or the related liability-dependent firm condition that would trigger default.

H in our model is the **ratio between the initial level of the default barrier and the initial value of the company assets**.

To take market uncertainty into account in a realistic and albeit simple manner, H **is replaced by a random variable assuming different values in different scenarios**.

Scenario Barrier Time-Varying Volatility (SBTV) Model

The risk neutral dynamics for the firm value is (like in the AT1P):

$$dV(t) = V(t)(r(t) - k(t))dt + V(t)\sigma_V(t)dW(t)$$

Now we introduce $I = 1, 2, \ldots, N$ **independent default scenarios** (indicating uncertainty on the capital situation of the firm), represented by different default barriers

$$\hat{H}^I(t) = H^I \exp\left[\int_0^t \left(r(s) - k(s) - B\sigma_V(s)^2\right) ds\right]$$

where each scenario I has probability $\mathbb{Q}(I = i) = p_i$ and, of course, $p_i \in [0, 1]$, $\sum_{i=1}^N p_i = 1$.

Also, what is important, each scenario I is **independent of the brownian motion** W.

Hence, the **SBTV model acts like a mixture of AT1P scenarios**.

Scenario Barrier Time-Varying Volatility (SBTV) Model

In this framework, using the **Iterated Expectaion Law**, for a given discounted payoff Π we have:

$$\mathbb{E}[\Pi] = \mathbb{E}\left[\mathbb{E}\left[\Pi | H^I\right]\right] = \sum_{i=1}^{N} p_i \mathbb{E}\left[\Pi | H^I = H_i\right]$$

In particular, for CDS with the SBTV model we have

$$SBTV\text{CDS}(t; T_a, T_b, \text{Rec}, R) = \sum_{i=1}^{N} p_i \cdot AT1P\text{CDS}\left(t; T_a, T_b, \text{Rec}, R, \boxed{H_i}\right)$$

where $AT1P$CDS is the CDS price computed according to the AT1P survival probability formula when H is set to H_i.

The SBTV Structural Model: CDS Calibration

Now we have

$$\hat{R}_{1y}, \hat{R}_{2y}, \dots, \hat{R}_{10y} \quad \longleftrightarrow \quad \hat{\sigma}_{1y}, \hat{\sigma}_{2y}, \dots, \hat{\sigma}_{10y}, H_1, H_2, \dots, H_N, p_1, p2, \dots, p_{N-1}, B$$

where $\sigma(t) = \hat{\sigma}_i$ for $\hat{T}_{i-1} \leq t < \hat{T}_i$ $(T_0 = 0)$ and $p_N = 1 - \sum_{i=1}^{N-1} p_i$.

This means that we have $n + 2N$ free parameters for n market quotes. We proceed in this way:

- Like in the AT1P we fix B before calibrating;
- Use $n = 5$ market quotes: the most liquid and used tenors are $\hat{T}_1 = 1y, \hat{T}_2 = 3y, \hat{T}_3 = 5y, \hat{T}_4 = 7y, \hat{T}_5 = 10y$, with CDS rates $\hat{R}_{1y}, \hat{R}_{3y}, \hat{R}_{5y}, \hat{R}_{7y}, \hat{R}_{10y}$;
- We use $N = 2$ scenarios (according to our experience, a larger value of N does not add much information);

We still have 5 market quotes with 8 free parameters. The **exact calibration**, hence, is run out in 2 steps.

The SBTV Structural Model: CDS Calibration, 1st Step

- We limit our analysis to the **first three CDS market quotes**: $\hat{T}_1 = 1y$, $\hat{T}_2 = 3y$, $\hat{T}_3 = 5y$; in this way $n = 3$;
- We set $\hat{\sigma}_{1y} = \hat{\sigma}_{3y} = \hat{\sigma}_{5y} = \bar{\sigma}$ where $\bar{\sigma}$ is a free value for volatility to be calibrated;
- We choose to fix H_1 (the lowest barrier) to 0.4 (as in the AT1P).
- Now the only free parameters are $H_2, p_1, \bar{\sigma}$;
- We calibrate them by minimization of the following target function

$$\sum_{k=1}^{3} \left[R_k^{\mathsf{MKT}} - R_k^{\mathsf{SBTV}} \right]^2$$

that is the sum of squared difference between CDS market rates and the rates obtained by the SBTV model.

The SBTV Structural Model: CDS Calibration, 2nd Step

- We now have calibrated $\bar{\sigma}, H_2, p_1$;
- We return to consider all quotes $\hat{R}_i, i = 1, \ldots, 5$;
- We consider 5 free volatility buckets $\hat{\sigma}_i, i = 1, \ldots, 5$;
- We run **exact calibration** to CDS market quotes.

Observe that from 1st step to 2nd step, we **re-calibrate the first three volatility buckets**. This is done in order to obtain **exact calibration** to market quotes;

Anyway, if the result of the first calibration is good, the refinement to $\hat{\sigma}_{i=1,2,3}$ due to second step calibration is negligible.

A Case Study with SBTV: Lehman Brothers CDS Calibration

Date: **July 10, 2007**

$\text{Rec} = 40\%$

$H = 0.4$ (AT1P)

$$\left.\begin{array}{ll} H_1 = 0.4000 & p_1 = 96.2\% \\ H_2 = 0.7313 & p_2 = 3.8\% \end{array}\right\} \text{SBTV}$$

$\hat{T}_i$	$\hat{R}_i$ (bps)
1y	16
3y	29
5y	45
7y	50
10y	58

$\hat{T}_i$	$\hat{\sigma}_i$	Surv (SBTV)	$\hat{\sigma}_i$	Surv (AT1P)
10 Jul 2007		100.0%		100.0%
1y	16.6%	99.7%	29.2%	99.7%
3y	16.6%	98.5%	14.0%	98.5%
5y	16.6%	96.1%	14.5%	96.1%
7y	12.6%	94.1%	12.0%	94.1%
10y	12.9%	90.2%	12.7%	90.2%

A Case Study with SBTV: Lehman Brothers CDS Calibration

Date: **March 18, 2008**

$\text{Rec} = 40\%$

$H = 0.4$ (AT1P)

$$\left.\begin{array}{ll} H_1 = 0.4000 & p_1 = 65.6\% \\ H_2 = 0.8006 & p_2 = 34.4\% \end{array}\right\} \text{SBTV}$$

$\hat{T}_i$	$\hat{R}_i$ (bps)
1y	409
3y	369
5y	309
7y	290
10y	264

$\hat{T}_i$	$\hat{\sigma}_i$	Surv (SBTV)	$\hat{\sigma}_i$	Surv (AT1P)
18 Mar 2008		100.0%		100.0%
1y	16.3%	93.4%	45.3%	93.3%
3y	16.3%	83.4%	24.7%	83.2%
5y	16.3%	78.1%	18.2%	77.9%
7y	18.1%	72.4%	19.6%	72.3%
10y	15.7%	66.5%	17.4%	66.4%

A Case Study with SBTV: Lehman Brothers CDS Calibration

Date: **June 12, 2008**

$\textsc{Rec} = 40\%$

$H = 0.4$ (AT1P)

$$\left.\begin{array}{ll} H_1 = 0.4000 & p_1 = 74.6\% \\ H_2 = 0.7971 & p_2 = 25.4\% \end{array}\right\} \text{SBTV}$$

$\hat{T}_i$	$\hat{R}_i$ (bps)
1y	397
3y	315
5y	277
7y	258
10y	240

$\hat{T}_i$	$\hat{\sigma}_i$	Surv (SBTV)	$\hat{\sigma}_i$	Surv (AT1P)
12 Jun 2008		100.0%		100.0%
1y	18.7%	93.6%	45.0%	93.5%
3y	18.7%	85.7%	21.9%	85.6%
5y	18.7%	80.1%	18.6%	79.9%
7y	17.4%	75.1%	18.1%	75.0%
10y	16.4%	68.8%	17.5%	68.7%

A Case Study with SBTV: Lehman Brothers CDS Calibration

Date: **September 12, 2008**

$\text{Rec} = 40\%$

$H = 0.4$ (AT1P)

$$\left.\begin{matrix} H_1 = 0.4000 & p_1 = 50.0\% \\ H_2 = 0.8427 & p_2 = 50.0\% \end{matrix}\right\} \text{SBTV}$$

$\hat{T}_i$	$\hat{R}_i$ (bps)
1y	1437
3y	902
5y	710
7y	636
10y	588

$\hat{T}_i$	$\hat{\sigma}_i$	Surv (SBTV)	$\hat{\sigma}_i$	Surv (AT1P)
12 Sep 2008		100.0%		100.0%
1y	19.6%	79.3%	62.2%	78.4%
3y	19.6%	66.2%	30.8%	65.5%
5y	19.6%	59.6%	24.3%	59.1%
7y	21.8%	52.9%	26.9%	52.5%
10y	23.7%	43.6%	29.5%	43.4%

AT1P and SBTV Structural Models: Comments

AT1P and SBTV:

- High fit to market quotes;
- Efficient formulas for calibration;
- Regular results;
- Link with firm economy;
- Equity like underlying dynamics;
- SBTV: more realistic results, more informative interpretation.

Possible developments:

- Natural extension to hybrid equity-credit pricing and counterparty risk valuation;
- SBTV: it is possible to add default contagion effects by using multivariate random variable $(H_{\text{ASSET1}}, H_{\text{ASSET2}})$.

Counterparty Risk in Equity Return Swap (ERS)

Consider an Equity Return Swap (ERS). We are a default-free company "A" entering a contract with counterparty "B". The reference underlying equity is a default-free company "C".

"A" and "B" agree on an amount K of stocks of "C" (with price S_0) to be taken as nominal ($N = KS_0$). The contract starts in $T_a = 0$ and has final maturity $T_b = T$.

At $t = 0$ there is no exchange of cash (alternatively, we can think that "B" delivers to "A" an amount K of "C" stock and receives a cash amount equal to KS_0).

At intermediate times "A" pays to "B" the dividend flows of the stocks (if any) in exchange for a periodic rate (e.g. semi-annual LIBOR or EURIBOR L) plus a spread X.

At final maturity $T = T_b$, "A" pays KS_T to "B" (or gives back the amount K of stocks) and receives a payment KS_0.

The price can be derived using risk neutral valuation, and the (fair) spread is chosen in order to obtain a contract whose value at inception is zero.

Counterparty Risk in Equity Return Swap (ERS)

Time 0: no flows, or

$A \longrightarrow KS_0^C$ cash $\longrightarrow$ B

$A \longleftarrow K$ equity of C $\longleftarrow$ B

Time T_i:

$A \longrightarrow$ equity dividends of C $\longrightarrow$ B

$A \longleftarrow$ Libor + Spread $\longleftarrow$ B

Time T_b:

$A \longrightarrow K$ equity of C $\longrightarrow$ B

$A \longleftarrow KS_0^C$ cash $\longleftarrow$ B

Counterparty Risk in Equity Return Swap (ERS)

We assume the underlying "C" to be default-free, or to have a much stronger credit quality than counterparty "B" (for example an equity index).

It can be proved that if "B" were default-free itself, the fair spread would be zero. It is then precisely counterparty risk that makes the spread non-zero.

If early default of "B" occurs at time $\tau = \tau_B$, the following happens.

Before τ payments go through normally, as before.

If $\tau \leq T$, compute the net present value (NPV) of the position at time τ.

If NPV is negative to us ("A"), then at τ we **fully** pay its opposite to "B". Instead, if it is positive to us ("A"), only a **recovery fraction** REC of that NPV is received.

It is clear that counterparty risk is bad to us ("A") when the market is good (large NPV).

Counterparty Risk in Equity Return Swap (ERS)

Price of the ERS to us ("A") results to be:

$$\mathsf{ERS}_0^D = \mathsf{NPV}(0,T_b) - \mathrm{L}_{\mathrm{GD}}\mathbb{E}_0\left[\mathbf{1}_{\{\tau\le T_b\}}D(0,\tau)\left(\mathsf{NPV}(\tau,T_b)\right)^+\right]$$

where

$$\begin{aligned}\mathsf{NPV}(t,T_b) &= \mathbb{E}_t\Big\{-k\mathsf{NPV}^{t,T_b}_{\mathsf{dividends}}(t) + KS_0\sum_{i=\beta(t)}^{b} D(t,T_i)\alpha_i\left(L(T_{i-1},T_i)+X\right)\\ &\quad + D(t,T_b)\left(KS_0 - KS_{T_b}\right)\Big\}\end{aligned}$$

We denoted by $\mathsf{NPV}^{s,t}_{\mathsf{dividends}}(u)$ the net present value of the dividend flows between s and t computed in u, α_i is the year fraction between T_{i-1} and T_i, and $\beta(t)$ is such that $t\in[T_{\beta(t)-1},T_{\beta(t)})$, i.e. $T_{\beta(t)}$ is the first date among the T_i's that follows t.

Counterparty Risk in Equity Return Swap (ERS)

A **key variable** for the valuation is the correlation ρ between counterparty V_B and underlying equity S_C.

Recall: Firm Value of the counterparty "B" (AT1P or SBVT) calibrated to CDS quotes of "B" and underlying equity of name "C" (Black Scholes) with $\sigma_C = 20\%$, $q = 8\%$ and r given by the zero curve:

$$\begin{aligned} dV_B(t) &= V_B(t)(r(t) - k(t))dt + V_B(t)\sigma_B(t)dW_B(t) \\ dS_C(t) &= S_C(t)(r(t) - q(t))dt + S_C(t)\sigma_C(t)dW_C(t) \\ \text{Correlation} &: dW_B dW_C = \rho_{B,C}dt \end{aligned}$$

$\rho_{B,C}$ can be estimated as historical correlation between daily returns of equities "B" and "C" or a view on it can be expressed.

We can find the value of X that makes the contract fair by iterating the MC simulation until the payoff value is sufficiently small.

Counterparty Risk in ERS: Results

Evaluation date: September 16, 2009.
ERS data: Tenor=$5y$, freq=semiannual, REC $= 40\%$.
Underlying equity data: $S_0 = 20$, $q = 0.8\%$, $\sigma_C = 20\%$.

Counterparty CDS spread

$\hat{T}_i$	$\hat{R}_i^{BID}$	$\hat{R}_i^{ASK}$
1y	25	31
3y	34	39
5y	42	47
7y	46	51
10y	50	55

ERS spread

$\rho_{B,C}$	spread (AT1P)	spread (SBTV)
-1.0	0.0*	0.0*
-0.2	3.0	3.6
0.0	5.5	5.5
0.5	14.7	11.4
1.0	24.9	17.9

Notice that AT1P and SBTV returns very similar results, even if not identical: this can be due to the fact that the two models are **calibrated exactly to CDS quotes**, but they do **not necessarily return the same term structure of default probability**, especially for dates not corresponding to CDS pillar tenors. Anyway, the differences between the two spreads are comparable with the **Bid-Ask spread** of CDS premia.

If we compute the fair spread with the intensity model, we find $X = 5.5$bps, coherent with the hypothesis of independence ($\rho = 0$) in both the AT1P and SBTV models.

SBTV Structural Model: Multiname

Consider two firms 1 and 2:

$$dV_1(t) = V_1(t)(r(t) - k_1(t))dt + V_1(t)\sigma_1(t)dW_1(t)$$
$$dV_2(t) = V_2(t)(r(t) - k_2(t))dt + V_2(t)\sigma_2(t)dW_2(t)$$

We take two scenarios on the initial level of the barrier

$$\mathsf{H}_1 = \begin{cases} H_1^h & \text{with prob } p_1^h \\ H_1^l & \text{with prob } p_1^l \end{cases} \qquad \mathsf{H}_2 = \begin{cases} H_2^h & \text{with prob } p_2^h \\ H_2^l & \text{with prob } p_2^l \end{cases}$$

There are **two** elements that control the default correlation of the two names.

- The **correlation between the two brownian motions** driving the firm asset's values: $\rho = \text{corr}[dW_1(t), dW_2(t)]$; this naturally represents the amount of default dependency coming from smoothly varying common variables (for example, the so-called cyclical default correlation due to macroeconomic variables, although it can include some company specific links).
- The **joint distribution of** $(\mathsf{H}_1, \mathsf{H}_2)$ that naturally represents the amount of default dependency coming from **"contagion" effects** specific of the two names.

SBTV Structural Model: Multiname

Three interesting scenarios on the barriers:

- Strong links: p^{hh} and p^{ll} very high, p^{hl} and p^{lh} very low (positive dependence);
- Scarcely related firms $p^{xy} = p_1^x p_2^y$ (independence);
- Competitors: p^{hh} and p^{ll} very low, p^{hl} and p^{lh} very high (negative dependence).

where $p^{xy} = \text{Prob}\left(\mathsf{H}_1 = H_1^x, \mathsf{H}_2 = H_2^y\right)$ $(x, y = h, l)$.

They match with the correlation cases: $\rho \approx 1$, $\rho \approx 0$, $\rho \approx -1$. It is important to notice that companies can have strong equity correlation while not experiencing contagion in case of default, and also the other way around. It becomes interesting to see which level of flexibility the models gives us in this cases, and what is the effect of the different assumptions.

A technical issue to consider is the design of the multivariate random barrier.

$$F_{\mathsf{H}_1,\mathsf{H}_2}(h_1, h_2) = \text{Prob}\left(\mathsf{H}_1 \le h_1, \mathsf{H}_2 \le h_2\right)$$

SBTV Structural Model: Multiname

$F_{\mathsf{H}_1,\mathsf{H}_2}(h_1,h_2)$	$h_2 < H_2^l$	$H_2^l \leq h_2 < H_2^h$	$h_2 \geq H_2^h$
$h_1 < H_1^l$	0	0	0
$H_1^l \leq h_1 < H_1^h$	0	p^{ll}	p_1^l
$h_1 \geq H_1^h$	0	p_2^l	1

So we can control the distribution just by controlling p^{ll}. We have to check the constraints:

1) Joint distribution can't be higher than marginals: $p^{ll} \leq \min\left(p_1^l, p_2^l\right)$;
2) All resulting probabilities must be non-negative, which means:
   - $p^{hl} = p_2^l - p^{ll} \geq 0$ (already guaranteed);
   - $p^{lh} = p_1^l - p^{ll} \geq 0$ (already guaranteed);
   - $p^{hh} = 1 - p_1^l - p_2^l + p^{ll} \geq 0$;

All the constraints summarize in:

$$\max\left(p_1^l + p_2^l - 1, 0\right) \leq p^{ll} \leq \min\left(p_1^l, p_2^l\right)$$

SBTV Structural Model: Multiname

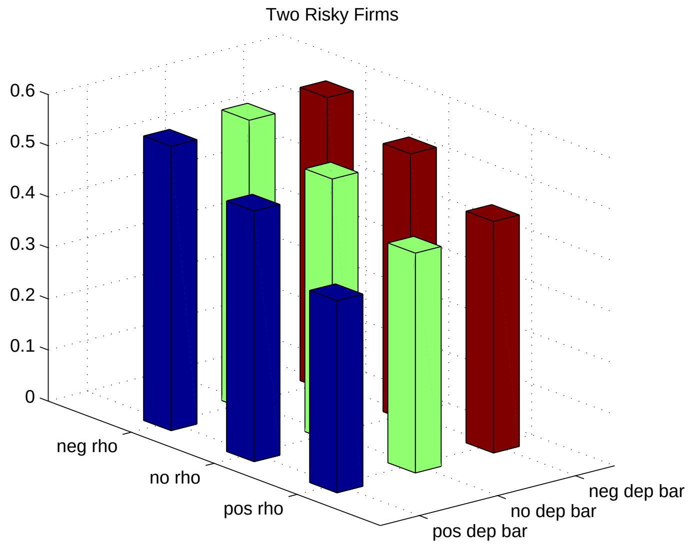
This also shows ho to design the distribution to have the desired features:

Maximum dependence	$\longrightarrow$	$p^{ll} = \min\left(p_1^l, p_2^l\right)$
Independence	$\longrightarrow$	$p^{ll} = p_1^l * p_2^l$
Minimum dependence	$\longrightarrow$	$p^{ll} = \max\left(p_1^l + p_2^l - 1, 0\right)$

In the following we see the result on the pricing of a multiname credit derivative (First To Default) considering different pairs of firms and assessing all 9 pivot configurations in terms of asset value correlation $\rho = \text{corr}\left[dW_1(t), dW_2(t)\right]$ and in terms of joint barrier distribution $\text{Prob}\left(\mathsf{H}_1 \leq h_1, \mathsf{H}_2 \leq h_2\right)$.

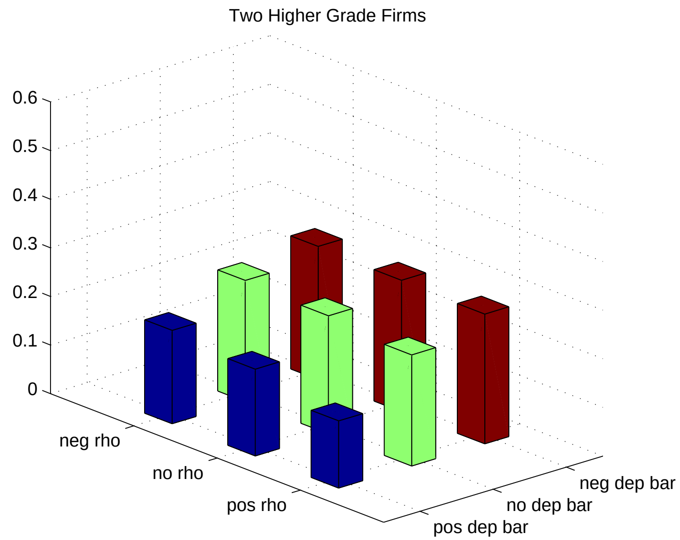
First To Default: Credit derivative protecting against the first default in a basket of credit entities. It works like a CDS: protection LGD is paid at $\tau = \min\left(\tau_1, \ldots \tau_n\right)$ (in our case $n = 2$).

SBTV Structural Model: Multiname



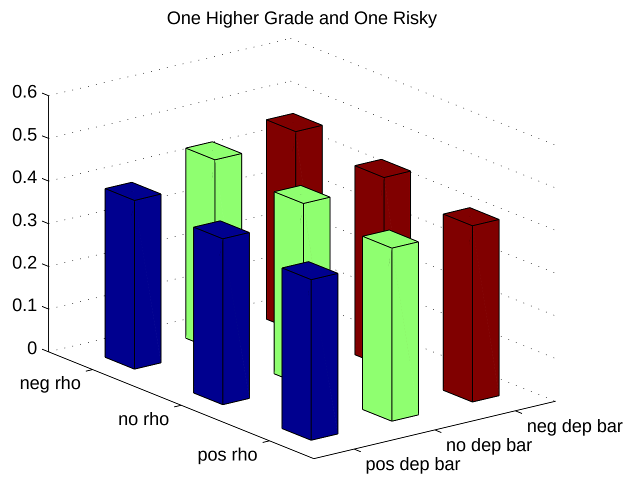
Here we consider **two risky firms**: their default barrier is certainly high, there is some uncertainty if it is so high to lead to close default or a bit lower. We are in a relatively high volatility context. The value of protection (risk of default of at least one name) grows when the dependency gets lower, either that we model dependency through smooth linear correlation of asset values or more abrupt scenario dependency. It is clear, however, that for this kind of firm the asset volatility matter most, so linear correlation has a dominant effect.

SBTV Structural Model: Multiname



We move to **two higher grade firms**, generally considered reliable but charged with short term credit spreads due to some uncertainty about their actual conditions. What matters most is the dependency or anti-dependency of the relative liability scenarios.

SBTV Structural Model: Multiname



When we take **two very different firms**, we see that this kind of dependency seem to have less importance. When two companies are so different they give little control on default correlation.

Conclusions

We introduced basic structural models: Merton and Black Cox. We explained the different philosophy with respect to intensity models.

We illustrated how our tractable first passage structural model AT1P is calibrated exactly to CDS data and showed a case study based on market data when the credit quality of the underlying name (Lehman) is deteriorating.

We extended the AT1P structural model using random parameters for the safety covenants barrier, leading to the SBTV model, and studied what changes in the calibration and the opportunities given by the new models.

We shortly analyzed the application of the two models to counterparty risk pricing in an Equity Return Swap.

We hinted at possible extensions to the multiname case.

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