VARIATION OF QUOTIENTS AND ETALE SLICES IN GEOMETRIC INVARIANT THEORY

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Introduction

This course is devoted to several topics in geometric invariant theory which were chosen to complement the basic course given by Professor Le Potier. These topics have a common theme that they are all related in some manner to the study of how quotients constructed via geometric invariant theory change when the stability condition (or linearized line bundle) varies. The subjects discussed include the following:

- At the beginning are several examples in which quotients $X^{L\text{-ss}}//G$ are computed explicitly by computing the rings of invariant global sections of the powers of L. A number of phenomena relating to variations of quotients are visible in these examples (variation of quotient morphisms, flips). These examples will recur throughout the course as local models for quotients and for variation of quotients.
- The Hilbert-Mumford numerical criterion for stability is discussed briefly. It is used at two later stages.
- King's construction [Ki] of moduli spaces of semistable representations of quivers is presented in order to illustrate the use of the Hilbert-Mumford numerical criterion. It also permits the discussion of a number of ideas related to moduli spaces of vector bundles with fewer technicalities (destabilizing subrepresentations, S-equivalence). Moreover, these spaces provide a large class of examples where variations of quotients can occur, for reasons very similar to variations of quotients in moduli of vector bundles. As will become evident later, moduli spaces of representations of (Ext) quivers can also help in the analysis of the local structure of variation of quotients.
- Chamber systems in variations of quotients are discussed (cf. [DH]).
- A brief introduction to étale slice theory is attempted. The emphasis is on the big ideas within and behind the theory: what is an étale neighborhood, locally isomorphic quotients, the slice, strongly étale G-equivariant morphisms, and the étale slice theorem as a G-equivariant étale tubular neighborhood theorem. Some of the technical cartesian diagrams are presented, since the true power of the étale slice theorem resides in them. For

Notes for a course for graduate students in algebra and geometry at Dyrkolbotn, Norway, on December 4–9, 1995. There were also courses by J. Le Potier (on basic geometric invariant theory), by D. Happel (on the geometry of representations of quivers), by S. A. Strømme (on the complete reducibility of representations of reductive groups), and by G. Ellingsrud (on Chow rings of quotient varieties).

further discussion of this important topic, see: [PV] §6 (also with many proofs missing), the relevant chapter in the DMV seminar notes [KSS] (which includes a proof of the étale slice theorem), or the original paper [Lu] (where the presentation often gets bogged down in étale technicalities). Mention is also made in one of the appendices of [MF].

- A method (taken from [KW2]) of applying étale slice theory to describe the local structure of variations of quotients is outlined. (The case of simple flips had already been treated by Dolgachev and Hu [DH] and by Thaddeus [T3] by sometimes different methods.)
- The use of Ext quivers for describing étale slices and local structure of quotients and of variation of quotients is discussed.

The bibliography includes several papers besides those mentioned above or in the course of the paper. The papers [BP], [H], [R], and [T2] deal with quotients and variation of quotients for actions by tori on algebraic varieties. Thaddeus' paper [T1] set off the current interest in variations of quotients.

Notation. For simplicity I will work over a algebraically closed base field of characteristic 0, to be denoted $\mathbb{C}$.[1] I will work always with varieties which are closed subvarieties $X \subset \mathbb{A}^N \times \mathbb{P}^M$. Such varieties are of the form $X = \operatorname{Proj} R$ where $R = \bigoplus_{d \geq 0} R_d$ is a graded $\mathbb{C}$-algebra which can be generated by a finite number of elements $x_1, \ldots, x_N \in R_0$ and $y_0, \ldots, y_M \in R_1$.[2]

Affine varieties fit into this picture as $\operatorname{Proj} R_0[T] = \operatorname{Spec} R_0$ where T is an indeterminate of degree at least 1. Projective varieties fit into the picture as $\operatorname{Proj} R$ with $R_0 = \mathbb{C}$ (i.e. all the generators of R are of positive degree). Geometrically this reads as saying that affine varieties are subvarieties of an $\mathbb{A}^N \times \mathbb{P}^0 \cong \mathbb{A}^N$, while projective varieties are subvarieties of an $\mathbb{A}^0 \times \mathbb{P}^M \cong \mathbb{P}^M$.

I will generally describe group actions in terms of the action of individual elements $g \in X$ on points $x \in X$ rather than abstractly using morphisms $G \times X \to X$. It is easy to pass between the two viewpoints. The orbit Gx of a point $x \in X$ will be denoted O_x.

1. Some examples of variation of quotients

Basic geometric invariant theory. To construct a quotient variety using geometric invariant theory one needs: an algebraic variety X, a reductive algebraic group G acting on X, and a G-linearized (or G-equivariant) ample line bundle L on X. My basic question will be the following:

Question. If one fixes X, G, and the action of G on X but lets the linearized line bundle L vary, how do the open loci $X^{L\text{-ss}} \subset X$ of L-semistable points and the quotients $X^{L\text{-ss}}//G$ vary?

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[1] This is not strictly necessary since geometric invariant theory works in characteristic p, although the basic theorems are much harder to prove than in characteristic 0.

[2] Actually if R is any finitely generated graded $\mathbb{C}$-algebra, then $\operatorname{Proj} R$ is isomorphic to a closed subvariety of an $\mathbb{A}^N \times \mathbb{P}^M$. This is because if the generators of R have degrees $0, \ldots, 0, n_1, \ldots, n_r$ with the $n_i \geq 1$, and if we set $m = \operatorname{lcm}(n_1, \ldots, n_r)$, then the "Veronese" subring $R_{(m)} := \bigoplus_{n \geq 0} R_{nm}$ is generated in degrees 0 and 1 but has $\operatorname{Proj} R_{(m)} = \operatorname{Proj} R$.

I will always use the word *line bundle* in the geometric sense meaning a morphism $p : L \to X$ with fibers $\mathbb{C}$ which is locally a product and such that the gluing data relating the local trivializations is linear. A *section* of L is therefore a map $s : X \to L$ such that $ps = 1_X$. A line bundle is *ample* if the graded ring

$$R_L := \bigoplus_{n \geq 0} \Gamma(X, L^{\otimes n})$$

of global sections of L and of its powers separates points (i.e. for any pair of points $x, y \in X$ with $x \neq y$ there exists an $s \in \Gamma(X, L^{\otimes n})$ such that $s(x) = 0$ but $s(y) \neq 0$). For a closed subvariety $X \subset \mathbb{A}^N \times \mathbb{P}^M$, if a line bundle L is ample on X, then the graded ring R_L above is finitely generated, and $X \cong \operatorname{Proj} R_L$. Giving an ample line bundle on X is essentially the same thing as giving a closed embedding $X \hookrightarrow \mathbb{A}^M \times \mathbb{P}^N$ (at least for the type of varieties we have restricted our attention to).

If G acts on X, then a *G-linearization* of $p : L \to X$ is an action $G \times L \to L$ which is compatible with the action of G on X, i.e. such that the following diagram is commutative:

$$\begin{array}{ccc} G \times L & \longrightarrow & L \\ {\scriptstyle 1\times p}\downarrow & & \downarrow{\scriptstyle p} \\ G \times X & \longrightarrow & X \end{array}$$

and such that for all $g \in G$ and $x \in X$ the map $g : L_x (\cong \mathbb{C}) \to L_{g(x)} (\cong \mathbb{C})$ is linear.

If L is a G-linearized ample line bundle on X, then G acts on R_L as follows. An element $g \in G$ acts on a section $s : X \to L^{\otimes n}$ (in $\Gamma(X, L^{\otimes n})$) by sending the subvariety $s(X) \subset L$ to the subvariety $gs(X) \subset L$. Because the action of g on L is compatible with its action on X, this means that $g(s(x))$ is the point of $gs(X)$ above $g(x) \in X$. Thus $g(s(x)) = gs(g(x))$ or after a substitution $(gs)(y) = g(s(g^{-1}(y)))$.

Giving a G-linearized ample line bundle on X is essentially equivalent to giving two finite-dimensional representations of G and a G-equivariant embedding $X \hookrightarrow V \times \mathbb{P}(W)$. One could then make the identification

$$R_L = \left[\operatorname{Sym}^*(V^*) \otimes \operatorname{Sym}^*(W^*)\right] / I(X)$$

where $\operatorname{Sym}^*(V^*)$ is the ring of polynomial functions on V (with the variables having degree 0 in the graded ring R_L), where $\operatorname{Sym}^*(W^*)$ is similar (but its variables having degree 1 in R_L), and where $I(X)$ is the ideal of polynomials vanishing on X. The action of G on R_L is then that induced by its action on V and W.

The invariant sections form a finitely generated subalgebra with an "irrelevant" ideal

$$R_L^G \supset I_L^G := \bigoplus_{n \geq 1} \Gamma(X, L^{\otimes n})^G.$$

The L-unstable locus is the Zariski closed subset

$$X^{L\text{-us}} := \{\text{common zeros of } I_L^G\} \subset X,$$

while the L-semistable locus is the complementary open subset $X^{L\text{-ss}} := X - X^{L\text{-us}}$. There is a quotient morphism $\pi_X^L : X^{L\text{-ss}} \to X^{L\text{-ss}}//G := \operatorname{Proj} R_L^G$. This morphism was shown to have several properties in Professor Le Potier's course:

1. The morphism $\pi_X^L : X^{L\text{-ss}} \to X^{L\text{-ss}}//G$ is G-invariant. In particular it is constant on orbits. Moreover, it is a *categorical quotient.* This means that any G-invariant morphism $X^{L\text{-ss}} \to Y$ factors uniquely as $X^{L\text{-ss}} \to X^{L\text{-ss}}//G \to Y$.
2. The quotient morphism sends each orbit in the open locus $X^{L\text{-s}} \subset X^{L\text{-ss}}$ of L-*stable points* to a different point of the quotient. But in $X^{L\text{-ss}} - X^{L\text{-s}}$ it may contract a complicated locus containing many different orbits onto a single point of the quotient.
3. Nevertheless, for any $\xi \in X^{L\text{-ss}}//G$ the locus $(\pi_X^L)^{-1}(\xi) \subset X^{L\text{-ss}}$ which gets contracted onto ξ contains a *unique closed orbit* O_x. Moreover, any $y \in X^{L\text{-ss}}$ is sent to the point of $X^{L\text{-ss}}//G$ corresponding to the unique orbit in $\overline{O_y} \cap X^{L\text{-ss}}$ which is closed in $X^{L\text{-ss}}$.

Affine examples. In the examples which follow the variety $X = \operatorname{Spec} A$ will be an affine space. The G-linearized line bundles will be G-linearizations of the trivial line bundle $X \times \mathbb{C}$. To give such a linearization one chooses a *character* of G, i.e. a morphism of algebraic groups $\chi : G \to \mathbb{C}^*$. The characters of G form a finitely generated abelian group $\operatorname{Char}(G)$ (of rank equal to $\dim Z(G)$ if the group G is reductive). I will usually write the character group additively even though the group law is $(\chi + \psi)(g) := \chi(g)\psi(g)$.

Having chosen χ, I let L_χ be "the trivial line bundle twisted by χ," i.e. the line bundle $X \times \mathbb{C}$ with action $g(x,a) := (g(x), \chi(g)a)$. The global sections of L_χ are the same as polynomial functions on X, i.e. $\Gamma(X, L_\chi) = A$, but the action of G is given by $(gf)(x) = \chi(g)f(g^{-1}x)$ for all $g \in G$ and $x \in X$. Then

$$\Gamma(X, L_\chi)^G = \{f \in A \mid f(gx) = \chi(g)f(x)\} =: \{\chi\text{-semi-invariants}\} =: A^{G,\chi}.$$

Hence $R_{L_\chi}^G = \bigoplus_{n \geq 0} A^{G,n\chi}$.

Example 1. Let $\mathbb{C}^*$ act on $\mathbb{C}^4$ via $t(x_1, x_2, y_1, y_2) = (tx_1, tx_2, t^{-1}y_1, t^{-1}y_2)$. The character group of $\mathbb{C}^*$ may be identified with $\mathbb{Z}$ by associating to an integer m the character $\chi_m : \mathbb{C}^* \to \mathbb{C}^*$ defined by $\chi_m(t) = t^m$. One may now impose a $\mathbb{Z}$-grading on the coodinate ring $A = \mathbb{C}[x_1, x_2, y_1, y_2]$ by assigning weights $1, 1, -1, -1$ to x_1, x_2, y_1, y_2, respectively. Then $A = \bigoplus_{m \in \mathbb{Z}} A_m$ with $A_m = A^{\mathbb{C}^*, \chi_m}$. One has

$$A_0 = \mathbb{C}[x_1y_1, x_2y_1, x_1y_2, x_2y_2] \cong \mathbb{C}[A, B, C, D]/(AD - BC).$$

If $m > 0$, then I have

$$A_m = x_1^m A_0 + x_1^{m-1}x_2 A_0 + \cdots + x_2^m A_0 \cong I^m,$$
$$A_{-m} = y_1^m A_0 + y_1^{m-1}y_2 A_0 + \cdots + y_2^m A_0 \cong J^m,$$

where $I = (A, B) \subset A_0$ and $J = (A, C) \subset A_0$ are ideals of A_0. I may now make the following table

	$m < 0$		$m = 0$		$m > 0$
graded ring	$A_0 \oplus J^{-m} \oplus J^{-2m} \oplus \cdots$		$A_0 \oplus A_0 \oplus \cdots$		$A_0 \oplus I^m \oplus I^{2m} \oplus \cdots$
unstable locus	$V_+ := \{y_1 = y_2 = 0\}$	$\supset$	$\varnothing$	$\subset$	$V_- := \{x_1 = x_2 = 0\}$
semistable locus	$\mathbb{C}^4 - V_+$	$\subset$	$\mathbb{C}^4$	$\supset$	$\mathbb{C}^4 - V_-$
	$\downarrow$		$\downarrow$		$\downarrow$
quotient	Q_-	$\rightarrow$	Q_0	$\leftarrow$	Q_+

where $Q_0 = \operatorname{Spec}(A_0)$ is the quadric cone $\{AD - BC = 0\}$ in $\mathbb{C}^4$, where Q_- is the blowup of Q_0 along the plane $\{A = C = 0\}$ and Q_+ is the blowup along the plane $\{A = B = 0\}$.

The *variation-of-quotient morphisms* $Q_- \to Q_0 \leftarrow Q_+$ exist because the composition $\mathbb{C}^4 - V_+ \hookrightarrow \mathbb{C}^4 \twoheadrightarrow \mathbb{C}^4/\!/\mathbb{C}^* = Q_0$ is $\mathbb{C}^*$-invariant and hence by the universal property above must factor through $Q_- = (\mathbb{C}^4 - V_+)/\!/\mathbb{C}^*$.

The fiber of a variation-of-quotients morphism $\rho_- : Q_- \to Q_0$ over a point $\xi \in Q_0$ may be computed as follows. Let $\pi_- : \mathbb{C}^4 - V_+ \to Q_-$ and $\pi_0 : \mathbb{C}^4 \to Q_0$ denote the quotient morphisms. To get the fiber $\rho_-^{-1}(\xi)$ one starts with the union $\pi_0^{-1}(\xi)$ of all orbits in $\mathbb{C}^4$ which get contracted onto ξ, one then throws out the points which are unstable for negative m (i.e. one considers $\pi_0^{-1}(\xi) \cap (\mathbb{C}^4 - V_+)$), and then one takes the image in Q_-. Formally this is $\rho_-^{-1}(\xi) = \pi_-(\pi_0^{-1}(\xi) \cap (\mathbb{C}^4 - V_+))$.

In the example if $\xi \in Q_0$ is any point other than the vertex of the cone, then the pullback of ξ to $\mathbb{C}^4$ consists of a single stable orbit which lies outside $V_+ \cup V_-$ and thus corresponds to a single point in Q_-. Thus ρ_- is locally an isomorphism away from the vertex of the cone. (The locus $\mathbb{C}^4 - (V_+ \cup V_-)$ is within the stable locus for all m and so maps onto isomorphic open sets in the three quotients.) But if ξ is the vertex of Q_0, then its pullback to $\mathbb{C}^4$ is the union of all orbits containing the origin $0 \in \mathbb{C}^4$ in their closure, i.e. $V_+ \cup V_-$. The part of this which lies in $\mathbb{C}^4 - V_+$ is simply $V_- - \{0\}$, which has image $(V_- - \{0\})/\mathbb{C}^* = \mathbb{P}V_- \cong \mathbb{P}^1$.

The other variation of quotients morphism $\rho_+ : Q_+ \to Q_0$ is also locally an isomorphism away from the vertex of the cone Q_0, and the fiber of ρ_+ over the vertex is $\mathbb{P}V_+ \cong \mathbb{P}^1$.

The diagram (where the dashed horizontal arrows represent rational maps)

$$\begin{tikzcd} Q_- \arrow[rr, dashed, shift left] \arrow[dr] & & Q_+ \arrow[ll, dashed, shift left] \arrow[dl] \\ & Q_0 & \end{tikzcd}$$

is called a *flip* in Mori theory. The singular threefold Q_0 has two distinct minimal desingularizations related by a flip.

Another way to factor the passage between Q_- and Q_+ is to "go over the top," i.e. to complete the flip diagram to a cartesian diagram

$$\begin{tikzcd} & \widetilde{Q} \arrow[dl] \arrow[dr] & \\ Q_- \arrow[rr, dashed, shift left] \arrow[dr] & & Q_+ \arrow[ll, dashed, shift left] \arrow[dl] \\ & Q_0 & \end{tikzcd}.$$

One may compute that $\widetilde{Q}$ is the blowup of Q_0 with center the vertex of the cone. Because of universal properties satisfied by blowups, this means that it is also the blowup of Q_- along $\mathbb{P}V_-$ (the inverse image of the vertex in Q_-) and the blowup of Q_+ along $\mathbb{P}V_+$. The exceptional divisor of the blowup $\widetilde{Q} \to Q_0$ is a $\mathbb{P}^1 \times \mathbb{P}^1$, and this exceptional divisor gets contracted to $* \times \mathbb{P}^1$ in Q_-, and to $\mathbb{P}^1 \times *$ in Q_+ (here $*$ denotes a point).

Example 2. Let n_+, n_-, and n_0 be nonnegative integers, and let $\mathbb{C}^*$ act on $\mathbb{C}^{n_+ + n_- + n_0}$ by

$$t(x_1, \dots, x_{n_+}, y_1, \dots, y_{n_-}, z_1, \dots, z_{n_0}) \\ = (tx_1, \dots, tx_{n_+}, t^{-1}y_1, \dots, t^{-1}y_{n_-}, z_1, \dots, z_{n_0}) \,.$$

One may analyze this example in the same way as the previous one. The results are similar at least if $n_+ \geq 2$ and $n_- \geq 2$. Indeed let $V_+ = \mathbb{C}^{n_+} = \{y_j = z_k = 0 \text{ for all } i, j\}$ and define $V_- = \mathbb{C}^{n_-}$ and $V_0 = \mathbb{C}^{n_0}$ similarly. Then the middle quotient Q_0 is a singular affine variety—indeed it is the product of $V_0 \times C$ with the C the cone over $\mathbb{P}V_- \times \mathbb{P}V_+$, and $\mathrm{Sing}(Q_0) = V_0 \times v$ where v is the vertex of the cone. The two other quotients Q_- and Q_+ are desingularizations of Q_0 where the $V_0 \times v$ is replaced by $V_0 \times \mathbb{P}V_-$ and $V_0 \times \mathbb{P}V_+$, respectively. The variety $\widetilde{Q}$ on top is the blowup of Q_0 along $V_0 \times v$ and also the blowups of Q_+ and of Q_- along the inverse images of the singular locus of Q_0. The passage from Q_+ to Q_- or vice-versa is still called a flip, indeed a *simple flip*.

If say $n_- = 1$ but $n_+ \geq 2$, then the picture is slightly different because $\mathbb{P}V_-$ is simply a point. Indeed one finds that Q_0 and Q_- are both the affine space $V_0 \times V_+$, but that Q_+ is the blowup of Q_0 along the subspace $V_0 \times \{0\}$. The passage from Q_+ to Q_- is called a *divisorial contraction* in this case.

If say $n_- = 0$, but $n_+ \geq 1$, then $Q_+ = V_0 \times \mathbb{P}V_+$, while $Q_0 = V_0$, and $Q_- = \varnothing$. The variation of quotients morphism $Q_+ \to Q_0$ is the projection $V_0 \times \mathbb{P}V_+ \to V_0$. It is a *fibration* in projective spaces.

These three phenomena—fibrations, divisorial contractions, and flips—are exactly the same kinds of phenomena which occur in the Mori's program for (nonsingular or mildly singular) minimal models for algebraic threefolds.

If one considers an general linear action of $\mathbb{C}^*$ on $\mathbb{C}^N$ with arbitrary integers as weights (not just 1, -1, and 0), then one gets the same kind of picture as above, only the fibers $\mathbb{P}V_+$ and $\mathbb{P}V_-$ would be weighted projective spaces.

See [R] for more on flips.

Example 3. Let r, m, and n be integers with $0 \leq r \leq m \leq n$. Let $X = \mathrm{Mat}_{m\times r}(\mathbb{C}) \times \mathrm{Mat}_{r\times n}(\mathbb{C})$ be the $\mathbb{C}^{mr+nr}$ parametrizing pairs of matrices (M, N) of the indicated sizes. Let $\mathrm{GL}(r)$ act on X via $g(M, N) = (Mg^{-1}, gN)$. The character group of $\mathrm{GL}(r)$ may again be identified with $\mathbb{Z}$ by associating to $m \in \mathbb{Z}$ the character $g \mapsto \det(g)^m$.

There are again three quotients Q_-, Q_0 and Q_+, depending on the sign of m. A study of these quotients as above by direct analysis of the semi-invariant functions gets a bit involved with Plücker coordinates and so forth. I will omit the details and give the answers.

For $m = 0$, all the pairs (M, N) are semistable, the quotient Q_0 is the subvariety $V_r \subset \mathrm{Mat}_{m\times n}(\mathbb{C})$ of $m \times n$ matrices of rank at most r, and the quotient map $\pi_0 : X \to Q_0$ is given by $(M, N) \mapsto MN$.

For $m < 0$, a pair (M, N) is semistable if and only if $\mathrm{rk}(M) = r$. The quotient Q_- is the subvariety of $V_r \times \mathrm{Grass}(m - r, m)$ parametrizing pairs (A, W) where A is an $m \times n$ matrix of rank at most r and $W \subset \mathbb{C}^m$ is an $(m-r)$-dimensional subspace contained in $\ker(A)$. The quotient map $\pi_- : X^{--\mathrm{ss}} \to Q_-$ is given by $(M, N) \mapsto (MN, \ker(M))$. The variation of quotients map $\rho_- : Q_- \to Q_0$ is given by $(A, W) \mapsto A$. The variety Q_- gives a resolution of the singularities of Q_0. A fiber $\rho_-^{-1}(A)$ is a Grassmannian $\mathrm{Grass}(m - r, m - \mathrm{rk}(A))$ which in general is not a projective space.

For $m > 0$ one has a symmetric situation.

The passage from Q_- to Q_+ or vice-versa is still considered a flip, but it is not a simple flip with projective space fibers as in the earlier examples.

2. The Hilbert-Mumford numerical criterion for stability

Although the "official" definitions of L-unstable, L-semistable, and L-stable points use invariant global sections of the $L^{\otimes n}$, actual geometric situations are often too complicated for it to be easy to compute the invariant global sections (or even all the global sections whether invariant or not). But Mumford kindly provided us with another method—the numerical criterion—which is proven in Professor Le Potier's course. But let me review it.

A *on-parameter subgroup* of G is a morphism of algebraic groups $\lambda : \mathbb{C}^* \to G$. Such a λ is *nontrivial* if $\lambda \not\equiv 1$.

If G acts on X, then any one-parameter subgroup induces an action of $\mathbb{C}^*$ on X via $t.x = \lambda(t)x$. A fixed point of λ is a point x such that $\lambda(t)x = x$ for all t.

Recall that if V is a G-equivariant vector bundle on X, and if $x \in X$, then any $g \in G$ induces a linear map between fibers $V_x \to V_{g(x)}$. We thus get the following lemma:

Lemma 1. *If V is a G-equivariant vector bundle of rank n on X, and $x \in X$ is a fixed point of G, then G acts linearly on the fiber $V_x \cong \mathbb{C}^n$. In particularly, the stabilizer G_x acts linearly on V_x for all $x \in X$.*

In particular, if x is a fixed point for λ, then λ acts as a group of automorphisms of the vector space $L_x \cong \mathbb{C}$. Then if $0 \neq a \in L_x$, there exists an integer $\mu \in \mathbb{Z}$ such that $\lambda(t)a = t^\mu a$. In order to show the dependence of this μ on various inputs, we write $\mu = \mu^L(x, \lambda)$.

If x is not a fixed point of λ but $x^0 := \lim_{t\to 0} \lambda(t)x$ exists, we set $\mu^L(x, \lambda) := \mu^L(x^0, \lambda)$. If the limit does not exist, one does not define $\mu^L(x, \lambda)$ (or one sets it equal to $+\infty$ or something arbitrary and positive).

The Hilbert-Mumford numerical criterion. *Let $X \subset \mathbb{A}^N \times \mathbb{P}^M$ be a closed subvariety (e.g. a projective or affine variety), let G be a reductive group acting on X, and let L be a G-linearized ample line bundle over X. Then a point $x \in X$ is L-(semi)stable if and only if for all nontrivial one-parameter subgroups $\lambda : \mathbb{C}^* \hookrightarrow G$ such that* $\lim_{t\to 0} \lambda(t)x$ *exists one has $\mu^L(x, \lambda) > 0$ (resp. $\mu^L(x, \lambda) \geq 0$).*

One may take powers of one-parameter subgroups using the formula $\lambda^m(t) = \lambda(t)^m$. Note that if $m > 0$, then $\mu(x, \lambda)$ and $\mu(x, \lambda^m)$ will always have the same sign, so we only have to look at one of λ and λ^m when applying the numerical criterion.

Example 1 (cont.). There are (up to positive powers) only two nontrivial one-parameter subgroups of $\mathbb{C}^*$, namely the identity λ_+ and the map λ_- defined by $t \mapsto t^{-1}$.

If $x \in \mathbb{C}^4$ is outside $V_+ \cup V_-$, then neither $\lim_{t\to 0} tx$ nor $\lim_{t\to 0} t^{-1}x$ exists, so these points are $+$-stable, 0-stable, and $-$-stable.

If $x \in V_+ - \{0\}$, then $\lim_{t\to 0} t^{-1}x$ does not exist, but $\lim_{t\to 0} tx = 0$ does, and $\mu^{\chi_m}(x, \lambda_+) = \mu^{\chi_m}(0, \lambda_+) = m$. Hence these points are $+$-stable, 0-semistable, and $-$-unstable.

If $x \in V_- - \{0\}$, then $\lim_{t\to 0} tx$ does not exist, but $\lim_{t\to 0} t^{-1}x = 0$ does, and $\mu^{\chi_m}(x, \lambda_-) = -m$. Hence these points are $+$-unstable, 0-semistable, and $-$-stable.

For $x = 0$, both limits exist, and $\mu^{\chi_m}(0, \lambda_+) = m$ while $\mu^{\chi_m}(0, \lambda_-) = -m$. Hence 0 is $+$-unstable, 0 -semistable, and $-$-unstable.

This agrees with what was calculated using invariant global sections.

Example 2 works in the same way.

3. Representation spaces of quivers

In this lecture I would like to describe King's construction [Ki] of moduli spaces of semistable representations of quivers which Professor Happel touched on in his lectures. Besides the intrinsic interest of these spaces, many things occur in these moduli spaces which also occur for moduli spaces of vector bundles but in circumstances where it is easier to give complete proofs. (Nevertheless I will cut some corners; for complete proofs, see [Ki]).

Recall that a *quiver* Q is given by a set of vertices $Q_0 = \{v_1, \dots, v_n\}$ and a set of arrows $Q_1 = \{a_1, \dots, a_\ell\}$ with arrow a_α starting at $v_{s(\alpha)}$ and ending at $v_{e(\alpha)}$. A *representation* $R = (R_i, \phi_\alpha)$ is given by specifying vector spaces $R_1, \dots, R_n$ for each vertex and linear maps $\phi_\alpha : R_{s(\alpha)} \to R_{e(\alpha)}$ for each arrow. I will call the R_i the components of R. The *dimension vector* of R is

$$\underline{\dim} R := (\dim R_1, \dots, \dim R_n) \in \mathbb{N}^n .$$

A *subrepresentation* $S \subset R$ is a system of subspaces $S_i \subset R_i$ such that $\phi_\alpha(S_{s(\alpha)}) \subset S_{e(\alpha)}$ for all α. A morphism of representations is defined similarly.

For $\underline{d} = (d_1, \dots, d_n) \in \mathbb{N}^n$, let

$$\mathrm{Rep}(Q, \underline{d}) := \prod_{\alpha=1}^{\ell} \mathrm{Mat}_{d_{e(\alpha)} \times d_{s(\alpha)}}(\mathbb{C})$$

be the *representation space* of Q with dimension vector $\underline{d}$ discussed in Professor Happel's lectures. It is an affine space parametrizing ℓ-tuples of matrices $M = (M_1, \dots, M_\ell)$ or equivalently representations $R_M = (\mathbb{C}^{d_1}, \dots \mathbb{C}^{d_n}; M_1, \dots, M_\ell)$.[3]

[3] I am thinking of the $\mathbb{C}^{d_i}$ as spaces of column vectors with the matrices acting on the left, while in Professor Happel's lectures they were spaces of row vectors with the matrices acting on the right. This leads to a number of minor differences.

The group

$$G := \prod_{i=1}^{n} \mathrm{GL}(d_i)$$

acts on $\mathrm{Rep}(Q,\underline{d})$ by changing the bases of the $\mathbb{C}^{d_i}$. This means explicitly that if $g=(g_i)\in G$, and $M=(M_\alpha)\in \mathrm{Rep}(Q,\underline{d})$, then $g.M=(g_{e(\alpha)}M_\alpha g_{s(\alpha)}^{-1})$. Two points $M,N\in \mathrm{Rep}(Q,\underline{d})$ are in the same G-orbit if and only if $R_M\cong R_N$.

The subgroup

$$\Delta=\{(t1_{d_1},\dots,t1_{d_n})\mid t\in\mathbb{C}^*\}\subset G$$

acts trivially on $\mathrm{Rep}(Q,\underline{d})$, so there is actually a well-defined action of the quotient group $\overline{G}:=G/\Delta$ on the affine space $\mathrm{Rep}(Q,\underline{d})$. I will now want to consider the quotients of $\mathrm{Rep}(Q,\underline{d})$ by $\overline{G}$.

To construct such a quotient I must pick a $\overline{G}$-linearized ample line bundle on $\mathrm{Rep}(Q,\underline{d})$. This means a twist L_χ of the trivial line bundle by a character χ of $\overline{G}$. Now the character group of G is isomorphic to $\mathbb{Z}^n$ with $\underline{\theta}:=(\theta_1,\dots,\theta_n)\in\mathbb{Z}^n$ corresponding to the character $\chi_{\underline{\theta}}:G\to\mathbb{C}^*$ given by $\chi_{\underline{\theta}}(g_1,\dots,g_n):=\det(g_1)^{\theta_1}\cdots\det(g_n)^{\theta_n}$. By pullback a character of $\overline{G}$ is precisely a character of G which is trivial on Δ, so we have

$$\mathrm{Char}(\overline{G})\cong\{\underline{\theta}\in\mathbb{Z}^n\mid\langle\underline{\theta},\underline{d}\rangle=0\}$$

where I write

$$\langle\underline{\theta},\underline{d}\rangle:=\sum\theta_i d_i,\qquad\text{and}\qquad\underline{\theta}(R):=\langle\underline{\theta},\underline{\dim}R\rangle.$$

Theorem 2 (King). *One has $M\in\mathrm{Rep}(Q,\underline{d})^{\underline{\theta}\text{-ss}}$ (resp. $M\in\mathrm{Rep}(Q,\underline{d})^{\underline{\theta}\text{-s}}$) if and only if $\underline{\theta}(R)=0$ and for all proper subrepresentations $0\subsetneq S\subsetneq R_M$ one has $\underline{\theta}(S)\geq 0$ (resp. $\underline{\theta}(S)>0$).*

Sketch of proof. One uses the Hilbert-Mumford numerical criterion. There are several steps. I will denote the components of the representation R_M by $R_i=\mathbb{C}^{d_i}$.

(1) Any one-parameter subgroup $\lambda:\mathbb{C}^*\to\overline{G}$ may be lifted to a one-parameter subgroup $\mathbb{C}^*\to G=\prod\mathrm{GL}(d_i)$.[4] We thus get an action of $\mathbb{C}^*$ on each component of the representation, inducing decompositions $R_i=\bigoplus_{w\in\mathbb{Z}}R_i^w$ where $\mathbb{C}^*$ acts on R_i^w via $t.r=t^wr$.

(2) Filter each R_i by subspaces $R_i^{\geq w}=\bigoplus_{u\geq w}R_i^u$. Then one shows (essentially by writing the M_α in bases compatible with the decompositions $R_i=\bigoplus_{w\in\mathbb{Z}}R_i^w$) that $\lim_{t\to0}\lambda(t)R_M$ exists if and only if $M_\alpha(R_{s(\alpha)}^{\geq w})\subset R_{e(\alpha)}^{\geq w}$ for all α and w. So one-parameter subgroups whose action on R_M tends to a limit correspond to decreasing filtrations of R_M by *subrepresentations*

$$\cdots\supset R_M^{\geq w}\supset R_M^{\geq w+1}\supset\cdots$$

[4]This is because $\lambda(\mathbb{C}^*)$ lies in a maximal torus of T of $\overline{G}$. Since all maximal tori are conjugate, T is conjugate to D/Δ where $D\subset G$ is the subgroup of n-tuples of diagonal matrices. It is easily seen that the surjection $D\twoheadrightarrow D/\Delta$ is split (i.e. has a right inverse), so D/Δ and therefore all maximal tori and also $\lambda(\mathbb{C}^*)$ lift to G.

such that $R_M^{\geq w} = R_M$ for $w \ll 0$ and $R_M^{\geq w} = 0$ for $w \gg 0$. Moreover, if λ is nontrivial, then there is a w such that $0 \neq R_M^{\geq w} \neq R_M$.

Looking at matrices again, one may see that $\lim_{t\to 0} \lambda(t) R_M$ corresponds to the representation

$$\operatorname{gr}(R_M) := \bigoplus_{w\in\mathbb{Z}} (R_M^{\geq w}/R_M^{\geq w+1})$$

(3) One computes

$$\mu^{\underline{\theta}}(R_M, \lambda) = \sum_{w,i} w\theta_i \dim R_i^w = \sum_w \underline{\theta}(R_M^{\geq w})$$

with the last sum finite since $\underline{\theta}(R_M^{\geq w}) = 0$ for $w \gg 0$ (since then $R_M^{\geq w} = 0$) and for $w \ll 0$ (since then $R_M^{\geq w} = R_M$ and $\underline{\theta}(R_M) = 0$).

(4) It now follows immediately that if $\underline{\theta}(S) > 0$ for all proper subrepresentations S of R_M, then $\mu^{\underline{\theta}}(R_M, \lambda) > 0$ for all nontrivial λ such that $\lim_{t\to 0} \lambda(t) R_M$ exists, and therefore that R_M is $\underline{\theta}$-stable. Conversely if there exists a proper subrepresentation such that $\underline{\theta}(S) \leq 0$, then one may use the filtration

$$\cdots \supset R_M \supset R_M \supset S \supset 0 \supset 0 \supset \cdots$$

to find a λ such that $\mu^{\underline{\theta}}(R_M, \lambda) \leq 0$ so that R_M is not $\underline{\theta}$-stable. Semistability works similarly. □

Theorem 2 motivates us to say that a representation R of Q is *$\underline{\theta}$-stable* (resp. *$\underline{\theta}$-semistable*) if $\underline{\theta}(R) = 0$ and if for all proper subrepresentations $0 \subsetneq S \subsetneq R$ one has $\underline{\theta}(S) > 0$ (resp. $\underline{\theta}(S) \geq 0$). The quotient

$$M^{\underline{\theta}}(Q, \underline{d}) := \operatorname{Rep}(Q, \underline{d})^{\underline{\theta}\text{-ss}} /\!/ \overline{G}$$

is then the *moduli space* of $\underline{\theta}$-semistable representations of Q with dimension vector $\underline{d}$. What do its points correspond to, and what points go where under the quotient map $\operatorname{Rep}(Q, \underline{d})^{\underline{\theta}\text{-ss}} \to M^{\underline{\theta}}(Q, \underline{d})$?

Lemma 3. *The category* $\operatorname{SSRep}(Q, \underline{\theta})$ *of $\underline{\theta}$-semistable representations of Q is an abelian category (i.e. images, kernels, and cokernels of maps between $\underline{\theta}$-semistable representations are $\underline{\theta}$-semistable). Its simple objects are the $\underline{\theta}$-stable representations.*

Proof. Exercises 2 and 3. □

Since every object of $\operatorname{SSRep}(Q, \underline{\theta})$ is artinian and noetherian, the Jordan-Hölder theorem holds for $\operatorname{SSRep}(Q, \underline{\theta})$:

Corollary 4. *Every $\underline{\theta}$-semistable representation R of Q has a maximal filtration*

$$0 \subset R_1 \subset \cdots \subset R_{s-1} \subset R_s = R$$

by $\underline{\theta}$-semistable subrepresentations. Furthermore, the associated graded object $\operatorname{gr}(R) := \bigoplus_{i=1}^s (R_i/R_{i-1})$ *is independent of the choice of maximal filtration (up to isomorphism).*

The representation $\mathrm{gr}(R)$ is a direct sum of $\underline{\theta}$-stable representations. Such a direct sum is called a *$\underline{\theta}$-polystable representation.* Two $\underline{\theta}$-semistable representations R and R' are called *S-equivalent* (or $\underline{\theta}$-S-equivalent) if their associated $\underline{\theta}$ -polystable representations satisfy $\mathrm{gr}(R) \cong \mathrm{gr}(R')$. Each $\underline{\theta}$-semistable representation is S-equivalent to a unique $\underline{\theta}$-polystable representation.

Theorem 5 (King [Ki]). *The points of the moduli space $M^{\underline{\theta}}(Q,\underline{d})$ correspond bijectively to the S-equivalence classes of $\underline{\theta}$-semistable representations of Q of dimension vector $\underline{d}$. Moreover, any $M \in \mathrm{Rep}(Q,\underline{d})^{\underline{\theta}\text{-ss}}$ is sent by the quotient map to the point of $M^{\underline{\theta}}(Q,\underline{d})$ corresponding to the S-equivalence class of R_M.*

King also showed that $M^{\underline{\theta}}(Q,\underline{d})$ is a coarse moduli space for S-equivalence classes of $\underline{\theta}$-semistable representations of Q of dimension vector $\underline{d}$. Moreover, if the dimension vector $\underline{d}$ is primitive (i.e. $\gcd(d_1,\dots,d_n)=1$) and $\underline{\theta}$ is sufficiently general so that all $\underline{\theta}$-semistable representations of Q of dimension vector $\underline{d}$ are actually $\underline{\theta}$-stable, then $M^{\underline{\theta}}(Q,\underline{d})$ is a *fine moduli space* over which there is a universal family of $\underline{\theta}$-stable representations.

Analogies with vector bundles. A vector bundle or sheaf is also (semi)stable if and only if it does not contain any (strictly) destabilizing subsheaves (in an appropriate sense). Moreover, S-equivalence for vector bundles or sheaves is defined in basically the same way as for representations, and points in coarse moduli spaces of semistable bundles (or sheaves) represent S-equivalence classes.[5]

Example 4. Let Q be the quiver below with two vertices and four arrows. Consider the dimension vector $\underline{d}=(1,1)$.

$$\begin{tikzcd}
\underset{-\theta}{\mathbb{C}} \arrow[r, bend left=50, "x_1"] \arrow[r, bend left=20, "x_2"'] & \underset{\theta}{\mathbb{C}} \arrow[l, bend left=20, "y_1"'] \arrow[l, bend left=50, "y_2"]
\end{tikzcd}$$

One has $\mathrm{Rep}(Q,\underline{d}) \cong \mathbb{C}^4$ with points $M=(x_1,x_2,y_1,y_2)$. The group $G = \mathbb{C}^* \times \mathbb{C}^*$ acts via

$$(r,s).(x_1,x_2,y_1,y_2) = (r^{-1}sx_1, r^{-1}sx_2, rs^{-1}y_1, rs^{-1}y_2).$$

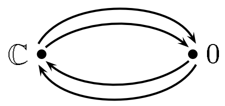
Then $\overline{G}=\mathbb{C}^*$, and it acts via

$$t(x_1,x_2,y_1,y_2) = (tx_1, tx_2, t^{-1}y_1, t^{-1}y_2)$$

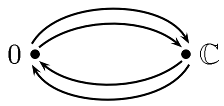
This is exactly the action of Example 1. Hence I should be able to identify the moduli spaces $M^{\underline{\theta}}(Q,\underline{d})$ (depending on $\underline{\theta}$) with the different quotients Q_+, Q_0, and Q_- of Example 1.

[5] The representation space $\mathrm{Rep}(Q,\underline{d})$ may also be viewed as a sort of Hilbert scheme, analogous with the schemes Quot^0 which are often used in the construction of moduli spaces of coherent sheaves. Indeed for each vertex v_i let P_i be the corresponding indecomposable projective representation of Q (whose j-th component P_{ij} is the vector space of formal linear combinations of paths from v_i to v_j, and an arrow a_α induces a linear map $P_{is(\alpha)} \to P_{ie(\alpha)}$ using composition of paths). Then $\mathrm{Rep}(Q,\underline{d})$ is an open subvariety of the smooth projective variety $\mathrm{Quot}(\bigoplus P_i^{d_i}, \underline{d})$ parametrizing quotients of $\bigoplus P_i^{d_i}$ of dimension vector $\underline{d}$. This perspective makes the calculations from the proof of [S] Theorem 5.2 seem to an algebraic geometer like dimension counts on Hilbert schemes and moduli spaces.

Now if a representation R_M has a proper subrepresentation S, then either $\underline{\dim} S = (1,0)$ or $\underline{\dim} S = (0,1)$. In other words S is one of the following two representations.



S_1



S_2

The loci V_+ and V_- appearing in Example 1 may be interpreted as:

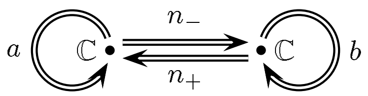
$$V_- = \{x_1 = x_2 = 0\} = \{M \mid 0 \subsetneq S_1 \subsetneq R_M\},$$
$$V_+ = \{y_1 = y_2 = 0\} = \{M \mid 0 \subsetneq S_2 \subsetneq R_M\}.$$

The origin of $\mathbb{C}^4$ corresponds to $S_1 \oplus S_2$, while the points of $\mathbb{C}^4 - (V_- \cup V_+)$ correspond to *simple representations* (i.e. representations R with no subrepresentations other than 0 and R).[6]

The $\underline{\theta}$ we are interested are orthogonal to the dimension vector $(1,1)$ and so are of the form $\underline{\theta} = (-\theta, \theta)$. So $\underline{\theta}(S_1) = -\theta$, and $\underline{\theta}(S_2) = \theta$. Hence we see that if $\theta < 0$, then $\mathrm{Rep}(Q,\underline{d})^{\underline{\theta}\text{-s}} = \mathbb{C}^4 - V_+$, and $\mathrm{Rep}(Q,\underline{d})^{\underline{\theta}\text{-us}} = V_+$, while $M^{\underline{\theta}}(Q,\underline{d}) = Q_-$. Similarly, if $\theta > 0$, then $M^{\underline{\theta}}(Q,\underline{d}) = Q_+$.

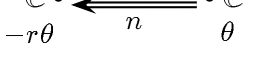
If $\theta = 0$, then all of $\mathbb{C}^4$ is semistable, but only $\mathbb{C}^4 - (V_+ \cup V_-)$ is stable. The nonstable representations are all either extensions of the form $0 \to S_2 \to R \to S_1 \to 0$ (in V_+) or extensions of the form $0 \to S_1 \to R \to S_2 \to 0$ (in V_-). They are all S-equivalent to the polystable representation $S_1 \oplus S_2$. Hence all of $V_+ \cup V_-$ gets contracted to a single point in $M^{\underline{\theta}}(Q,\underline{d}) = Q_0$.

Example 5. The action of Example 2 of $\mathbb{C}^*$ on $\mathbb{C}^{n_+ + n_- + n_0}$ is the same as the action of $\overline{G}$ on $\mathrm{Rep}(Q,\underline{d})$ for the quiver below with dimension vector $\underline{d} = (1,1)$.



The doubled arrows signify that there are n_- arrows from v_1 (on the left) to v_2 (on the right), there are n_+ arrows in the other direction, there are a loops attached at v_1, and b loops attached at v_2 with $n_0 = a + b$. (See Exercise 4.)

Example 6. The action of Example 3 of $\mathrm{GL}(r)$ on $\mathrm{Mat}_{m\times r}(\mathbb{C}) \times \mathrm{Mat}_{r\times n}(\mathbb{C})$ is the same as the action of $\overline{G}$ on $\mathrm{Rep}(Q,\underline{d})$ for the quiver below with dimension vector $\underline{d} = (1, r)$. (See Exercise 5.)



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[6] Since this quiver has oriented cycles, it has simple representations other than S_1 and S_2, cf. [LP].

Miscellaneous interesting facts. King showed that moduli spaces of representations of a quiver Q are projective if Q has no oriented cycles.

There are a number of classical computations of stable points for group actions which have traditionally been done on a case-by-case basis but which can be done more systematically using quivers and King's theorem. These involve varieties of the form

$$Z := \prod_{i=1}^{m} \mathrm{Grass}(d_i, U_i \otimes H) \times \prod_{i=m+1}^{n} \mathrm{Grass}(d_i, U_i \otimes H^*)$$

(where H and the U_i are vector spaces). The group $\mathrm{PGL}(H)$ acts naturally on all the factors simultaneously. The linearized ample line bundles are of the form $L_{\underline{a}} = \mathcal{O}_Z(a_1, \ldots, a_n)$ with all the $a_i > 0$.

The quiver Q associated to such a Z has $n+1$ vertices $v_0, v_1, \ldots, v_n$. For $i = 1, \ldots, m$ it has $\dim U_i$ arrows from v_i to v_0, while if $i = m+1, \ldots, n$, it has $\dim U_i$ arrows from v_0 to v_i. There are no other arrows. The associated dimension vector is $\underline{d} = (d_0, d_1, \ldots, d_n)$ where $d_0 = \dim H$. The associated character is given by $\underline{\theta} = (\theta_0, -a_1, \ldots, -a_m, a_{m+1}, \ldots, a_n)$ where θ_0 is determined by the other invariants and the condition $\langle \underline{\theta}, \underline{d} \rangle = 0$. One may construct $M^{\underline{\theta}}(Q, \underline{d})$ by first quotienting an open subset of stable points of $\mathrm{Rep}(Q, \underline{d})$ by the normal subgroup $\prod_{i=1}^{n} \mathrm{GL}(d_i) \subset \overline{G}$ to get Z, and then by the quotient subgroup $\mathrm{PGL}(H)$. Thus $Z^{L_{\underline{a}}\text{-ss}} /\!/ \mathrm{PGL}(H) \cong M^{\underline{\theta}}(Q, \underline{d})$.

For the case of $\mathrm{PGL}(2)$ acting on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ see Exercise 7. The case of $\mathrm{PGL}(H)$ acting on $\mathrm{Grass}(d, U \otimes H)$ came up in Professor Le Potier's course, and it is interesting to redo it with the Kronecker quiver

$$\mathbb{C}^d \bullet \xRightarrow{\;u\;} \bullet\, H$$

where there are $u = \dim U$ arrows. The case where $m = n = 1$ and $\dim U_1 = \dim U_2 = 3$ came up in [Le]. It corresponds to the following quiver (cf. [KW1] §4).

$$\bullet \;\Rrightarrow\; \bullet \;\Rrightarrow\; \bullet$$

Products of flag varieties may be treated by attaching chains of vertices. For instance $\mathrm{Flag}(r_1, r_2, r_3, U_1 \otimes H) \times \mathrm{Flag}(s_1, s_2, U_2 \otimes H^*)$ would use the quiver

$$\begin{tikzcd} & & & \overset{H}{\bullet} \arrow[dr, Rightarrow, "u_2"] & & \\ \overset{\mathbb{C}^{r_1}}{\bullet} \arrow[r] & \overset{\mathbb{C}^{r_2}}{\bullet} \arrow[r] & \overset{\mathbb{C}^{r_3}}{\bullet} \arrow[ur, Rightarrow, "u_1"] & & \overset{\mathbb{C}^{s_2}}{\bullet} \arrow[r] & \overset{\mathbb{C}^{s_1}}{\bullet} \end{tikzcd}$$

with two arrows of multiplicities u_1 and u_2 and three arrows of multiplicity 1.

4. Where do stability quotients live?

Recall that I asked the following question a while ago:

Question. If one fixes X, G, and the action of G on X but lets the linearized line bundle L vary, how do the open loci $X^{L\text{-ss}} \subset X$ of L-semistable points and the quotients $X^{L\text{-ss}} /\!/ G$ vary?

I will begin by presenting Dolgachev and Hu's answer (from [DH]) to the following more specific problem:

Problem. Describe the sets of L for which $X^{L\text{-ss}}$ are constant.

Two G-linearized line bundles L_1 and L_2 over X are *algebraically equivalent* if there exists a connected algebraic variety S containing two points s_1 and s_2 and a G-linearized line bundle $\mathcal{L}$ over $X \times S$ compatible with the action of G on $X \times S$ given by $g(x,s) = (g(x), s)$ such that $\mathcal{L}|_{X\times\{s_i\}} \cong L_i$ for $i = 1, 2$.

Lemma 6. *Suppose G is a reductive group acting on a closed subvariety $X \subset \mathbb{A}^M \times \mathbb{P}^N$. If L_1 and L_2 are algebraically equivalent G-linearized line bundles on X, then $X^{L_1\text{-ss}} = X^{L_2\text{-ss}}$, and $X^{L_1\text{-s}} = X^{L_2\text{-s}}$ and also $X^{L_1\text{-ss}}//G = X^{L_2\text{-ss}}//G$.*

Proof. For fixed x and λ, the invariants $\mu^L(x,\lambda)$ depend continuously on L but take integer values. So any connected algebraic family of G-linearized line bundles will generate the same functions $\mu^L(x,\lambda)$ and hence, according to the Hilbert-Mumford numerical criterion, the same open subsets $X^{L\text{-s}}$ and $X^{L\text{-ss}}$ and the same quotients $X^{L\text{-ss}}//G$. □

The equivariant Néron-Severi group. The group of algebraic equivalence clases of G-linearized line bundles on X is called the *G-equivariant Néron-Severi group of X*, written $\mathrm{NS}^G(X)$. The group $\mathrm{NS}^G(X)$ is a finitely generated abelian group because (1) the ordinary Néron-Severi group $\mathrm{NS}(X)$ of algebraic equivalence classes of line bundles is known to be finitely generated (this requires some work), (2) the character group of G is also finitely generated, and (3) there is an exact sequence

$$0 \to \mathrm{Char}(G) \to \mathrm{NS}^G(X) \to \mathrm{NS}(X).$$

Moreover, for any positive integer n, a linearized ample line bundle L and its power $L^{\otimes n}$ would define exactly the same semistable loci and quotients. (See Exercise 6.) As a result I may ignore any torsion in $\mathrm{NS}^G(X)$ and also extend notions of $X^{L\text{-ss}}$ and $X^{L\text{-ss}}//G$ to any L in the $\mathbb{Q}$-vector space $\mathrm{NS}^G_{\mathbb{Q}}(X)$ by interpolation.

Note that by what I said previously $\mathrm{NS}^G_{\mathbb{Q}}(X)$ is a finite-dimensional vector space.

Within $\mathrm{NS}_{\mathbb{Q}}(X)$ is an open convex cone $\mathrm{NA}_{\mathbb{Q}}(X)$ of (positive rational multiples of) algebraic equivalence classes of ample line bundles called the ample cone of X. (See Exercise 8.) The inverse image in $\mathrm{NS}^G_{\mathbb{Q}}(X)$ of the ample cone of X is the *equivariant ample cone of X*, denoted $\mathrm{NA}^G_{\mathbb{Q}}(X)$. This equivariant ample cone is in principle where the stability conditions on X live.

For affine varieties as in the earlier examples, all line bundles are ample. So for affine varieties $\mathrm{NA}^G_{\mathbb{Q}}(X)$ is all of $\mathrm{NS}^G_{\mathbb{Q}}(X)$. But for nonaffine varieties there is a strict inclusion, since the trivial bundle is not ample if X is not affine.

Chamber systems. A finite set of hyperplanes $H_1, \ldots, H_r$ in a finite dimensional $\mathbb{Q}$-vector space V defines a finite *chamber system* on V. The chambers are the connected components of $V - (\bigcup H_i)$. More generally, the *faces* of the chamber system are defined by saying that ℓ_1 and ℓ_2 lie in the same face if for each hyperplane the ℓ_1 and ℓ either both lie on the same side of the hyperplane or both lie in the hyperplane.[7] A *wall* is a codimension-one face. Two chambers are *adjacent* if they have a common wall in their closures.

Theorem 7 (Dolgachev-Hu). *There is a finite chamber system in* $\mathrm{NA}^G_{\mathbb{Q}}(X)$ *such that* $X^{L_1\text{-ss}} = X^{L_2\text{-ss}}$ *if and only if* L_1 *and* L_2 *lie in the same face of the chamber system.*

Dolgachev and Hu proved Theorem 7 by showing that the orbit structure for the action of G on X must contain a certain type of substructure whenever there are points which are semistable for some L_1 but not for another L_2. The substructure was such that general finiteness principles for algebraic varieties implied that only a finite number of such substructures could exist in a single algebraic variety. Their particular substructure was defined using Kempf's theory of maximally destabilizing one-parameter subgroups, which is a topic which will not be covered in this course. But a variant proof based on the étale slice theorem might be possible.

For Examples 1, 2, and 3, we have $\mathrm{NA}^G_{\mathbb{Q}}(X) = \mathbb{Q}$, the wall W is at the origin, and it separates two chambers C_+ and C_-.

Warning. $\mathrm{NA}^G_{\mathbb{Q}}(X)$ is not the same thing as the locus of L such that $X^{L\text{-ss}} \neq \varnothing$ (or even the interior of that locus). It is the locus of L such that the *not necessarily invariant* sections of L and its powers separate all points of X. For such an L if the invariant sections of the powers of L do not separate orbits, it is for reasons related to the orbit structure which may be analyzed using geometric invariant theory. (For other L one would have to worry about what the noninvariant sections fail to separate before starting in with the numerical criterion.) But there may certainly be L in $\mathrm{NA}^G_{\mathbb{Q}}(X)$ for which $X^{L\text{-ss}} = \varnothing$. For instance if $\mathbb{C}^*$ acts on $\mathbb{C}^N$ via $t(x_1, \dots, x_N) = (tx_1, \dots, tx_N)$, then $\mathrm{NA}^G_{\mathbb{Q}} = \mathbb{Q}$, but the quotient is empty in the negative chamber. (This is a case of Example 2 with $n_- = 0$.)

Representations of quivers. For Q a quiver with n vertices, $\underline{d}$ a *sincere* dimension vector (i.e. all $d_i > 0$), and $X = \mathrm{Rep}(Q, \underline{d})$, one has $\mathrm{NA}^G_{\mathbb{Q}}(X) = \mathbb{Q}^{n-1}$. Any dimension vector $\underline{e}$ of possible subrepresentations of representations of dimension vector $\underline{d}$ generates a wall

$$W_{\underline{e}} = \{\underline{\theta} \mid \langle \underline{\theta}, \underline{d} \rangle = 0 \text{ and } \langle \underline{\theta}, \underline{e} \rangle = 0\}$$

(unless $\underline{e}$ and $\underline{d}$ are proportional). On one side of this wall (where $\langle \underline{\theta}, \underline{e} \rangle < 0$) those representations containing subrepresentations of dimension vector $\underline{e}$ are $\underline{\theta}$-unstable. On the other side those representations with quotients of dimension vector $\underline{e}$ are $\underline{\theta}$-unstable. Hence we may distinguish two kinds of walls. There are *exterior walls* where the "nonempty-quotient cone" $\{\underline{\theta} \mid M^{\underline{\theta}}(Q, \underline{d}) \neq \varnothing\}$ lies all on one side of the wall—these are $W_{\underline{e}}$ such that all representations contain a

[7] Sometimes one runs into chamber systems which are only locally finite, i.e. which are defined by an infinite number of hyperplanes but such that every point of the space (or of an open part with which one is concerned, e.g. an ample cone) has an open neighborhood meeting only a finite number of hyperplanes. For instance chamber systems for moduli spaces of vector bundles on surfaces are often only locally finite. But our chamber systems will be finite.

subrepresentation of dimension vector $\underline{e}$ or $\underline{d} - \underline{e}$. There are also *interior walls* corresponding to $\underline{e}$ such that some representations have subrepresentations of dimension vector $\underline{e}$ and others do not.

Some of the $W_{\underline{e}}$ may be redundant and just do things like dividing up chambers with empty quotients into smaller chambers.

Schofield's paper [S] discusses the dimension vectors $\underline{e}$ of subrepresentations of a general representation of dimension vector $\underline{d}$. This corresponds to finding the exterior walls of the "nonempty-quotient cone."

5. Etale neighborhoods

The rest of this course is devoted to local questions about orbit structures for groups acting on varieties and for quotients.

To begin with, we must deal with the word "local." The problem is that the usual topology of algebraic geometers, the Zariski topology, is extremely coarse. A Zariski neighborhood in a variety X is almost all of X—and almost global can be a bad approximation for local. For instance a smooth variety of dimension n should look locally like $\mathbb{A}^n$, but this is untrue in the Zariski topology unless the variety is globally almost the same as $\mathbb{A}^n$ (i.e. birational to $\mathbb{A}^n$). This is usually not the case. However, in the classical (or analytic) topology for complex algebraic varieties, smooth varieties do look locally like $\mathbb{C}^n$. In the complex analytic topology one may also take roots of functions. For instance if f is holomorphic in a neighborhood of x, and $f(x) \neq 0$, then $\sqrt[m]{f}$ is holomorphic (in a possibly smaller neighborhood). This does not work with polynomials and the Zariski topology.

There is more than one way of resolving these problems:

- Formal power series and formal neighborhoods (but these are really germs rather than neighborhoods and are too local for certain applications),
- Convergent power series (over the complex numbers) and the complex analytic topology (but this involves analysis), and
- Algebraic functions and étale neighborhoods.

Recall that a morphism $f : Z \to X$ of algebraic varieties is *étale* if it is smooth of relative dimension 0. Equivalently f is étale if and only if for all $z \in Z$ it induces an isomorphism $f^* : \widehat{\mathcal{O}}_{X,f(z)} \to \widehat{\mathcal{O}}_{Z,z}$ of complete local rings. Over the complex numbers f is étale if and only if it is locally biholomorphic. An *étale neighborhood* of $x \in X$ is a diagram

$$\begin{array}{ccc} z & \in & Z \\ \downarrow & & \downarrow f \\ x & \in & X \end{array}$$

such that f is étale. One might write that (Z, z) is an étale neighborhood of (X, x).

Zariski neighborhoods are étale neighborhoods (because open inclusions $U \subset X$ are étale morphisms). But many étale neighborhoods are not Zariski neighborhoods. Consider for instance unramified covers of a variety or of a Zariski open subset of a variety.

One should think of étale neighborhoods as being neighborhoods in a topology which is finer than the Zariski topology but coarser than the complex analytic topology. I do not want to get into further details of this étale topology (e.g. étale covers, overlaps of étale neighborhoods, descent and sheaf theory, etc.) partially because it is not a topology in the usual sense but a Grothendieck topology. Nevertheless, I will use phrases such as "locally in the étale topology," and the meaning should usually be obvious.

It may not seem intuitive to regard étale neighborhoods as neighborhoods, but doing so actually resolves a number of difficulties we had with the Zariski topology.

- In the étale topology any smooth variety X of dimension n is locally isomorphic to $\mathbb{A}^n$. I.e. for any $x \in X$, one can find a common étale neighborhood for (X, x) and $(\mathbb{A}^n, 0)$. How? Embed an affine open neighborhood $U \ni x$ into $\mathbb{A}^N$ and take a linear projection of U onto $T_x U \cong \mathbb{A}^n$. Throwing out the ramification locus of this projection map, one gets a smaller open neighborhood V of x on which projection is étale. This makes (V, x) a Zariski open neighborhood of (X, x) and an étale open neighborhood of $(\mathbb{A}^n, 0)$.
- If f is a function defined on a Zariski open neighborhood U of x with $f(x) \neq 0$, then there is a smaller neighborhood $x \in V \subset U$ which is affine (with $V = \operatorname{Spec}(A)$) and on which f is everywhere nonzero. Let $Z = \operatorname{Spec}(A[T]/(T^m - f))$. Then Z is an étale neighborhood of x on which $\sqrt[m]{f}$ exists.
- More generally, the implicit and inverse function theorems work locally in the étale topology but not in the Zariski topology. Since these tools are used constantly in analytic geometry, it is nice to have them around.

Etale neighborhoods for varieties with group action. For varieties with group action we have the following stronger definition. If $f : X \to Y$ is a G-equivariant morphism between varieties on which G acts for which categorical quotients $X//G$ and $Y//G$ exist, then f is *strongly étale* if $f//G : X//G \to Y//G$ is étale and the following diagram is cartesian (a fiber product diagram)

$$\begin{array}{ccc} X & \to & Y \\ \downarrow & \square & \downarrow \\ X//G & \to & Y//G \end{array}$$

This means that (1) f is étale, but also that (2) for any $\xi \in X//G$ mapping to $\eta \in Y//G$, the locus $\pi_X^{-1}(\xi)$ of all orbits in X which collapse to ξ is mapped isomorphically onto the locus $\pi_Y^{-1}(\eta)$ of all orbits in Y which collapse onto η. In particular (3) any orbit in X is mapped isomorphically onto its image in Y.

6. Etale slices

Let $Y \subset X$ be a smooth closed subvariety of a smooth variety. Recall that the *normal bundle* of Y in X is the vector bundle on Y defined by

$$0 \to T_Y \to T_X|_Y \to N_{Y/X} \to 0.$$

The rank of $N_{Y/X}$ is $\operatorname{codim}_X Y$. The points of $N_{Y/X}$ parametrize normal vectors to Y in X, i.e. the component "perpendicular to Y" of the tangent vectors to X along Y.

Now let G be a reductive group acting on X, and let L be a G-linearized ample line bundle on X. Suppose that $Y = O_x \subset X^{L\text{-ss}}$ is a closed semistable orbit along which X is smooth. Then G acts on O_x, and T_{O_x}, $T_X|_{O_x}$, and $N_{O_x/X}$ are G-equivariant vector bundles. By Lemma 1 this implies that G_x acts on the fibers of these bundles over X and in particular on $N_{O_x/X,x}$. We will write

$$N := N_{O_x/X} \qquad \text{and} \qquad N_x := N_{O_x/X,x}.$$

Two proofs of the following theorem may be found in [Lu], pp. 84–86.

Theorem 8 (Matsushima). *If G is a reductive group acting on X, and $O_x \subset X^{L\text{-ss}}$ is a closed semistable orbit, then the stabilizer G_x is reductive.*

Luna's étale slice theorem then says the following:

Etale Slice Theorem (weak form). *If $O_x \subset X^{L\text{-ss}}$ is a closed semistable orbit along which X is smooth and if ξ is the corresponding point of the quotient, then the pointed varieties $(X^{L\text{-ss}}//G, \xi)$ and $(N_x//G_x, \overline{0})$ are locally isomorphic in the étale topology.*

Tubular neighborhoods. In topology, if Y is a smooth closed submanifold of a smooth manifold X, then there exists a neighborhood U of Y in X such that U is diffeomorphic to $N_{Y/X}$ (with $Y \subset U$ being identified with the zero section of $N_{Y/X}$). This U is called a *tubular neighborhood* of Y in X.

The strong form of the étale slice theorem is a G-EQUIVARIANT STRONGLY ÉTALE TUBULAR NEIGHBORHOOD THEOREM for a closed semistable orbit $O_x \subset X^{L\text{-ss}}$. Its precise statement is somewhat complicated, however, because it involves an identification of the common étale open neighborhood of $O_x \subset X$ and of the zero section $O_x \subset N$.

Fibers and slices. Part of the geometric structure of the normal bundle N is its fibers, and in particular the fiber N_x over x. This is a smooth subvariety transversal to the zero section at x and G_x-invariant (globally not pointwise).

According to the tubular neighborhood philosophy this N_x should correspond to a "normal slice" $V_x \subset X^{L\text{-ss}}$ transversal to O_x at x and G_x-invariant.

In addition the G-equivariant vector bundle N may be recovered from the fiber N_x and the action of G_x on N_x via the formula

$$N = G \times^{G_x} N_x := (G \times N_x)/G_x$$

where G_x acts on $G \times N_x$ via $h.(g,n) = (gh^{-1}, hn)$.[8] The projection of the vector bundle onto its base $N \to G/G_x \cong O_x$ is induced from the first projection

[8] There is no universally accepted notation for the construction which I have written as $G \times^{G_x} N_x$. It is sometimes written $G \times_{G_x} N_x$ in analogy with the tensor product, but unfortunately this makes it look like a fiber product which it definitely is not. The notation $G \wedge^{G_x} N_x$ also exists.

$G \times N_x \to G$ by quotienting by G_x. The action of G on N is similarly induced by the left translation action of G on the first factor of $G \times N_x$.[9]

Finally there is a natural G-equivariant map $G \times^{G_x} V_x \to X^{L\text{-ss}}$ given by $[(g, v)] \mapsto gv$.

Etale Slice Theorem (strong form). *Suppose $O_x \subset X^{L\text{-ss}}$ is a closed semistable orbit along which $X^{L\text{-ss}}$ is smooth. Then a G_x-invariant normal slice V_x (as above) exists together with a G_x-equivariant strongly étale morphism $(V_x, x) \to (N_x, 0)$ such that the morphisms $G \times^{G_x} V_x \to G \times^{G_x} N_x = N$ and $G \times^{G_x} V_x \to X^{L\text{-ss}}$ are both G-equivariant and strongly étale.*

Note that the theorem says strongly étale. This means not only that the morphisms are G-equivariant and étale but that the squares is the following diagram are cartesian squares (which is very powerful):

$$\begin{array}{ccccccccc}
O_x \subset & X^{L\text{-ss}} & \xleftarrow{\text{étale, } G} & G \times^{G_x} V_x & \xrightarrow{\text{étale, } G} & N & \supset \text{ zero section} \\
& \downarrow & \square & \downarrow & \square & \downarrow & \\
\xi \in & X^{L\text{-ss}}//G & \xleftarrow[\text{étale}]{} & (G \times^{G_x} V_x)//G & \xrightarrow[\text{étale}]{} & N//G & \cong N_x//G_x \ni \overline{0}
\end{array}$$

(For the isomorphism $N//G \cong N_x//G_x$ see Exercise 9.)

The weak form of the étale slice theorem is simply the bottom line of the diagram. The strong form (both lines) says in addition that the G-orbit structure of $X^{L\text{-ss}}$ near O_x is locally the same as the G-orbit structure of N near the zero section which in turn is the same as the G_x-orbit structure of N_x near the origin. For instance one may deduce that the locus $\pi_X^{-1}(\xi)$ which is collapsed onto the point $\xi \in X^{L\text{-ss}}//G$ is related to the locus $\pi_{N_x}^{-1}(\overline{0})$ of G_x-orbits in N_x which contain the origin in their closure via the formula $\pi_X^{-1}(\xi) \cong G \times^{G_x} \pi_{N_x}^{-1}(\overline{0})$.

The pair (G_x, N_x) the *model* for the action of G around O_x.

Construction of the étale slice. The construction of the étale slice is not difficult but requires that G_x be *linearly reductive.* One begins by finding a G_x-invariant decomposition $T_x X = T_x O_x \oplus N_x$. The linear equations defining the normal space then span a G_x-submodule $N_x^\perp \subset T_x^* X \cong \mathfrak{m}_x/\mathfrak{m}_x^2$ where $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is the maximal ideal of functions vanishing at x within the local ring of rational functions on X which are defined at x. By linear reductivity one may lift these equations to a G_x-submodule of $\mathfrak{m}_x$ which maps isomorphically onto $N_x^\perp$. The étale slice is defined locally by these lifted equations. The proof that the various maps in the étale slice theorem are strongly étale is rather more technical but is done well in [KSS].

In characteristic 0 linear reductivity does not pose a problem because the group G_x is reductive by Theorem 8, and by Professor Strømme's course, reductive groups in characteristic 0 are linearly reductive. But in characteristic p, semisimple algebraic groups are not linearly reductive, so G_x is linearly reductive if and only if the identity component G_x^0 is a torus, and the finite group G_x/G_x^0 is of order prime to p.

[9] This construction is part of a general correspondence between G-equivariant bundles on homogeneous spaces G/H and representations of H.

7. Examples of $N_x//G_x$

Finite groups. Suppose G is a finite group acting linearly on a finite dimensional vector space V.

Suppose $x \in V$ with image $\xi \in V/G$. The orbit $O_x = \{g(x) \mid g \in G\}$ is a discrete finite subset of V. Hence $N_x = T_xV \cong V$ itself. An element $g \in G_x$ acts on $T_xV \cong V$ by sending

$$v \mapsto \frac{d}{dt}\big[g(x+tv)\big]\bigg|_{t=0} = g(v)$$

Thus the action of the stabilizer G_x on $N_x = V$ is the restriction of the action of G. Hence $\xi \in V/G$ looks like $\overline{0} \in V/G_x$ locally in the étale topology.

If the action of G on $V - 0$ is free (i.e. only the identity has a nonzero fixed point, a common but not universal situation), then $\overline{0} \in V/G$ is the only singularity. But if the action is not free, then there may be other singularities.

Representations of quivers. For constructions of moduli spaces, the closed semistable orbit O_x corresponds to a stable or polystable object (representation, curve, vector bundle, ...) R_x. Then usually $G_x = \mathrm{Aut}(R_x)$ or $\mathrm{Aut}(R_x)/\mathbb{C}^*$, while N_x is the tangent space of the moduli functor at R_x and is usually given by some deformation theory formula.

For representations of quivers, a closed $\underline{\theta}$-semistable orbit corresponds to a $\underline{\theta}$-polystable representation which one may write as $R_M \cong \bigoplus R_j^{n_j}$ with the R_j $\underline{\theta}$-stable and pairwise nonisomorphic. Then

$$G_M \cong \mathrm{Aut}(R_M)/\mathbb{C}^* \cong \big(\prod_j \mathrm{GL}(n_j)\big)/\mathbb{C}^*$$

and $N_M \cong \mathrm{Ext}^1(R_M, R_M)$. The group $\mathrm{Aut}(R_M)$ acts on $\mathrm{Ext}^1(R_M, R_M)$ via $g.\xi = g\xi g^{-1}$. (This much happens also with moduli of vector bundles.)

Let me do some special cases.

Example 7. Suppose $R_M \cong R_1 \oplus R_2$ with R_1 and R_2 $\underline{\theta}$-stable and nonisomorphic. Then

$$N_M = \mathrm{Ext}^1(R_1, R_2) \oplus \mathrm{Ext}^1(R_2, R_1) \oplus \mathrm{Ext}^1(R_1, R_1) \oplus \mathrm{Ext}^1(R_2, R_2).$$

The subspace $\mathrm{Ext}^1(R_1, R_2)$ corresponds to representations R which are extensions of the form $0 \to R_2 \to R \to R_1 \to 0$. The subspace $\mathrm{Ext}^1(R_2, R_1)$ corresponds to extensions the other way. The subspace $\mathrm{Ext}^1(R_1, R_1) \oplus \mathrm{Ext}^1(R_2, R_2)$ corresponds to representations of the form $R_1' \oplus R_2'$ with each R_i' a deformation of R_i.

The group $\mathrm{Aut}(R_M) \cong \mathbb{C}^* \times \mathbb{C}^*$ acts on N_M with the first $\mathbb{C}^*$ acting by homotheties on R_1 and the second on R_2. So $\mathrm{Aut}(R_M)$ acts via

$$(s,t).(e_{12}, e_{21}, e_{11}, e_{22}) = (ts^{-1}e_{12}, st^{-1}e_{21}, e_{11}, e_{22}).$$

The group $G_M \cong \mathbb{C}^*$ acts via $\lambda(e_{12}, e_{21}, e_{11}, e_{22}) = (\lambda e_{12}, \lambda^{-1}e_{21}, e_{11}, e_{22})$. This is exactly the action of Example 2 with

$$n_+ = \dim \mathrm{Ext}^1(R_1, R_2), \qquad n_- = \dim \mathrm{Ext}^1(R_2, R_1),$$
$$n_0 = \dim\left(\mathrm{Ext}^1(R_1, R_1) \oplus \mathrm{Ext}^1(R_2, R_2)\right).$$

The singularity of $M^{\underline{\theta}}(Q,\underline{d})$ at the point considered is locally isomorphic in the étale (or classical) topology to that of the Q_0 of Example 2, i.e. to $\mathbb{A}^{n_0} \times C$ where C is the cone over $\mathbb{P}^{n_+-1} \times \mathbb{P}^{n_--1}$. Its local equations are

$$\mathbb{C}[X_1,\ldots,X_{n_0},Y_{11},\ldots,Y_{1n_-},Y_{21},\ldots Y_{2n_-},\ldots,Y_{n_+n_-}]/I$$

where I is the ideal generated by all the two-by-two minors

$$\begin{vmatrix} Y_{ij} & Y_{i\ell} \\ Y_{kj} & Y_{k\ell} \end{vmatrix}$$

of the matrix $Y = (Y_{ij})$.

Example 8. $R_M = R_1 \oplus R_2^r \cong R_1 \oplus (H \otimes_{\mathbb{C}} R_2)$ where H is a vector space of dimension r over $\mathbb{C}$. Then

$$N_M \cong (H \otimes E_{12}) \oplus (H^* \otimes E_{21}) \oplus E_{11} \oplus (H \otimes H^* \otimes E_{22})$$

where $E_{ij} := \mathrm{Ext}^1(R_i,R_j)$. The group $G_M = (\mathbb{C}^* \times \mathrm{GL}(H))/\mathbb{C}^* \cong \mathrm{GL}(H)$ acts via the natural actions of $\mathrm{GL}(H)$ on H, on H^* and on $H \otimes H^*$.

To simplify things let us assume $E_{22} = \mathrm{Ext}^1(R_2,R_2) = 0$, i.e. that that R_2 is *rigid*. Then we are essentially in the situation of Example 3. So let Q_0 be the middle quotient of Example 3, i.e. let $Q_0 = V_r \subset \mathrm{Mat}_{m\times n}(\mathbb{C})$ be the subvariety of $m \times n$ matrices of rank at most r where $m = \dim E_{12}$, and $n = \dim E_{21}$. Then locally the singularity of $M^{\underline{\theta}}(Q,\underline{d})$ at the point is isomorphic to $Q_0 \times \mathbb{A}^p$ where $p = \dim E_{11}$.

Ext quivers. Let me go back to the general situation with $\underline{\theta}$-polystable representations of quivers. We have a representation $R_M = \bigoplus_{j=1}^s R_j^{n_j}$, and we have the group $G_M = \left(\prod GL(n_i)\right)/\mathbb{C}^*$ acting on

$$N_M = \mathrm{Ext}(R_M,R_M) = \bigoplus_{i,j} \left(\mathrm{Mat}_{n_i\times n_j}(\mathbb{C}) \otimes \mathrm{Ext}(R_i,R_j)\right)$$

via the natural action on the $\mathrm{Mat}_{n_i\times n_j}(\mathbb{C})$. But this is exactly like the construction of a moduli space for a quiver, although for a different quiver and a different dimension vector.

The new quiver Δ has s vertices $r_1,\ldots,r_s$ corresponding to the s nonisomorphic stable components of R_M and $\dim \mathrm{Ext}^1(R_i,R_j)$ arrows from r_i to r_j. This new quiver is the *Ext quiver* of R_M or of the indecomposables $R_1,\ldots,R_s$. The new dimension vector is $\underline{n} = (n_1,\ldots,n_s)$, the multiplicities of the R_i in R_M. The quotient $N_M//G_M$ is $M^{\underline{0}}(\Delta,\underline{n})$, the moduli space of $\underline{0} = (0,\ldots,0)$-semistable representations of $\Delta,\underline{n}$. Thus our singularity of $M^{\underline{\theta}}(Q,\underline{d})$ is locally isomorphic to the "origin" in another moduli space $M^{\underline{0}}(\Delta,\underline{n})$ of the Ext quiver of R_M. This is from [LP].

The Ext quiver for Example 7 is exactly the quiver of Example 5, while the Ext quiver for Example 8 is exactly the quiver of Example 6 (if R_2 is rigid).

8. Etale slices and variation of quotients

Suppose that a reductive group G acts on a closed subvariety $X \subset \mathbb{A}^N \times \mathbb{P}^M$. Recall from an earlier lecture that there is a finite chamber system in the equivariant ample cone $\mathrm{NA}^G_{\mathbb{Q}}(X)$ (which is an open convex cone in a finite-dimensional $\mathbb{Q}$-vector space) such that $X^{L_1\text{-ss}} = X^{L_2\text{-ss}}$ if and only if $[L_1]$ and $[L_2]$ lie in the same face of the chamber system. I neglected to mention it the other day, but if C and W are faces such that $W \subset \overline{C}$, then $X^{C\text{-ss}} \subset X^{W\text{-ss}}$. The composition $X^{C\text{-ss}} \hookrightarrow X^{W\text{-ss}} \to X^{W\text{-ss}}//G$ is then G-invariant, so it must factor through the categorical quotient $X^{C\text{-ss}}//G$. In other words, one has a diagram

$$\begin{array}{ccc} X^{C\text{-ss}} & \subset & X^{W\text{-ss}} \\ \pi_C \downarrow & & \downarrow \pi_W \\ X^{C\text{-ss}}//G & \xrightarrow{\rho} & X^{W\text{-ss}}//G \end{array}$$

with ρ a variation-of-quotients map. We saw several such diagrams in the examples the first day.

There are more bad (unstable) points to throw away in C than in W, and there are also more good (stable) points. But there are fewer mediocre (semistable but nonstable) points in C than in W.

There are usually many C-semistable orbits which get identified in $X^{W\text{-ss}}//G$ but which stay separated in $X^{C\text{-ss}}//G$. This is because the W-semistable orbit that they had in their closure which caused the identification in $X^{W\text{-ss}}//G$ is C-unstable.

For example consider $\mathbb{C}^*$ acting on $\mathbb{C}^n$ via $t(x_1, \dots, x_n) = (tx_1, \dots, tx_n)$. At the wall W where all points are semistable, the semistability of the origin causes all the lines to be collapsed together with the origin into a single point. But in the positive chamber, the origin is unstable and all other points are stable. So the images of the lines remain separated in the quotient $\mathbb{P}^{n-1}$ for the positive chamber.

Strategy for studying local structure variations of quotients. Take a point $\xi \in X^{W\text{-ss}}//G$ and the corresponding closed W -semistable orbit $O_x \subset X^{W\text{-ss}}$. Then try to see which orbits near O_x remain C-semistable and which do not, in order to see which orbits which get collapsed onto $\xi \in X^{W\text{-ss}}$ stay separate in $X^{C\text{-ss}}//G$. More formally, study

$$\rho^{-1}(\xi) = \pi_C\left(\pi_W^{-1}(\xi) \cap X^{C\text{-ss}}\right).$$

Tool we want to use to study orbits near O_x. The étale slice theorem, because it says that the G-orbit structure near O_x is essentially the same as the G_x-orbit structure near $0 \in N_x$.

Question. C-semistability for the G-action on X should correspond to which stability condition for the G_x-action on N_x? I.e. pick a G-linearized line bundle L on X with $[L] \in C$. This restricts to a G_x-linearized line bundle $L|_{V_x}$ on the slice V_x. For what character χ of G_x is the twisted trivial line bundle L_χ on N_x similar to $L|_{V_x}$?

Plausible answer. Note that G_x fixes x. This means that G_x acts on the fiber $L|_x \cong \mathbb{C}$ by a lemma I proved earlier. This action is via a character $\iota_x(L) \in \mathrm{Char}(G_x)$.

The point $x \in V_x$ is like $0 \in N_x$. The action of G_x on $L_\chi|_0 \cong \mathbb{C}$ is via $g(0,a) = (0, \chi(g)a)$, i.e. via the character χ. So one should have $\chi = \iota_x(L)$.

Theorem 9 ([KW2], but cf. [T3] [DH] for $G_x = \mathbb{C}^*$). *With the above notation the following diagrams are cartesian with horizontal arrows étale and vertical arrows variation of quotients morphisms:*

$$\begin{array}{ccccccccc} & & X^{C\text{-ss}}//G & \leftarrow & (G \times^{G_x} V_x)^{C\text{-ss}}//G & \rightarrow & N_x^{\iota_x(L)\text{-ss}}//G_x & & \\ & & \downarrow & \Box & \downarrow & \Box & \downarrow & & \\ \xi & \in & X^{W\text{-ss}}//G & \leftarrow & (G \times^{G_x} V_x)^{W\text{-ss}}//G & \rightarrow & N_x//G_x & \ni & \overline{0} \end{array}$$

This means that, locally in the étale topology, all variations of quotients can be calculated in the Luna model, i.e. with varying stability conditions for G_x acting on N_x.

Example 9. Suppose that you have a $\underline{\theta}_0$-polystable representation of the form $R_M = R_1 \oplus R_2$ with $\underline{\dim} R_M = \underline{d}$ not proportional to $\underline{\dim} R_1 = \underline{e}$. Then $\underline{\theta}_0$ lies on the wall

$$W_{\underline{e}} = \{\underline{\theta} \mid \langle \underline{\theta}, \underline{e} \rangle = \langle \underline{\theta}, \underline{d} \rangle = 0\}$$

defined several lectures ago. On one side of the wall are $\underline{\theta}_+$ such that $\langle \underline{\theta}_+, \underline{e} \rangle > 0$ for which deformations R of R_M which are extensions $0 \to R_2 \to R \to R_1 \to 0$ are unstable. On the other side one has $\underline{\theta}_-$ such that $\langle \underline{\theta}_-, \underline{e} \rangle < 0$ for which extensions the other way round are unstable. We have already seen in Example 7 that on the wall the quotient $M^{\underline{\theta}_0}(Q, \underline{d})$ looks locally like the quotient Q_0 of Example 2. But now we see that the variation-of-quotients map $M^{\underline{\theta}_+}(Q, \underline{d}) \to M^{\underline{\theta}_0}(Q, \underline{d})$ looks locally like $Q_+ \to Q_0$ and similarly in the other chamber. So $M^{\underline{\theta}}$ goes through a simple flip as $\underline{\theta}$ goes through the wall $W_{\underline{e}}$, at least locally around our point.

Example 10. If $R_M = R_1 \oplus R_2^r$ with R_2 rigid and if $\underline{\dim} R_1 = \underline{e}$ and $\underline{d}$ are not proportional, then you get the flip of Example 3.

One may do flips (for moduli spaces of representations of quivers, vector bundles, etc.) locally using Ext quivers, provided transforms the character $\underline{\theta}$ for the original quiver (or a stability condition on vector bundles, etc.) into a character $\iota_M(\underline{\theta})$ for the Ext quiver Δ. For an Ext quiver with s vertices corresponding to nonisomorphic $\underline{\theta}$-stable representations $R_1, \ldots, R_s$, one sets $\iota_M(\underline{\theta}) = (\underline{\theta}(R_1), \underline{\theta}(R_2), \ldots, \underline{\theta}(R_s))$. (Exercise: verify this).

Note that you can fix W and then use the same étale slice machinery to look at all neighboring faces C with $W \subset \overline{C}$. As you do this the characters $\iota_x(L)$ vary in a chamber system in $\mathrm{NS}^{G_x}_{\mathbb{Q}}(N_x) = \mathrm{Char}(G)_{\mathbb{Q}}$. This is useful if you want to pass from one chamber C_- to a chamber C_+ through a wall.

Fibrations and strata. The G_x-module N_x may be split into an invariant part and a complementary submodule as $N_x = N_x^{G_x} \oplus N_1$. Then the variation of quotients map becomes

$$\begin{array}{c} N_x^{G_x} \times \left(N_1^{\iota_x(L)\text{-ss}}//G_x \right) \\ \downarrow \\ N_x^{G_x} \times (N_1//G_x) \end{array}$$

Thus there is a smooth stratum S of dimension equal to $\dim N_x^{G_x}$ through $\xi \in X^{W\text{-ss}}//G$ along which the model (G_x, N_x) is constant and along which the variation of quotients morphism is locally a product. This implies for instance that if $F \subset N_1^{\iota_x(L)\text{-ss}}//G_x$ is the fiber over $\overline{0} \in N_1//G_x$, then $\rho^{-1}(S) \to S$ is a fibration with fiber F which is locally trivial in the étale topology. Thaddeus showed that if $G_x = (\mathbb{C}^*)^t$ is a torus and if the weights of the action of G_x on N_x generate the lattice $\mathrm{Char}(G_x) \cong \mathbb{Z}^t$, then this fibration $\rho^{-1}(S) \to S$ is locally trivial in the Zariski topology. This result of Thaddeus applies for instance with simple flips with the simple flips.

9. Exercises

Exercise 1. Apply the Hilbert-Mumford criterion to Example 3 to find out which pairs of matrices (M, N) are $+$-(semi)stable, 0-(semi)stable, and $-$-(semi)stable.

Exercise 2. (a) Show that if $0 \to S \to R \to F \to 0$ is a short exact sequence of representations of a quiver Q with n vertices, and if $\underline{\theta} \in \mathbb{Z}^n$, then $\underline{\theta}(R) = \underline{\theta}(S) + \underline{\theta}(F)$.

(b) Show that representation R of Q is $\underline{\theta}$-stable (resp. $\underline{\theta}$-semistable) if and only if $\underline{\theta}(R) = 0$, and for all quotient representations F of R such that $0 \neq F \neq R$, one has $\underline{\theta}(F) < 0$ (resp. $\underline{\theta}(F) \leq 0$).

Exercise 3. Prove Lemma 3.

Exercise 4. Let Q be the quiver of Example 5, and let $\underline{d} = (1, 1)$.

(a) Show that the action of $\overline{G}$ on the representation space $\mathrm{Rep}(Q, \underline{d})$ is indeed the same as the action of $\mathbb{C}^*$ on $\mathbb{C}^{n_+ + n_- + n_0}$ of Example 2.

(b) Analyze the representations corresponding to the three subspaces V_+, V_-, and V_0 of $\mathbb{C}^{n_+ + n_- + n_0}$ defined in Example 2.

Exercise 5. Let Q be the quiver of Example 6 with $0 \leq r \leq m \leq n$, and let $\underline{d} = (1, r)$.

(a) Show that the action of $\overline{G}$ on the representation space $\mathrm{Rep}(Q, \underline{d})$ is indeed the same as the action of $\mathrm{GL}(r)$ on $\mathrm{Mat}_{m\times r}(\mathbb{C}) \times \mathrm{Mat}_{r\times n}(\mathbb{C})$ of Example 3.

(b) Let S_2 be the unique representation of Q of dimension vector $(0, 1)$. Show that if a representation R of dimension vector $(1, r)$ is not simple, then R either has a subrepresentation or a quotient representation isomorphic to S_2.

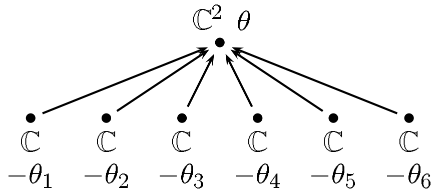
(c) Suppose $\underline{\theta} = (-r\theta, \theta)$ satisfies $\langle \underline{\theta}, \underline{d} \rangle = 0$. Show that if $\theta > 0$, then a representation $R_{(M,N)}$ corresponding to a pair of matrices (M, N) is $\underline{\theta}$-(semi)stable if and only if $\mathrm{rk}(M) = r$. Show also that if $\theta < 0$, then $R_{(M,N)}$ is $\underline{\theta}$-(semi)stable if and only if $\mathrm{rk}(N) = r$.

Exercise 6. Suppose that G is a reductive algebraic group acting on a closed subvariety $X \subset \mathbb{A}^N \times \mathbb{P}^M$ and that L is a G-linearized ample line bundle on X. Let m be a positive integer.

(a) Prove that $X^{L\text{-ss}} = X^{L^{\otimes m}\text{-ss}}$ and that $X^{L\text{-ss}}//G = X^{L^{\otimes m}\text{-ss}}//G$ using invariant global sections.

(b) Prove the same thing using the Hilbert-Mumford numerical criterion for stability.

Exercise 7. Describe the $\underline{\theta}$-polystable points for the representations of the following quiver with the dimension vector $\underline{d}$ and character $\underline{\theta}$ as marked



Exercise 8. Use the fact that $\mathrm{NS}(X)$ is finitely generated together with general properties of ample line bundles to show that $\mathrm{NA}_{\mathbb{Q}}(X) \subset \mathrm{NS}_{\mathbb{Q}}(X)$ is an open convex cone.

Exercise 9. (a) Suppose that H is a normal algebraic subgroup of an algebraic group G and that H and G are reductive. Suppose that G acts on an affine variety X and that L is a G linearized ample line bundle on X. Show that G/H acts on $X//H$, and that $(X//H)//(G/H) \cong X//G$.

(b) Suppose that K is a reductive subgroup of a reductive algebraic group. Suppose that K acts on an affine variety Y, so that G acts on $G \times^K Y$. Show that $(G \times^K Y)//G \cong Y//K$.

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