ENRIQUES SURFACES AND OTHER NON-PFAFFIAN SUBCANONICAL SUBSCHEMES OF CODIMENSION 3

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Abstract. We give examples of subcanonical subvarieties of codimension 3 in projective n-space which are not *Pfaffian*, i.e. defined by the ideal sheaf of submaximal Pfaffians of an alternating map of vector bundles. This gives a negative answer to a question asked by Okonek [29].

Walter [36] had previously shown that a very large majority of subcanonical subschemes of codimension 3 in $\mathbb{P}^n$ are Pfaffian, but he left open the question whether the exceptional non-Pfaffian cases actually occur. We give non-Pfaffian examples of the principal types allowed by his theorem, including (Enriques) surfaces in $\mathbb{P}^5$ in characteristic 2 and a smooth 4-fold in $\mathbb{P}^7_{\mathbb{C}}$.

These examples are based on our previous work [14] showing that any strongly subcanonical subscheme of codimension 3 of a Noetherian scheme can be realized as a locus of degenerate intersection of a pair of Lagrangian (maximal isotropic) subbundles of a twisted orthogonal bundle.

There are many relations between the vector bundles $\mathcal{E}$ on a nonsingular algebraic variety X and the subvarieties $Z \subset X$. For example, a globalized form of the Hilbert-Burch theorem allows one to realize any codimension 2 locally Cohen-Macaulay subvariety as a degeneracy locus of a map of vector bundles. In addition the Serre construction gives a realization of any subcanonical codimension 2 subvariety $Z \subset X$ as the zero locus of a section of a rank 2 vector bundle on X.

The situation in codimension 3 is more complicated. In the local setting, Buchsbaum and Eisenbud [5] described the structure of the minimal free resolution of a Gorenstein (i.e. subcanonical) codimension 3 quotient ring of a regular local ring. Their construction can be globalized: If $\phi : \mathcal{E} \to \mathcal{E}^*(L)$ is an alternating map from a vector bundle $\mathcal{E}$ of **odd** rank $2n+1$ to a twist of its dual by a line bundle L, then the $2n \times 2n$ Pfaffians of ϕ define a degeneracy locus $Z = \{x \in X \mid \dim\ker(\phi(x)) \geq 3\}$. If X is nonsingular and $\operatorname{codim}(Z) = 3$, the largest possible value, (or more generally if X is locally Noetherian and $\operatorname{grade}(Z) = 3$), then, after twisting $\mathcal{E}$ and L appropriately, $\mathcal{O}_Z$ has a symmetric locally free resolution

$$0 \to L \xrightarrow{p^*} \mathcal{E} \xrightarrow{\phi} \mathcal{E}^*(L) \xrightarrow{p} \mathcal{O}_X \to \mathcal{O}_Z \to 0 \tag{0.1}$$

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with p locally the vector of submaximal Pfaffians of ϕ and with Z subcanonical with $\omega_Z \cong \omega_X(L^{-1})|_Z$. (The reader will find a general discussion of *subcanonical* subschemes in the introduction of our paper [14] and in Section 2 below; for the purpose of this introduction it may suffice to know that a codimension 3 subvariety Z of $\mathbb{P}^n$ is *subcanonical* if it is (locally) Gorenstein with canonical line bundle $\omega_Z = \mathcal{O}_Z(d)$.)

Okonek [29] called such a Z a *Pfaffian subvariety*, and he asked:

Question (Okonek). *Is every smooth subcanonical subvariety of codimension* 3 *in a smooth projective variety a Pfaffian subvariety*?

This paper is one of a series which we have written in response to Okonek's question. In the first paper (Walter [36]) one of us found necessary and sufficient numerical conditions for a codimension 3 subcanonical subscheme $X \subset \mathbb{P}^N_k$ to be Pfaffian, but was unable to give subcanonical schemes failing the numerical conditions. In the second paper [14] we found a construction for codimension 3 subcanonical subschemes which is more general than that studied by Okonek, and proved that this construction gives all subcanonical subschemes of codimension 3 satisfying a certain necessary lifting condition. (This lifting condition always holds if the ambient space is $\mathbb{P}^n$ or a Grassmannian of dimension at least 4.)

The construction is as follows: Let $\mathcal{E}, \mathcal{F}$ be a pair of Lagrangian subbundles of a twisted orthogonal bundle with $\dim_{k(x)}\bigl[\mathcal{E}(x) \cap \mathcal{F}(x)\bigr]$ always odd. There is a natural scheme structure on the degeneracy locus

$$Z := \{x \in X \mid \dim_{k(x)}\bigl[\mathcal{E}(x) \cap \mathcal{F}(x)\bigr] \geq 3\} \tag{0.2}$$

which is subcanonical if $\operatorname{grade}(Z) = 3$; the structure theorem asserts that every strongly subcanonical codimension 3 subscheme (in the sense of the definition in the introduction of our paper [14]) arises in this way.

Another more complicated but also more explicit description of the structure theorem may be given as follows (see our paper [14] for more details): If one has a subcanonical subvariety with $\omega_Z \cong L|_Z$, then the isomorphism $\mathcal{O}_Z \cong \omega_Z(L^{-1})|_Z \cong \mathcal{E}xt^3_{\mathcal{O}_X}(\mathcal{O}_Z, \omega_X(L^{-1}))$ is a sort of symmetry in the derived category. Let $L_1 := \omega_X(L^{-1})$. Then $\mathcal{O}_Z$ should have a locally free resolution which is symmetric in the derived category. This is the case if $\mathcal{O}_Z$ has a locally free resolution with a symmetric quasi-isomorphism into the twisted shifted dual complex:

$$\begin{tikzcd}
0 \arrow[r] & L_1 \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r, "\psi"] \arrow[d, "\phi"'] & \mathcal{G} \arrow[r] \arrow[d, "\phi^*"] & \mathcal{O}_X \arrow[r] \arrow[d, equal] & \mathcal{O}_Z \arrow[r] \arrow[d, "\cong"', "\eta"] & 0 \\
0 \arrow[r] & L_1 \arrow[r] & \mathcal{G}^*(L_1) \arrow[r, "-\psi^*"] & \mathcal{E}^*(L_1) \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{E}xt^3_{\mathcal{O}_X}(\mathcal{O}_Z, L_1) \arrow[r] & 0
\end{tikzcd} \tag{0.3}$$

If ϕ^* is the identity, then we get the Pfaffian resolution of (0.1). But in general it is not. However, the resolutions in (0.3) mean that Z can be interpreted as the degeneracy locus of (0.2) for the Lagrangian subbundles $\mathcal{E}, \mathcal{G}^*(L) \subset \mathcal{G} \oplus \mathcal{G}^*(L)$.

In this paper we give examples of non-Pfaffian codimension 3 subcanonical subvarieties. Thus we give an answer "No" to Okonek's question. Because of the numerical conditions of [36], our examples fall into four types:

1. Surfaces $S \subset \mathbb{P}^5_k$ with $\mathrm{char}(k) = 2$ and with $\omega_S \cong \mathcal{O}_S(2d)$ such that $h^1(\mathcal{O}_S(d))$ is odd.
2. Fourfolds $Y \subset \mathbb{P}^7$ with $\omega_Y \cong \mathcal{O}_Y(2d)$ such that $h^2(\mathcal{O}_Y(d))$ is odd.
3. Fourfolds $Z \subset \mathbb{P}^7_{\mathbb{R}}$ such that $\omega_Z \cong \mathcal{O}_Z(2d)$ such that the cup product pairing on $H^2(\mathcal{O}_Z(d))$ is positive definite (or otherwise not hyperbolic).
4. Examples in ambient varieties other than $\mathbb{P}^N$.

Our examples include among others: codimension 3 Schubert subvarieties of orthogonal Grassmannians, non-Pfaffian fourfolds of degree 336 in $\mathbb{P}^7$, and nonclassical Enriques surfaces in $\mathbb{P}^5$ in characteristic 2.

Structure of the paper. In §1 we review well known about quadratic forms, orthogonal Grassmannians and Pfaffian line bundles. In §2 and §3 we review the machinery and results of our previous papers [14] and [36]. The rest of the paper is devoted to the examples of non-Pfaffian subcanonical subschemes. The first example, in §4, is a codimension 3 Schubert variety in $\mathbb{OG}_{2n}$. In §5 we construct a smooth non-Pfaffian subcanonical fourfold of degree 336 in $\mathbb{P}^7$ with $K = 12H$. In §6 we construct some subcanonical fourfolds in $\mathbb{P}^7_{\mathbb{R}}$ which are not Pfaffian over $\mathbb{R}$ but become Pfaffian over $\mathbb{C}$. In §7 we analyze Reisner's example in characteristic 2 of the union of 10 coordinate 2-planes in $\mathbb{P}^5$.

In the last two sections §§8,9 we analyze nonclassical Enriques surfaces in characteristic 2 and their Fano models. As a consequence we obtain a (partial) moduli description of Fano-polarized unnodal nonclassical Enriques surfaces as a "quotient" of $\mathbb{OG}_{20}$ modulo SL_6 (Corollary 8.9). In §9 we identify the closure of the locus of $\boldsymbol{\alpha}_2$-surfaces within $\mathbb{OG}_{20}$ by calculating the action of Frobenius on our resolutions.

The philosophy that (skew)-symmetric sheaves should have locally free resolutions that are (skew)-symmetric up to quasi-isomorphism is also pursued in [15] and [37]. The former deals primarily with methods for constructing explicit locally free resolutions for (skew)-symmetric sheaves on $\mathbb{P}^n$. The latter investigates the obstructions (in Balmer's derived Witt groups [1]) for the existence of a genuinely (skew)-symmetric resolution.

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1. Orthogonal Grassmannians

In this section we bring together a number of facts about orthogonal Grassmannians which we will need throughout the paper. All the results are well known, so we omit proofs.

Quadratic forms [23] [31]. Let V be vector space of even dimension $2n$ over a field k. (We impose no restrictions on k; it need not be algebraically closed nor of characteristic $\neq 2$.) A quadratic form on V is a homogeneous quadratic polynomial in the linear forms on V, i.e. a member $q \in S^2(V^*)$. The symmetric bilinear form $b : V \times V \to k$ associated to q is given by the formula

$$b(x,y) := q(x+y) - q(x) - q(y). \tag{1.1}$$

The quadratic form q is *nondegenerate* if b is a perfect pairing.

A *Lagrangian subspace* $L \subset (V,q)$ is a subspace of dimension n such that $q|_L \equiv 0$. There exist such subspaces if and only if the quadratic form q is *hyperbolic*. By definition, this means that there exists a coordinate system on V in which one can write $q = \sum_{i=1}^n x_i x_{n+i}$. Lagrangian subspaces L satisfy $L = L^\perp$, and the converse is true if $\mathrm{char}(k) \neq 2$.

If X is an algebraic variety over the field k, then the quadratic form q makes $V \otimes \mathcal{O}_X$ into an orthogonal bundle. A *Lagrangian subbundle* $\mathcal{E} \subset V \otimes \mathcal{O}_X$ is a subbundle of rank n such that all fibers $\mathcal{E}(x) \subset V$ are Lagrangian with respect to q.

Orthogonal Grassmannians [27] [28] [32]. Let V be a vector space of even dimension $2n$ with the hyperbolic quadratic form $q = \sum_{i=1}^n x_i x_{n+i}$.

The Lagrangian subspaces of (V,q) form two disjoint families such that $\dim(L \cap W) \equiv n \pmod 2$ if L and W are Lagrangian subspaces in the same family, while $\dim(L \cap W) \equiv n+1 \pmod 2$ otherwise.

Each family of Lagrangian subspaces is parametrized by the k-rational points of an orthogonal Grassmannian (or spinor variety) $\mathbb{OG}_{2n}$ of dimension $n(n-1)/2$. Let U (resp. W) be a Lagrangian subspace such that $\dim(L \cap U)$ is even (resp. $\dim(L \cap W)$ is odd) for all L in the first family. Then $\mathbb{OG}_{2n}$ contains the following Schubert varieties:

$$\sigma_2(U) := \{L \mid \dim(L \cap U) \geq 2\}, \tag{1.2}$$

$$\sigma_3(W) := \{L \mid \dim(L \cap W) \geq 3\}, \tag{1.3}$$

$$\sigma_5(W) := \{L \mid \dim(L \cap W) \geq 5\}. \tag{1.4}$$

We have the following basic results about these subvarieties:

Lemma 1.1. (a) *The subvariety $\sigma_2(U) \subset \mathbb{OG}_{2n}$ is of codimension* 1, *and its complement* (*parametrizing Lagrangian subspaces complementary to U*) *is an affine space $\mathbb{A}_k^{n(n-1)/2}$.*

(b) *The subvarieties $\sigma_3(W)$ and $\sigma_5(W)$ are of codimensions* 3 *and* 10 *in $\mathbb{OG}_{2n}$, respectively. Moreover, $\sigma_3(W)$ is nonsingular outside $\sigma_5(W)$.*

The reason behind Lemma 1.1(a) is that if one fixes a Lagrangian subspace U^* complementary to U, then the Lagrangian subspaces complementary to U are the graphs of alternating maps $U \to U^*$. This fact also underlies the following lemma.

Lemma 1.2. *Suppose $L \subset (V, q)$ is a Lagrangian subspace. Then there is a natural identification $T_{[L]}(\mathbb{OG}_{2n}) \cong \Lambda^2 L^*$. Moreover, if $\dim(L \cap W) = 3$, then there is a natural identification*

$$T_{[L]}(\sigma_3(W)) \cong \{f \in \Lambda^2 L^* \mid f|_{L\cap W} \equiv 0\}.$$

Pfaffian line bundles [27] [32]. The Picard group of $\mathbb{OG}_{2n}$ is isomorphic to $\mathbb{Z}$. Its positive generator $\mathcal{O}(1)$ is the *universal Pfaffian line bundle*. The divisors $\sigma_2(U)$ of Lemma 1.1(a) are zero loci of particular sections of $\mathcal{O}(1)$. The canonical line bundle of $\mathbb{OG}_{2n}$ is $\mathcal{O}(2-2n)$.

On the orthogonal Grassmannian $\mathbb{OG}_{2n}$ there is a universal short exact sequence of vector bundles

$$0 \to \mathcal{S} \to V \otimes \mathcal{O}_{\mathbb{OG}_{2n}} \to \mathcal{S}^* \to 0 \tag{1.5}$$

with $\mathcal{S}$ the *universal Lagrangian subbundle* of rank n. The determinant line bundle related to the Plücker embedding is $\det(\mathcal{S}^*) = \mathcal{O}(2)$.

The universal Lagrangian subbundle has the following universal property: If X is an algebraic variety and $\mathcal{E} \subset V \otimes \mathcal{O}_X$ is a Lagrangian subbundle over X whose fibers lie in the family parametrized by $\mathbb{OG}_{2n}$, then there exists a unique map $f : X \to \mathbb{OG}_{2n}$ such that the natural exact sequence $0 \to \mathcal{E} \to V \otimes \mathcal{O}_X \to \mathcal{E}^* \to 0$ is the pullback along f of the universal exact sequence (1.5). The *Pfaffian line bundle* of $\mathcal{E}$ is then $\mathrm{Pf}(\mathcal{E}) := f^*\big(\mathcal{O}(-1)\big)$. It is a canonically defined line bundle on X such that $\mathrm{Pf}(\mathcal{E})^{\otimes 2} \cong \det(\mathcal{E})$.

2. The Lagrangian subbundle construction

Let X be a nonsingular algebraic variety over a field k, and let $Z \subset X$ be a codimension 3 subvariety. In [14] we called $Z \subset X$ *subcanonical* if the sheaf $\mathcal{O}_Z$ has local projective dimension over $\mathcal{O}_X$ equal to its grade (in this case 3), and the relative canonical sheaf $\omega_{Z/X} := \mathcal{E}xt^3_{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_X)$ is isomorphic to the restriction to Z of a line bundle L on X. We called Z *strongly subcanonical* if in addition the map

$$\mathrm{Ext}^3_{\mathcal{O}_X}(\mathcal{O}_Z, L^{-1}) \to \mathrm{Ext}^3_{\mathcal{O}_X}(\mathcal{O}_X, L^{-1}) = H^3(X, L^{-1}).$$

induced by the surjection $\mathcal{O}_X \to \mathcal{O}_Z$ is zero. The last condition is immediate if $X = \mathbb{P}^n$, for $n \geq 4$, as the right cohomology group is always zero. Thus all subcanonical subvarieties of $\mathbb{P}^n$ are strongly subcanonical.

The method we use to construct subcanonical subvarieties of codimension 3 was developed in [14]. In fact, all our examples can be constructed using the following result, which is a special case of [14] Theorem 3.1.

Theorem 2.1. *Let V be a vector space of even dimension $2n$ over a field k, and $q = \sum_{i=1}^n x_i x_{n+i}$ a hyperbolic quadratic form on V. Suppose that X is a* *nonsingular algebraic variety over k, and that $\mathcal{E} \subset V \otimes \mathcal{O}_X$ is a Lagrangian subbundle with Pfaffian line bundle $L := \mathrm{Pf}(\mathcal{E})$ satisfying $L^{\otimes 2} \cong \det(\mathcal{E})$. Let $W \subset (V, q)$ be a Lagrangian subspace such that $\dim_{k(x)} [\mathcal{E}(x) \cap W]$ is odd for all x. If W is sufficiently general, then*

$$Z_W = \{x \in X \mid \dim_{k(x)} (\mathcal{E}(x) \cap W) \geq 3\}, \tag{2.1}$$

is a (strongly) subcanonical subvariety of codimension 3 in X with

$$\omega_{Z_W} \cong (\omega_X \otimes \det(\mathcal{E}^*))\,|_{Z_W}$$

and with symmetrically quasi-isomorphic locally free resolutions

$$\begin{tikzcd} 0 \arrow[r] & L^{\otimes 2} \arrow[r] \arrow[d, equal] & \mathcal{E}(L) \arrow[r, "\psi"] \arrow[d, "\phi"'] & W^* \otimes \mathcal{O}_X(L) \arrow[r] \arrow[d, "\phi^*"] & \mathcal{O}_X \arrow[r, two heads] \arrow[d, equal] & \mathcal{O}_{Z_W} \arrow[d, "\cong"', "\eta"] \\ 0 \arrow[r] & L^{\otimes 2} \arrow[r] & W \otimes \mathcal{O}_X(L) \arrow[r, "-\psi^*"] & \mathcal{E}^*(L) \arrow[r] & \mathcal{O}_X \arrow[r, two heads] & \mathcal{E}xt^3_{\mathcal{O}_X}(\mathcal{O}_{Z_W}, L^{\otimes 2}) \end{tikzcd} \tag{2.2}$$

Moreover, if W is sufficiently general and $\mathrm{char}(k) = 0$, *then* $\mathrm{Sing}(Z_W)$ *is of codimension 10 in X.*

According to Fulton-Pragacz [16] (6.5), the fundamental class of Z_W in the Chow ring of X is $\frac{1}{4}(c_1c_2 - 2c_3)$, where $c_i := c_i(\mathcal{E}^*)$.

To check the smoothness of Z_W in characteristic 0 we proceed as follows. The universal property of the orthogonal Grassmannian gives us a morphism $f : X \to \mathbb{OG}_{2n}$. The group SO_{2n} acts on $\mathbb{OG}_{2n}$, and it translates the Schubert varieties into other Schubert varieties: $g \cdot \sigma_i(\Lambda) = \sigma_i(g\Lambda)$. Since the characteristic is 0, Kleiman's theorem on the transversality of the general translate [22] applies. So there exists a nonempty Zariski open subset $U \subset \mathrm{SO}_{2n}$ such that if $g \in U$, then $\sigma_5(g\Lambda)$ intersects $f(X)$ transversally in codimension 10, and $\sigma_3(g\Lambda)$ intersects $f(X)$ transversally in codimension 3. Moreover, since SO_{2n} is a rational variety and k is an infinite field, U contains k-rational points. So if $W := g\Lambda$ with g a k-rational point in U, then $W \subset (V, q)$ is a Lagrangian subspace defined over k such that $Z_W := f^{-1}(\sigma_3(W))$ is of codimension 3 and is smooth outside $f^{-1}(\sigma_5(W))$, which is of codimension 10 in X.

In characteristic p, it will be more complicated to prove that Z_W is smooth (see Lemma 5.2 below).

3. Pfaffian subschemes

Theorem 2.1 allows us to construct subcanonical subvarieties $Z \subset X$ of codimension 3 with resolutions which are not symmetric. Such a Z might nevertheless possess other locally free resolutions which are symmetric (i.e. Pfaffian). However, the third author gave in [36] a necessary and sufficient condition for a subcanonical subscheme of codimension 3 in projective space to be Pfaffian.

So suppose $Z \subset \mathbb{P}^{n+3}$ is a subcanonical subscheme of codimension 3 and dimension n such that $\omega_Z \cong \mathcal{O}_Z(\ell)$. If n and ℓ are both even, then Serre duality defines a nondegenerate bilinear form

$$H^{n/2}(\mathcal{O}_Z(\ell/2)) \times H^{n/2}(\mathcal{O}_Z(\ell/2)) \xrightarrow{\cup} H^n(\mathcal{O}_Z(\ell)) \cong k. \tag{3.1}$$

This bilinear form is symmetric if $n \equiv 0 \pmod 4$, and it is skew-symmetric if $n \equiv 2 \pmod 4$. In characteristic 2 it is symmetric in both cases. The result proven was the following.

Theorem 3.1 (Walter [36]). *Suppose that $Z \subset \mathbb{P}^{n+3}$ is a subcanonical subscheme of codimension 3 and dimension $n \geq 1$ over the field k such that $\omega_Z \cong \mathcal{O}_Z(\ell)$. Then Z is Pfaffian if and only at least one of the following conditions holds*: (i) *n or ℓ is odd,* (ii) *$n \equiv 2 \pmod 4$ and* $\mathrm{char}(k) \neq 2$, *or* (iii) *n and ℓ are even, and $H^{n/2}(\mathcal{O}_Z(\ell/2))$ is of even dimension and contains a subspace which is Lagrangian with respect to the cup product pairing of* (3.1).

Another way of stating the theorem is that a subcanonical subscheme $Z \subset \mathbb{P}^{n+3}$ of dimension n with $\omega_Z \cong \mathcal{O}_Z(\ell)$ is not Pfaffian if and only either

(a) $n \equiv 0 \pmod 4$, or else
(b) $n \equiv 2 \pmod 4$ and $\mathrm{char}(k) = 2$;

and at the same time either

(c) ℓ is even and $H^{n/2}(\mathcal{O}_X(\ell/2))$ is odd-dimensional, or else
(d) ℓ is even and $H^{n/2}(\mathcal{O}_X(\ell/2))$ is even-dimensional but has no subspace which is Lagrangian with respect to the cup product pairing of (3.1).

The simplest cases where non-Pfaffian subcanonical subschemes of dimension n in $\mathbb{P}^{n+3}_k$ could exist are therefore

1. Surfaces $S \subset \mathbb{P}^5_k$ with $\mathrm{char}(k) = 2$ and with $\omega_S \cong \mathcal{O}_S(2d)$ such that $h^1(\mathcal{O}_S(d))$ is odd.
2. Fourfolds $Y \subset \mathbb{P}^7$ with $\omega_Y \cong \mathcal{O}_Y(2d)$ such that $h^2(\mathcal{O}_Y(d))$ is odd.
3. Fourfolds $Z \subset \mathbb{P}^7_{\mathbb{R}}$ such that $\omega_Z \cong \mathcal{O}_Z(2d)$ such that the cup product pairing on $H^2(\mathcal{O}_Z(d))$ is positive definite (or otherwise not hyperbolic).

4. The codimension 3 Schubert variety in $\mathbb{OG}_{2n}$

Our first examples of non-Pfaffian subcanonical subschemes of codimension 3 are the codimension 3 Schubert varieties of $\mathbb{OG}_{2n}$.

Theorem 4.1. *Let $n \geq 4$, and let $\sigma_3(W) \subset \mathbb{OG}_{2n}$ be one of the codimension 3 Schubert varieties of Lemma* 1.1(b). *Then Z is* (*strongly*) *subcanonical with $\omega_Z \cong \mathcal{O}_Z(4-2n)$ but is not Pfaffian.*

Proof. We apply Theorem 2.1 with $X := \mathbb{OG}_{2n}$ and $\mathcal{E} := \mathcal{S}$ the universal Lagrangian subbundle. The degeneracy locus is $Z := \sigma_3(W)$ and it has resolutions

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-2) \arrow[r] \arrow[d, equal] & \mathcal{S}(-1) \arrow[r, "\psi"] \arrow[d, "\phi"] & W^* \otimes \mathcal{O}(-1) \arrow[r] \arrow[d, "\phi^*"] & \mathcal{O} \arrow[r, two heads] \arrow[d, equal] & \mathcal{O}_Z \arrow[d, "\cong"', "\eta"] \\
0 \arrow[r] & \mathcal{O}(-2) \arrow[r] & W \otimes \mathcal{O}(-1) \arrow[r, "-\psi^*"] & \mathcal{S}^*(-1) \arrow[r] & \mathcal{O} \arrow[r, two heads] & \mathcal{E}xt^3(\mathcal{O}_Z, \mathcal{O}(-2))
\end{tikzcd} \tag{4.1}$$

Moreover, $\omega_Z \cong \omega_{\mathbb{OG}_{2n}}(2)|_Z \cong \mathcal{O}_Z(4-2n)$ since $\omega_{\mathbb{OG}_{2n}} \cong \mathcal{O}(2-2n)$.

It remains to show that Z is not Pfaffian. Let U be a general totally isotropic subspace of dimension $n-3$, and let $i : \mathbb{OG}_6(U^\perp/U) \hookrightarrow \mathbb{OG}_{2n}$ be the natural embedding. If Z were Pfaffian, its symmetric resolution would be of the form

$$0 \to \mathcal{O}(-2) \to \mathcal{E}(-1) \to \mathcal{E}^*(-1) \to \mathcal{O} \to \mathcal{O}_Z \to 0 \tag{4.2}$$

(cf. (0.1) in the introduction). It would thus pull back to a symmetric resolution of $i^{-1}(Z)$ on $\mathbb{OG}_6(U^\perp/U)$ of the same form.

Now $\mathbb{OG}_6$ parametrizes one family of $\mathbb{P}^2$'s contained in a smooth hyperquadric in $\mathbb{P}^5$. Thus $\mathbb{OG}_6 \cong \mathbb{P}^3$ (see Jessop [21] or Griffiths-Harris [18]), and $i^{-1}(Z)$ is a codimension 3 Schubert subvariety of $\mathbb{OG}_6 \cong \mathbb{P}^3$ and indeed a point Q. If we pull the resolution (4.2) back to $\mathbb{P}^3$ and twist, we get a resolution

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3) \to i^*\mathcal{E}(-2) \to i^*\mathcal{E}^*(-2) \to \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_Q(-1) \to 0.$$

This gives $1 = \chi(\mathcal{O}_Q(-1)) = 2\chi(i^*\mathcal{E}(-2))$, which is a contradiction. This completes the proof. □

The contradiction obtained at the end of the proof above is of the same nature as the obstructions in Theorem 3.1 (see [36]). The contradiction can also be derived from [14] Theorem 7.2.

5. A non-Pfaffian subcanonical 4-fold in $\mathbb{P}^7$

In this section we use the construction of Theorem 2.1 to give an example of a smooth non-Pfaffian subcanonical 4-fold in $\mathbb{P}^7$ over an arbitrary infinite field k.

Let $V = H^0(\mathcal{O}_{\mathbb{P}^7}(1))^*$, so that $\mathbb{P}^7$ is the projective space of lines in V. We fix an identification $\Lambda^8 V \cong k$. The 70-dimensional vector space $\Lambda^4 V$ has the quadratic form $q(u) = u^{(2)}$, the divided square in the exterior algebra. (If $\mathrm{char}(k) \neq 2$, then $u^{(2)} = \frac{1}{2} u \wedge u$.) The associated symmetric bilinear form is $b(u,v) = u \wedge v$. This quadratic form is hyperbolic because it has Lagrangian subspaces, as we shall see in a moment.

We now look for a Lagrangian subbundle of $\Lambda^4 V \otimes \mathcal{O}_{\mathbb{P}^7}$. On $\mathbb{P}^7$ there is a canonical map $\mathcal{O}_{\mathbb{P}^7} \to V \otimes \mathcal{O}_{\mathbb{P}^7}(1)$. Using this, one constructs a Koszul complex of which one part is:

$$\cdots \to \Lambda^3 V \otimes \mathcal{O}_{\mathbb{P}^7}(-1) \xrightarrow{d} \Lambda^4 V \otimes \mathcal{O}_{\mathbb{P}^7} \to \Lambda^5 V \otimes \mathcal{O}_{\mathbb{P}^7}(1) \to \cdots \tag{5.1}$$

The image and cokernel of d fit into an exact sequence

$$0 \to \Omega^4_{\mathbb{P}^7}(4) \to \Lambda^4 V \otimes \mathcal{O}_{\mathbb{P}^7} \to \Omega^3_{\mathbb{P}^7}(4) \to 0. \tag{5.2}$$

The fiber of $\Omega^4_{\mathbb{P}^7}(4)$ over a point $\overline{\xi} \in \mathbb{P}^7$, viewed as a subspace of $\Lambda^4 V$, is the image over $\overline{\xi}$ of the map d of (5.1), namely $W_\xi := \xi \wedge \Lambda^3 V$. These spaces are all totally isotropic of dimension 35, half the dimension of $\Lambda^4 V$. Hence $\Omega^4_{\mathbb{P}^7}(4)$ is a Lagrangian subbundle of $\Lambda^4 V \otimes \mathcal{O}_{\mathbb{P}^7}$.

The fibers W_ξ of $\Omega^4_{\mathbb{P}^7}(4)$ are all members of the same family of Lagrangian subspaces of the hyperbolic quadratic space $\Lambda^4 V$. Let W be a general member of that family, and (to abuse notation) W^* a general member of the opposite family of Lagrangian subspaces of $\Lambda^4 V$. Then $\Lambda^4 V = W \oplus W^*$, and the symmetric bilinear form on $\Lambda^4 V$ induces a natural isomorphism between W^* and the dual of W. Finally, let

$$Z_W := \{\overline{\xi} \in \mathbb{P}^7 \mid \dim(W_\xi \cap W) \geq 3\}.$$

We now apply Theorem 2.1 and obtain the following result.

Theorem 5.1. *If the base field k is infinite, and if $W \subset \Lambda^4 V$ is a Lagrangian subspace in the same family as the W_ξ which is sufficiently general, then Z_W is a nonsingular subvariety of $\mathbb{P}^7$ of dimension 4 which is subcanonical but not Pfaffian. It is of degree 336 and has $\omega_{Z_W} \cong \mathcal{O}_{Z_W}(12)$, and its structural sheaf has locally free resolutions (where we write $\mathcal{O} := \mathcal{O}_{\mathbb{P}^7}$)*

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-20) \arrow[r] \arrow[d, equal] & \Omega^4_{\mathbb{P}^7}(-6) \arrow[r, "\psi"] \arrow[d, "\phi"'] & W^* \otimes \mathcal{O}(-10) \arrow[r] \arrow[d, "\phi^*"] & \mathcal{O} \arrow[r, two heads] \arrow[d, equal] & \mathcal{O}_{Z_W} \arrow[d, "\cong"', "\eta"] \\
0 \arrow[r] & \mathcal{O}(-20) \arrow[r] & W \otimes \mathcal{O}(-10) \arrow[r, "-\psi^*"] & \Omega^3_{\mathbb{P}^7}(-6) \arrow[r] & \mathcal{O} \arrow[r, two heads] & \omega_{Z_W}(-12)
\end{tikzcd} \tag{5.3}$$

Proof. The existence of Z_W and the resolutions follow from Theorem 2.1, using the fact that $\det(\Omega^4_{\mathbb{P}^7}(4)) = \mathcal{O}_{\mathbb{P}^7}(-20)$. The subcanonical subvariety Z_W is not Pfaffian because $\omega_Z = \mathcal{O}_Z(12)$ and $h^2(\mathcal{O}_Z(6)) = 1$ (as one sees from (5.3)). Using the resolutions and a computer algebra package, one computes the Hilbert polynomial of Z_W as:

$$336\binom{n+3}{4} - 2520\binom{n+2}{3} + 9814\binom{n+1}{2} - 25571n + 49549$$

So the degree of Z_W is 336. In characteristic 0 the smoothness of the general Z_W follows from Kleiman's transversality theorem as explained after Theorem 2.1. In characteristic p the smoothness follows from Lemma 5.2 below. □

The free resolution of R/I_{Z_W} is of the form

$$\begin{aligned} 0 \to R(-14) \to R(-13)^8 \to R(-12)^{28} \oplus R(-20) \\ \to R(-11)^{56} \to R(-10)^{35} \to R \to R/I_{Z_W} \to 0. \end{aligned} \tag{5.4}$$

Lemma 5.2. *In Theorem 5.1, if k is an infinite field and W is general, then Z_W is smooth.*

Proof. Let

$$\begin{aligned} \Upsilon := \{W \in \mathbb{OG}_{70} \mid \mathbb{P}^7 \cap \sigma_5(W) = \varnothing, \\ \text{and } \mathbb{P}^7 \cap \sigma_3(W) = Z_W \text{ is nonsingular of dimension } 4\} \end{aligned}$$

We will show that Υ is nonempty without invoking Kleiman's theorem on the transversality of a general translate. We will do this by showing that the complement has dimension at most $\dim \mathbb{OG}_{70} - 1$.

Let $n := 35$, and let $\mathrm{Gr}(3, 2n)$ be the Grassmannian parametrizing three-dimensional subspaces of $\Lambda^4 V$. Let

$$Y := \{(\xi, F, W) \in \mathbb{P}^7 \times \mathrm{Gr}(3, 2n) \times \mathbb{OG}_{2n} \mid F \subset W_\xi \cap W\}.$$

To choose a point of Y, one first chooses $\xi \in \mathbb{P}^7$, then F in a fiber of a Grassmannian bundle with fiber $\mathrm{Gr}(3, n)$, then W in a fiber of an orthogonal Grassmannian bundle with fiber $\mathbb{OG}_{2(n-3)}$. We see that Y is nonsingular with $\dim(Y) = \dim \mathbb{OG}_{2n} + 4$. A similar computation shows that

$$Y_1 := \{(\xi, F, W) \in Y \mid \dim(W_\xi \cap W) \geq 5\}$$

is of dimension equal to $\dim \mathbb{OG}_{2n} - 3$. So the image of Y_1 under the projection $\pi : Y \to \mathbb{OG}_{2n}$ is not dominant. Over a point $W \notin \pi(Y_1)$, one has $\pi^{-1}(W) = Z_W$. Hence it is enough to show that the dimension of

$$Y_2 := \{(\xi, F, W) \in Y \mid d\pi : T_{(\xi,F,W)} Y \to T_W \,\mathbb{OG}_{2n} \text{ is not surjective}\}$$

is at most $\dim \mathbb{OG}_{2n} - 1$. The essential question is: if $(\xi, F, W) \in Y - Y_1$, so that $\pi^{-1}(W) = Z_W$, when is the dimension of $T_\xi Z_W = \ker(d\pi)_{(\xi,F,W)}$ equal to 4, and when is it more?

Now $T_\xi Z_W = T_{W_\xi}(\mathbb{P}^7 \cap \sigma_3(W))$. In order to identify this space we must identify $T_{W_\xi}(\mathbb{OG}_{2n})$ and the two subspaces $T_{W_\xi}\mathbb{P}^7$ and $T_{W_\xi}(\sigma_3(W))$.

There is a well known isomorphism $T_{W_\xi}(\mathbb{OG}_{2n}) \cong \Lambda^2 W_\xi^*$. Essentially, given a first-order deformation of W_ξ as a Lagrangian subspace of $\Lambda^4 V$, any vector $w \in W_\xi$ deforms within the deforming subspace as a $w + \varepsilon u(w)$ with $u(w)$ well-defined modulo W_ξ. This gives a map $W_\xi \to \Lambda^4 V / W_\xi \cong W_\xi^*$. The vector $w + \varepsilon u(w)$ remains isotropic if and only if u is alternating. To describe u as an alternating bilinear form on W_ξ, one looks at

$$\langle w_1, w_2 + \varepsilon u(w_2)\rangle = \varepsilon \langle w_1, u(w_2)\rangle.$$

If $W \cap W_\xi$ is a space F of dimension 3, then $T_{W_\xi}(\sigma_3(W)) = \{u \in \Lambda^2 W_\xi^* \mid u|_{F\times F} \equiv 0\}$ according to Lemma 1.2. I.e. $T_{W_\xi}(\sigma_3(W))$ is the kernel of the natural map $\Lambda^2 W_\xi^* \to \Lambda^2 F^*$.

Now let $V_\xi := V/\langle \xi \rangle$. Then there is a natural isomorphism $T_\xi \mathbb{P}^7 \cong V_\xi$. There is also a natural isomorphism $W_\xi = \xi \wedge \Lambda^3 V \cong \Lambda^3 V_\xi$. A vector $\theta \in T_\xi \mathbb{P}^7 = V_\xi$ corresponds to a first-order infinitesimal deformation $\xi + \varepsilon\theta$ of ξ, and hence to the first-order infinitesimal deformation $(\xi + \varepsilon\theta) \wedge \Lambda^3 V$ of W_ξ. The corresponding alternating bilinear form on $W_\xi = \xi \wedge \Lambda^3 V$ is computed by

$$\langle \xi \wedge \alpha, (\xi + \varepsilon\theta) \wedge \beta \rangle = \varepsilon(\xi \wedge \alpha \wedge \theta \wedge \beta) \in \varepsilon(\Lambda^8 V).$$

If we use the isomorphism $W_\xi \cong \Lambda^3 V_\xi$, then the corresponding alternating bilinear form on $\Lambda^3 V_\xi$ is therefore

$$g_\theta(\alpha, \beta) = -\alpha \wedge \beta \wedge \theta \in \Lambda^7 V_\xi.$$

Hence if we think of $F := W \cap W_\xi$ as a three-dimensional subspace of $\Lambda^3 V_\xi \cong W_\xi$, then $T_{W_\xi}(\mathbb{P}^7 \cap \sigma_3(W)) \subset V_\xi$ is the orthogonal complement of the image of $\Lambda^2 F$ in $\Lambda^6 V_\xi \cong V_\xi^*$. As a result $(\xi, F, W) \in Y_2$ if and only if the map $\Lambda^2 F \to \Lambda^6 V_\xi$ is not injective.

Every vector in $\Lambda^2 F$ is of the form $\alpha \wedge \beta$. We therefore look at the set

$$T_1(\xi) := \{(\alpha, \beta) \in (\Lambda^3 V_\xi - \{0\})^2 \mid \alpha \wedge \beta = 0\}.$$

We stratify this locus according to the rank of the map $m_\alpha : V_\xi \to \Lambda^4 V_\xi$ defined by multiplication by α. Generically this map is injective, and the dual multiplication map $m_\alpha^* : \Lambda^3 V_\xi \to \Lambda^6 V_\xi$ is surjective. For (α, β) to be in $T_1(\xi)$, β must be in the codimension-7 subspace $\ker(m_\alpha^*) \subset \Lambda^3 V_\xi$. Hence $T_1(\xi)$ contains a stratum of dimension $2n - 7$ where m_α is injective.

The locus $T_1(\xi)$ contains two other strata, but they are of smaller dimension. In one of them the α have the form $\alpha' \wedge \alpha''$ with $\alpha' \in V_\xi$ and $\alpha'' \in \Lambda^2 V_\xi$ indecomposable. Since α' is really only well-defined up to a scalar multiple, and α'' is really well-defined only in $\Lambda^2(V_\xi/\langle\alpha'\rangle)$, these form a locus of dimension $21 = n - 14$ in $\Lambda^3 V$. For these α the rank of m_α is 6, so this stratum in $T_1(\xi)$ has dimension $2n - 20$.

In the final stratum the α are of the form $\alpha_1 \wedge \alpha_2 \wedge \alpha_3$, a locus of dimension $13 = n - 22$. For these α the rank of m_α is 4, so this stratum of $T_1(\xi)$ has dimension $2n - 26$.

Thus $\dim T_1(\xi) = 2n - 7$. As a result the locus

$$T_2(\xi) := \{F_0 \in \mathrm{Gr}(2, W_\xi) \mid F_0 = \mathrm{Span}(\alpha, \beta) \text{ with } \alpha \wedge \beta = 0\}$$

is of dimension $2n - 11$. Consequently the locus

$$T_3(\xi) := \{F \in \mathrm{Gr}(3, W_\xi) \mid F \supset F_0 \text{ for some } F_0 \text{ in } T_2(\xi)\}$$

is of dimension at most $\dim T_2(\xi) + n - 3 = 3n - 14$.

For each $F \in T_3(\xi)$, to give a $W \in \mathbb{OG}_{2n}$ such that $F \subset W \cap W_\xi$ is equivalent to giving a member of one of the families of Lagrangian subspaces of $F^\perp/F$. Consequently, such W form a family of dimension $(n-3)(n-4)/2$. As a result, the locus

$$T_4(\xi) := \{(F, W) \in \mathrm{Gr}(3, W_\xi) \times \mathbb{OG}_{2n} \mid F \subset W \cap W_\xi \text{ and } F \in T_3(\xi)\}$$

is of dimension at most $(3n-14) + (n-3)(n-4)/2 = n(n-1)/2 - 8$. Finally

$$T_5 := \{(\xi, F, W) \mid (F, W) \in T_4(\xi)\}$$

is of dimension at most $n(n-1)/2 - 1$. But the part of T_5 where $F = W \cap W_\xi$ is exactly the part of Y_2 lying in $Y - Y_1$. So $\dim Y_2 < n(n-1)/2 = \dim(\mathbb{OG}_{2n})$.

As a result Y_2 does not dominate $\mathbb{OG}_{2n}$, and $\Upsilon \subset \mathbb{OG}_{2n}$ is a nonempty open subset. As before, Υ must then contain a rational point because the base field is infinite, and we conclude. $\square$

6. Extension of scalars and non-Pfaffian examples

In this section we prove the following extension of Theorem 5.1. It will allow us to construct subcanonical subschemes of codimension 3 in $\mathbb{P}_k^N$ which are not Pfaffian over k but are Pfaffian over a finite extension of k.

Theorem 6.1. *Let k be an infinite field, let E be a k-vector space of dimension $d \geq 1$, and let $b : E \times E \to k$ be a nondegenerate symmetric bilinear form. There exists a nonsingular subcanonical fourfold $X \subset \mathbb{P}^7_k$ of degree $(1000d^3 + 8d)/3$ with $K_X = (20d-8)H$ such that the cup product pairing (3.1) on the middle cohomology*

$$H^2(\mathcal{O}_X(10d-4)) \times H^2(\mathcal{O}_X(10d-4)) \to H^4(\mathcal{O}_X(20d-8)) \cong k$$

may be identified with $b : E \times E \to k$.

Proof. If $e_1, \ldots, e_d$ is a basis of E and $b_{ij} := b(e_i, e_j)$, then the quadratic form on $E \otimes \Lambda^4 V$ given by

$$Q\left(\sum_i e_i \otimes u_i\right) := \sum_i b_{ii}\, u_i^{(2)} + \sum_{i<j} b_{ij}\, u_i \wedge u_j$$

is nondegenerate and has the property that if L is a Lagrangian subspace of $(\Lambda^4 V, q)$, then $E \otimes L$ is a Lagrangian subspace of $(E \otimes \Lambda^4 V, Q)$. Consequently $E \otimes \Omega^4_{\mathbb{P}^7}(4)$ is a Lagrangian subbundle of $(E \otimes \Lambda^4 V \otimes \mathcal{O}_{\mathbb{P}^7}, Q)$. Also, if we make a standard identification $\Lambda^4 V^* \cong \Lambda^4 V$, then the symmetric bilinear form associated to Q is $b \otimes 1 : E \otimes \Lambda^4 V \to E^* \otimes \Lambda^4 V$.

We now choose a general Lagrangian subspace $W \subset (E \otimes \Lambda^4 V, Q)$ and apply the construction of Theorem 2.1. This produces a subcanonical fourfold $X \subset \mathbb{P}^7_k$ with $K_X = (20d-8)H$. The degree of X can be computed from the formula after Theorem 2.1. If $\operatorname{char}(k) = 0$, then X is nonsingular by Theorem 2.1. If k is infinite of positive characteristic, X is nonsingular by an argument similar to Lemma 5.2 whose details we leave to the reader.

Let $N := 10d - 4$. The isomorphism $\eta : \mathcal{O}_X(N) \to \omega_X(-N)$ induces an isomorphism

$$H^2(\mathcal{O}_X(N)) \xrightarrow{\sim} H^2(\omega_X(-N)) \cong H^2(\mathcal{O}_X(N))^* \tag{6.1}$$

which corresponds to the cup product pairing of Serre duality (3.1). Our problem is to identify this map.

Now twisting the diagram (2.2) of Theorem 2.1 produces a commutative diagram with exact rows (where we write $\mathcal{O} := \mathcal{O}_{\mathbb{P}^7}$ to keep things within the margins):

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-N-8) \arrow[r, "f"] \arrow[d, equal] & E \otimes \Omega^4_{\mathbb{P}^7} \arrow[r, "\psi"] \arrow[d, "\phi"] & W^* \otimes \mathcal{O}(-4) \arrow[r, "g"] \arrow[d, "\phi^*"] & \mathcal{O}(N) \arrow[r, two heads] \arrow[d, equal] & \mathcal{O}_X(N) \arrow[d, equal] \\
0 \arrow[r] & \mathcal{O}(-N-8) \arrow[r, "g^*"] & W \otimes \mathcal{O}(-4) \arrow[r, "-\psi^*"] & E^* \otimes \Omega^3_{\mathbb{P}^7} \arrow[r, "f^*"] & \mathcal{O}(N) \arrow[r, two heads] & \omega_X(-N)
\end{tikzcd} \tag{6.2}$$

The rows of the diagram produce natural isomorphisms $H^2(\mathcal{O}_X(N)) \cong E$ and $H^2(\omega_X(-N)) \cong E^*$. Moreover, the Serre duality pairing

$$H^2(\mathcal{O}_X(N)) \times H^2(\omega_X(-N)) \to H^4(\omega_X) \xrightarrow{\text{tr}} k$$

corresponds to the canonical pairing $E \times E^* \to k$ since the bottom resolution is dual to the top resolution, so the hypercohomology of the bottom resolution is naturally Serre dual to the hypercohomology of the top resolution.

It remains to identify the map $H^2(\mathcal{O}_X(N)) \to H^2(\omega_X(-N))$ with the map $b : E \to E^*$.

We do this by using the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-N-8) \arrow[r, "f"] \arrow[d, equal] & E \otimes \Omega^4_{\mathbb{P}^7} \arrow[r, "\psi"] \arrow[d, "\phi"'] & W^* \otimes \mathcal{O}(-4) \arrow[r, "g"] \arrow[d, "\alpha"] & \mathcal{O}(N) \arrow[r, two heads] \arrow[d, equal] & \mathcal{O}_X(N) \arrow[d, equal] \\
0 \arrow[r] & \mathcal{O}(-N-8) \arrow[r] \arrow[d, equal] & W \otimes \mathcal{O}(-4) \arrow[r, "-\beta"] \arrow[d, equal] & E \otimes \Omega^3_{\mathbb{P}^7} \arrow[r] \arrow[d, "b\otimes 1"] & \mathcal{O}(N) \arrow[r, two heads] \arrow[d, equal] & \mathcal{O}_X(N) \arrow[d, "\eta", "\cong"'] \\
0 \arrow[r] & \mathcal{O}(-N-8) \arrow[r, "g^*"] & W \otimes \mathcal{O}(-4) \arrow[r, "-\psi^*"] & E^* \otimes \Omega^3_{\mathbb{P}^7} \arrow[r, "f^*"] & \mathcal{O}(N) \arrow[r, two heads] & \omega_X(-N)
\end{tikzcd} \tag{6.3}$$

The maps α and β in this diagram come from the identification $E \otimes \Lambda^4 V \cong W^* \oplus W$, which yields a short exact sequence

$$0 \to E \otimes \Omega^4_{\mathbb{P}^7} \xrightarrow{\binom{\psi}{\phi}} (W^* \otimes \mathcal{O}(-4)) \oplus (W \otimes \mathcal{O}(-4)) \xrightarrow{(\alpha\ \beta)} E \otimes \Omega^3_{\mathbb{P}^7} \to 0. \tag{6.4}$$

The exactness of (6.4) implies that the top middle square of diagram (6.3) commutes, and that ϕ induces an isomorphism $\ker(\psi) \xrightarrow{\sim} \ker(\beta)$, while α induces an isomorphism $\operatorname{coker}(\psi) \xrightarrow{\sim} \operatorname{coker}(\beta)$. As a result the second row of the diagram is exact, and the squares in the top row are all commutative.

To see that the squares in the bottom row are all commutative, recall how the maps α, β, ϕ^* and ψ^* are defined. The inclusions of W^* and W in $W^* \oplus W \cong E \otimes \Lambda^4 V$, together with the bilinear form and the Koszul complex, give rise to a diagram with a commutative square

$$\begin{tikzcd}
W \otimes \mathcal{O}(-4) \arrow[r, hook, "i"] & E \otimes \Lambda^4 V \otimes \mathcal{O}(-4) \arrow[r, "b\otimes\mathbf{1}"] \arrow[d, "\pi"] & E^* \otimes \Lambda^4 V \otimes \mathcal{O}(-4) \arrow[d, "\pi'"] \\
W^* \otimes \mathcal{O}(-4) \arrow[ur, hook, "i'"'] & E \otimes \Omega^3_{\mathbb{P}^7} \arrow[r, "b\otimes 1_\Omega"] & E^* \otimes \Omega^3_{\mathbb{P}^7}
\end{tikzcd} \tag{6.5}$$

such that

$$\alpha = \pi i', \qquad \beta = \pi i, \qquad \psi^* = \pi'(b \otimes \mathbf{1}) i, \qquad \phi^* = \pi'(b \otimes \mathbf{1}) i'.$$

Since the square in (6.5) is commutative, we get $\psi^* = (b \otimes 1_\Omega)\beta$ and $\phi^* = (b \otimes 1_\Omega)\alpha$. The first equality means that the bottom set of squares in diagram (6.3) commutes. The second equality means that the composite of the two chain maps of diagram (6.3) is the chain map of diagram (6.2).

Now the first and second rows of diagram (6.3) produce compatible identifications $H^2(\mathcal{O}_X) \cong E$. And the second and third rows of diagram (6.3) show that the map $H^2(\mathcal{O}_X(N)) \to H^2(\omega_X(-N))$ induced by η may be identified with $b : E \to E^*$. This completes the proof of the theorem. ☐

As an example, we may apply Theorem 6.1 with b a positive definite symmetric bilinear form on $\mathbb{R}^2$. There are no Lagrangian subspaces of $(\mathbb{R}^2, b)$, but there are Lagrangian subspaces of $(\mathbb{C}^2, b_{\mathbb{C}})$. So by the Pfaffianness criterion of Theorem 3.1, we get the following corollary.

Corollary 6.2. *There exists a nonsingular subcanonical fourfold $Y_{\mathbb{R}} \subset \mathbb{P}^7_{\mathbb{R}}$ of degree 2672 with $K_Y = 32H$ which is not Pfaffian over $\mathbb{R}$ but whose complexification $Y_{\mathbb{C}} \subset \mathbb{P}^7_{\mathbb{C}}$ is Pfaffian over $\mathbb{C}$.*

7. Reisner's example in $\mathbb{P}^5$

Another example of a codimension 3 subcanonical subscheme $Z \subset \mathbb{P}^5$ is provided by the Stanley-Reisner ring of the minimal triangulation of the real projective plane $\mathbb{RP}^2$ in characteristic 2. This is Reisner's original example of a monomial ideal whose minimal free resolution depends on the characteristic of the base field. See Hochster [20], Reisner [33], Stanley [34].

Recall that if Δ is a simplicial complex on the vertex set $E = \{e_0, \dots e_n\}$, the corresponding face variety $X(\Delta) \subset \mathbb{P}^n_k$ in the sense of Stanley, Hochster and Reisner (see [34] for more details) is the locus defined by the ideal I_Δ generated by monomials corresponding to the simplexes (faces) not contained in Δ. Geometrically, $X(\Delta)$ is a "projective linear realization" of Δ. It is the union of linear subspaces of $\mathbb{P}^n_k$ corresponding to the simplices in Δ, where for each m-simplex $\{e_{i_0}, \dots, e_{i_m}\}$ one includes the m-plane spanned by the vertices $P_{i_0}, \dots, P_{i_m}$ of the standard coordinate system on $\mathbb{P}^n_k$.

If the topological realization $|\Delta|$ is a manifold, then $X(\Delta)$ is a locally Gorenstein scheme with $\omega^{\otimes 2}_{X(\Delta)} \cong \mathcal{O}_{X(\Delta)}$. Moreover, $\omega_{X(\Delta)}$ is trivial if and only if $|\Delta|$ is orientable over the field k. The cohomology of $X(\Delta)$ is given by the formula $H^i(\mathcal{O}_{X(\Delta)}) = H^i(\Delta, k)$. The Hilbert polynomial of $X(\Delta)$, which coincides with the Hilbert function for strictly positive values, is completely determined by the combinatorial data (see [34] for more details).

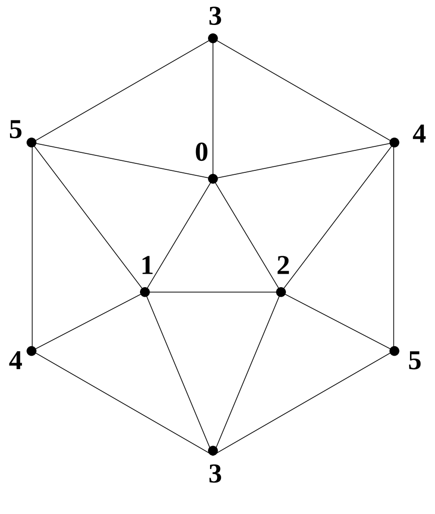

Figure 1. The Minimal Triangulation of $\mathbb{RP}^2$

We apply this theory with Δ the triangulation of $\mathbb{RP}^2$ given in Figure 1, and with k a field of characteristic 2. Then $X := X(\Delta)$ is a union of ten

2-planes in $\mathbb{P}^5_k$ corresponding to the ten triangles. Moreover, $\omega_X \cong \mathcal{O}_X$ since $\mathbb{RP}^2$ is orientable in characteristic 2. The cohomology is given by

$$h^0(\mathcal{O}_X) = 1, \qquad h^1(\mathcal{O}_X) = 1, \qquad h^2(\mathcal{O}_X) = 1,$$

since this is the cohomology of $\mathbb{RP}^2$ over a field of characteristic 2. We may now prove

Proposition 7.1 (char 2). *Let $X \subset \mathbb{P}^5$ be the face variety corresponding to the triangulation of $\mathbb{RP}^2$ given in Figure* 1. *Then X is subcanonical of codimension* 3 *but not Pfaffian.*

Proof. The subvariety X is a locally Gorenstein surface with $\omega_X = \mathcal{O}_X$ but $h^1(\mathcal{O}_X) = 1$. Thus it is subcanonical but not Pfaffian by Theorem 3.1. □

In addition, $X \subset \mathbb{P}^5$ is linearly normal (all $X(\Delta)$ are) and is not contained in a hyperquadric (since all pairs of vertices are joined by an edge in Δ). This will allow us to conclude below that X is a degenerate member of a family of nonclassical Enriques surfaces (see Proposition 8.7 and (8.5)). Over the complex numbers this is a Type III degeneration in Morrison [26] Corollary 6.2 (iii a). See also Symms [35] and Altmann-Christophersen (in preparation) for the deformation theory of this scheme.

8. Enriques surfaces in $\mathbb{P}^5$

In their extension to characteristic p of the Enriques-Castelnuovo-Kodaira classification of surfaces not of general type, Bombieri and Mumford [4] characterize Enriques surfaces as nonsingular projective surfaces X with numerically trivial canonical class $K_X \equiv 0$ and with $\chi(\mathcal{O}_X) = 1$. In characteristic 2, Enriques surfaces are divided into three types:

	$h^1(\mathcal{O}_X)$	$h^2(\mathcal{O}_X)$	canonical class	$F : H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_X)$
classical	0	0	$2K_X = 0$, $K_X \neq 0$	—
$\boldsymbol{\mu}_2$	1	1	$K_X = 0$	injective
$\boldsymbol{\alpha}_2$	1	1	$K_X = 0$	0

The map $F : H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_X)$ is the action of Frobenius on the cohomology. There is a voluminous literature on Enriques surfaces which we do not cite here. Our Pfaffianness criterion Theorem 3.1 yields immediately

Proposition 8.1 (char 2). *If $X \subset \mathbb{P}^5$ is a nonclassical Enriques surface, then X is subcanonical but not Pfaffian.*

A *Fano polarization* on an Enriques surface X is a numerically effective divisor H such that $H^2 = 10$ and such that $H \cdot F \geq 3$ for every numerically effective divisor F with $F^2 = 0$. It is well known that every Enriques surface admits a Fano polarization, and that for any Fano polarization H the linear system $|H|$ is base-point-free and defines a map $\varphi_H : X \to \mathbb{P}^5$ which is birational onto its image. The image $\varphi_H(X) \subset \mathbb{P}^5$ is called a *Fano model* of X. A surface $Y \subset \mathbb{P}^5$ is a Fano model of an Enriques surface if and only if it is of degree 10 with at worst rational double points as singularities and has $\chi(\mathcal{O}_Y) = 1$ and numerically trivial canonical class $K_Y \equiv 0$.

An Enriques surface is called *nodal* if it contains a smooth rational curve C with $C^2 = -2$. Fano models of unnodal Enriques surfaces are necessarily smooth. The following theorem was proven in characteristic $\neq 2$ by Cossec [8]. Igor Dolgachev has informed us that the theorem also holds in characteristic 2 using arguments similar to Dolgachev-Reider [13].

Theorem 8.2. *Let H be a Fano polarization of an Enriques surface X. Then the following are equivalent*: (a) *X is nodal, and* (b) *one of the Fano models $\varphi_H(X)$ or $\varphi_{H+K_X}(X)$ is contained in a hyperquadric.*

We will now give an algebraic method for realizing Fano models of unnodal nonclassical Enriques surfaces as degeneracy loci associated to Lagrangian subbundles of an orthogonal bundle on $\mathbb{P}^5$. We will need the following technical lemmas about divided squares. Both lemmas hold for even m in all characteristics.

Lemma 8.3. *Suppose that R is a commutative algebra over a field of characteristic* 2*, that $m \geq 3$ is an odd integer, and that E a free R-module of rank $2m$. Then there exists a well-defined divided square operation $\Lambda^m E \to \Lambda^{2m} E \cong R$ whose formula, with respect to any basis $e_1, \dots, e_m$, is given by*

$$\Big(\sum x_{i_1 \dots i_m} e_{i_1 \dots i_m}\Big)^{(2)} := \sum_{\substack{i_1 < \cdots < i_m \\ j_1 < \cdots < j_m \\ i_1 < j_1 \\ \{i_k\} \cap \{j_\ell\} = \varnothing}} x_{i_1 \cdots i_m} x_{j_1 \cdots j_m} e. \tag{8.1}$$

Proof. It is enough to prove the lemma for R a field, since one can reduce to the case where the coefficients $x_{i_1 \dots i_m}$ are independent indeterminates, and R is the field of rational functions in these indeterminates.

We have to show that (8.1) is invariant under changes of basis in $\mathrm{GL}(E) = \mathrm{GL}_n(R)$. But since R is now assumed to be a field, $\mathrm{GL}_n(R)$ is generated by three special kinds of changes of basis: permutations of the basis elements, operations of the form $e_i \rightsquigarrow \lambda e_i$, and operations of the form $e_j \rightsquigarrow e_j + \alpha e_i$. The formula (8.1) is almost trivially invariant under the first two kinds of operations, and it is invariant under the third kind of operation because the portions

$$x_{ijk_1 \dots k_{m-2}} x_{\ell_1 \dots \ell_m} \qquad \text{and} \qquad x_{ik_1 \dots k_{m-1}} x_{j\ell_1 \dots \ell_{m-1}} + x_{i\ell_1 \dots \ell_{m-1}} x_{jk_1 \dots k_{m-1}}$$

do not change. ∎

Lemma 8.4. *Suppose that $m \geq 3$ is an odd integer, that V is a vector space of dimension $2m$ over a field of characteristic* 2*, and that Q is a nonzero quadratic form on $\Lambda^m V$. Then $Q(x \wedge w) = 0$ for all $x \in V$ and $w \in \Lambda^{m-1} V$ if and only if Q is a constant multiple of the divided-square quadratic form of Lemma* 8.3.

Proof. Suppose first that Q is a multiple of the divided-square quadratic form, and that $0 \neq x \in V$ and $w \in \Lambda^{m-1}V$. Choosing a basis of V of the form $e_1 = x, e_2, \ldots, e_m$ and applying formula (8.1), one sees easily that $Q(x \wedge w) = 0$.

Conversely, suppose that $Q(x \wedge w) = 0$ for all $x \in V$ and $w \in \Lambda^{m-1}V$. Write $Q(\sum y_I e_I) := \sum_{I \preceq J} \alpha_{IJ} y_I y_J$. The hypothesis implies that $Q(e_I) = \alpha_{II} = 0$ for all I. It also implies that each $Q(e_{iA} + e_{iB}) = \alpha_{iA,iB} = 0$. Hence $\alpha_{IJ} = 0$ if the multi-indices I and J are not disjoint. Finally $Q((e_i + e_j) \wedge (e_A + e_B)) = \alpha_{iA,jB} + \alpha_{iB,jA} = 0$. In other words $\alpha_{IJ} = \alpha_{KL}$ if I and J are disjoint, and K and L are disjoint, and I and K contain exactly $m-1$ common indices. One then easily deduces that all the α_{IJ} with I and J disjoint must be equal. But then Q is the same as the quadratic form of (8.1) up to a constant. □

Now let $V = H^0(\mathcal{O}_{\mathbb{P}^5}(1))^*$, and consider the exact sequence of vector bundles on $\mathbb{P}^5$.

$$0 \to \Omega^3_{\mathbb{P}^5}(3) \to \Lambda^3 V \otimes \mathcal{O}_{\mathbb{P}^5} \to \Omega^2_{\mathbb{P}^5}(3) \to 0. \tag{8.2}$$

As in (5.2), if $\xi \in V$, then the fiber of $\Omega^3_{\mathbb{P}^5}(3)$ over $\overline{\xi} \in \mathbb{P}^5 = \mathbb{P}(V)$ may be identified with the subspace $\xi \wedge \Lambda^2 V \subset \Lambda^3 V$. Therefore Lemmas 8.3 and 8.4 have the following corollary.

Corollary 8.5 (char 2)**.** *There exists a nondegenerate quadratic form Q on $\Lambda^3 V$, unique up to a constant multiple, such that $\Omega^3_{\mathbb{P}^5}(3)$ is a Lagrangian subbundle of the orthogonal bundle $(\Lambda^3 V \otimes \mathcal{O}_{\mathbb{P}^5}, Q)$.*

We now apply the construction of Theorem 2.1 to the Lagrangian subbundle $\Omega^3_{\mathbb{P}^5}(3) \subset \Lambda^3 V \otimes \mathcal{O}_{\mathbb{P}^5}$. We get the following result.

Theorem 8.6 (char 2)**.** *If $W \subset (\Lambda^3 V, Q)$ is a general Lagrangian subspace in the family opposite to the one containing the fibers of $\Omega^3_{\mathbb{P}^5}(3)$, then the degeneracy locus*

$$Z_W := \{x \in \mathbb{P}^5 \mid \dim\left[\Omega^3_{\mathbb{P}^5}(3)(x) \cap W\right] \geq 3\} \tag{8.3}$$

is a smooth Fano model of a nonclassical unnodal Enriques surface with symmetrically quasi-isomorphic resolutions

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^5}(-6) \arrow[r] \arrow[d, equal] & \Omega^3_{\mathbb{P}^5} \arrow[r, "\psi"] \arrow[d, "\phi"] & W^* \otimes \mathcal{O}_{\mathbb{P}^5}(-3) \arrow[r] \arrow[d, "\phi^*"] & \mathcal{O}_{\mathbb{P}^5} \arrow[r, two heads] \arrow[d, equal] & \mathcal{O}_{Z_W} \arrow[d, "\eta", "\cong"'] \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^5}(-6) \arrow[r] & W \otimes \mathcal{O}_{\mathbb{P}^5}(-3) \arrow[r, "-\psi^*"] & \Omega^2_{\mathbb{P}^5} \arrow[r] & \mathcal{O}_{\mathbb{P}^5} \arrow[r, two heads] & \omega_{Z_W}
\end{tikzcd} \tag{8.4}$$

Proof. The general Z_W is a subcanonical surface of degree 10 with $\omega_{Z_W} \cong \mathcal{O}_{Z_W}$ and with the resolutions (8.4) according to Theorem 2.1. It is smooth by an argument similar to Lemma 5.2. Since $K_{Z_W} = 0$ and $h^i(\mathcal{O}_{Z_W}) = 1$ for $i = 0, 1, 2$, it is a nonclassical Enriques surface by the Bombieri-Mumford classification. Since the degree is 10, it is a Fano model. Moreover, according to the resolution, it is not contained in a hyperquadric. Hence it is unnodal by Theorem 8.2. □

The theorem has the following converse.

Proposition 8.7 (char 2). *Let $X \subset \mathbb{P}^5$ be a locally Gorenstein subscheme of dimension 2 and degree 10, with $\omega_X \cong \mathcal{O}_X$, with $h^i(\mathcal{O}_X) = 1$ for $i = 0, 1, 2$, and which is linearly normal and not contained in a hyperquadric. Let $V = H^0(\mathcal{O}_{\mathbb{P}^5}(1))^*$, and let Q be the divided square on $\Lambda^3 V$. Then there exists a unique Lagrangian subspace $W \subset (\Lambda^3 V, Q)$ such that $X = Z_W$ (cf. Theorem 8.6) with locally free resolutions as in (8.4).*

For example, the face variety $X = X(\Delta)$ of Proposition 7.1 satisfies all the hypotheses of the proposition and is defined as $X = Z_W$ for the Lagrangian subspace $W \subset \Lambda^3 V$ spanned by the exterior monomials

$$e_{013},\ e_{014},\ e_{023},\ e_{025},\ e_{045},\ e_{124},\ e_{125},\ e_{135},\ e_{234},\ e_{345}. \tag{8.5}$$

Any Z_W in (8.3) which is of dimension 2 satisfies all the hypotheses of Proposition 8.7.

The following refinement of the Castelnuovo-Mumford Lemma is needed for the proof of Proposition 8.7. Its proof is left to the reader.

Lemma 8.8. *Let $\mathcal{M}$ be a coherent sheaf on $\mathbb{P}^n$ with no nonzero skyscraper subsheaves, and let $0 \le q \le n-1$ and m be integers. Suppose that $\mathcal{M}$ is $(m+1)$-regular, that $H^i(\mathcal{M}(m-i)) = 0$ for $i \ge n-q$, and that the comultiplication maps*

$$H^i(\mathcal{M}(m-1-i)) \to V \otimes H^i(\mathcal{M}(m-i))$$

are surjective for $1 \le i < n-q$. Then for any $0 \le j \le q$, the module of j-th syzygies of $H^0_*(\mathcal{M})$ is generated in degrees $\le m+j$.

Proof of Proposition 8.7. The Hilbert polynomial of X is $\chi(\mathcal{O}_X(t)) = 5t^2+1$ by Riemann-Roch. The reader may easily verify that the sheaves $\mathcal{O}_X(t)$ have cohomology as given in the following table:

	$h^0(\mathcal{O}_X(t))$	$h^1(\mathcal{O}_X(t))$	$h^2(\mathcal{O}_X(t))$
$t > 0$	$5t^2+1$	0	0
$t = 0$	1	1	1
$t < 0$	0	0	$5t^2+1$

Now according to the table $\mathcal{O}_X$ is 3-regular; the only group obstructing its 2-regularity is $H^2(\mathcal{O}_X)$; and the comultiplication $H^2(\mathcal{O}_X(-1)) \to V \otimes H^2(\mathcal{O}_X)$ is dual to the multiplication $V^* \otimes H^0(\mathcal{O}_X) \to H^0(\mathcal{O}_X(1))$ and thus is an isomorphism. So we may apply Lemma 8.8 to see that all the generators of $H^0_*(\mathcal{O}_X)$ are in degrees ≤ 2, and all the relations are in degrees ≤ 3. But for all $t \le 2$ the restriction maps $H^0(\mathcal{O}_{\mathbb{P}^5}(t)) \to H^0(\mathcal{O}_X(t))$ are bijections. So the generators and relations of $H^0_*(\mathcal{O}_{\mathbb{P}^5})$ and of $H^0_*(\mathcal{O}_X)$ in degrees ≤ 2 are the same—one generator in degree 0 and no relations. Hence $H^0_*(\mathcal{O}_X)$ has only a single generator in degree 0, and its relations are all in degree 3. In other words $H^1_*(\mathcal{I}_X) = 0$, and the homogeneous ideal $I(X)$ is generated by a 10-dimensional vector space W^* of cubic equations.

We now go through the argument in [14] Theorem 6.2 which constructs a pair of symmetrically quasi-isomorphic locally free resolutions of a subcanonical subscheme of codimension 3. We have an exact sequence

$$0 \to \mathcal{K} \to W^* \otimes \mathcal{O}_{\mathbb{P}^5}(-3) \to \mathcal{O}_{\mathbb{P}^5} \to \mathcal{O}_X \to 0.$$

There is a natural isomorphism

$$\operatorname{Ext}^1_{\mathcal{O}_{\mathbb{P}^5}}(\mathcal{K}, \mathcal{O}_{\mathbb{P}^5}(-6)) \cong H^4(\mathcal{K})^* \cong H^2(\mathcal{O}_X)^* \cong k,$$

and a nonzero member of this group induces an extension which we can attach to give a locally free resolution

$$0 \to \mathcal{O}_{\mathbb{P}^5}(-6) \to \mathcal{F} \to W^* \otimes \mathcal{O}_{\mathbb{P}^5}(-3) \to \mathcal{O}_{\mathbb{P}^5} \to \mathcal{O}_X \to 0.$$

The intermediate cohomology of $\mathcal{F}$ comes from the intermediate cohomology of X and is given by $H^i_*(\mathcal{F}) = 0$ for $i = 1, 2, 4$ and $H^3_*(\mathcal{F}) \cong H^1_*(\mathcal{O}_X) = k$. According to Horrocks' Theorem [30], this means that $\mathcal{F} \cong \Omega^3_{\mathbb{P}^5} \oplus \bigoplus \mathcal{O}_{\mathbb{P}^5}(a_i)$. Since $\mathcal{F}$ and $\Omega^3_{\mathbb{P}^5}$ are both of rank 10, they are isomorphic. This gives the top locally free resolution of (8.4). The rest of diagram (8.4) is now constructed as in the proof of [14] Theorem 6.2.

It remains to identify the quadratic spaces $W^* \oplus W$ and $\Lambda^3 V$, and to show that W is uniquely determined by X. However, (8.4) gives us an exact sequence

$$0 \to \Omega^3_{\mathbb{P}^5}(3) \xrightarrow{\binom{\psi}{\phi}} (W^* \oplus W) \otimes \mathcal{O}_{\mathbb{P}^5} \xrightarrow{(\phi^* \ \psi^*)} \Omega^2_{\mathbb{P}^5}(3) \to 0.$$

This is the unique nontrivial extension of $\Omega^2_{\mathbb{P}^5}(3)$ by $\Omega^3_{\mathbb{P}^5}(3)$, and thus coincides with the truncated Koszul complex (8.2). There is therefore a natural identification of $W^* \oplus W$ with $\Lambda^3 V$ which identifies the hyperbolic quadratic form on $W^* \oplus W$ with a quadratic form Q on $\Lambda^3 V$ for which $\Omega^3_{\mathbb{P}^5}(3) \subset \Lambda^3 V \otimes \mathcal{O}_{\mathbb{P}^5}$ is a Lagrangian subbundle. By Corollary 8.5, Q is necessarily the divided square quadratic form.

The identification of the quadratic spaces $W^* \oplus W$ and $\Lambda^3 V$ is unique up to homothety, once the morphisms ψ, ϕ are chosen. However, the diagram (8.4) is unique up to an alternating homotopy $\alpha : W^* \otimes \mathcal{O}_{\mathbb{P}^5} \to W \otimes \mathcal{O}_{\mathbb{P}^5}$. One may check that this alternating homotopy does not change the Lagrangian subspace $W \subset \Lambda^3 V$ but only the choice of the Lagrangian complement W^* (cf. [14] §5). Thus W is uniquely determined by X. This completes the proof. □

Corollary 8.9 (char 2). *There is a universal family of Fano models in $\mathbb{P}^5 = \mathbb{P}(V)$ of unnodal nonclassical Enriques surfaces parametrized by an $\mathrm{SL}(V)$-invariant Zariski open subset $U \subset \mathbb{OG}_{20}(\Lambda^3 V)$. The $\mathrm{SL}(V)$-orbits correspond to isomorphism classes of Fano-polarized unnodal nonclassical Enriques surfaces (X, H).*

In particular, unnodal nonclassical Enriques surfaces form an irreducible family. We will see below that its general member is a $\boldsymbol{\mu}_2$-surface (Proposition 9.1).

The universal family of unnodal nonclassical Enriques surfaces has the following universal locally free resolution over $\mathbb{G} := \mathbb{OG}_{20}(\Lambda^3 V)$. Let $\mathcal{S} \subset \Lambda^3 V \otimes \mathcal{O}_{\mathbb{G}}$ by the universal Lagrangian subbundle on $\mathbb{G}$. According the machinery of [14] Theorem 2.1, the two subbundles $p^*\Omega^3_{\mathbb{P}^5}(3)$ and $q^*\mathcal{S}$ on $\mathbb{P}^5 \times \mathbb{G}$ (where p and q are the two projections) define a degeneracy locus Z, which one can verify has the expected codimension 3 using an incidence variety argument. Therefore $Z \subset \mathbb{P}^5 \times \mathbb{G}$ has a locally free resolution

(8.6)

$$0 \to \mathcal{O}(-6,-2) \to (p^*\Omega^3_{\mathbb{P}^5})(0,-1) \to (q^*\mathcal{S}^*)(-3,-1) \to \mathcal{O} \to \mathcal{O}_Z \to 0.$$

Remark 8.10. (a) Barth and Peters have shown that a generic Enriques surface over $\mathbb{C}$ has exactly $2^{14} \cdot 3 \cdot 17 \cdot 31$ distinct Fano polarizations ([2] Theorem 3.11).

(b) The fact that the Fano model of a unnodal Enriques surface is defined by 10 cubic equations was first observed by F. Cossec (cf. [8] [9]).

(c) (char 2) The following geometrical construction of a general Fano-polarized unnodal $\boldsymbol{\mu}_2$-surface is inspired by Mori-Mukai ([25] Proposition 3.1). Let E be an elliptic curve with Hasse invariant one, and let $i : E \to E$ be the involution given by translating by the unique nontrivial 2-torsion point. Let $\pi : E \to E' := E/\langle i \rangle$ be the quotient map. Let D' be a divisor of degree 3 on E' and let $D := \pi^*(D')$. The linear system $|D|$ gives an embedding $E \hookrightarrow \mathbb{P}^5$ with image an elliptic normal sextic curve. The involution i on E extends to an involution of $\mathbb{P}^5$. As in Mori-Mukai [25], (3.1.1) we may choose three general i-invariant hyperquadrics Q_1, Q_2, Q_3 containing E, such that $X := Q_1 \cap Q_2 \cap Q_3 \subset \mathbb{P}^5$ is a smooth K3 surface, such that $i|_X$ has no fixed points. Let L be the hyperplane divisor class on X. Then $S := X/\langle i \rangle$ is an Enriques surface of type $\boldsymbol{\mu}_2$ (since it has an étale double cover by a K3 surface, cf. [4]). The divisor class $L + E$ descends to a Fano polarization H on S, and the elliptic curve $E \subset X$ descends to $E' \subset S$. By [7] or [13] Theorem 1, (which Igor Dolgachev informs us also holds in characteristic 2), S is unnodal if and only if $|H - 2E'| = \emptyset$. But this follows since $|L - E| = \emptyset$ because $E \subset \mathbb{P}^5$ is nondegenerate. An easy dimension count shows that this construction gives rises to 10 moduli of Fano polarized unnodal $\boldsymbol{\mu}_2$ Enriques surfaces, and thus to a general one.

9. Calculating the action of Frobenius

We would like to give a method for identifying when a Fano model of a nonclassical Enriques surface constructed in Theorem 8.6 is of type $\boldsymbol{\mu}_2$ or type $\boldsymbol{\alpha}_2$. This involves computing explicitly the action of Frobenius on the resolutions. We follow the method used for cubic plane curves in [19] Proposition IV.4.21.

Let $\mathcal{O} := \mathcal{O}_{\mathbb{P}^5}$. Then $\mathcal{O}$ has a Frobenius endomorphism given on local sections by $x \mapsto x^p$. If the ideal sheaf $\mathcal{I}_X$ is locally generated by sections $f_1, \dots, f_r$, then $F(\mathcal{I}_X)\mathcal{O}$ is locally generated by $f_1^p, \dots f_r^p$. Let $F(X)$ denote the subscheme of $\mathbb{P}^5$ defined by the ideal sheaf $F(\mathcal{I}_X)\mathcal{O}$. Then $\mathcal{I}_{F(X)} \subset \mathcal{I}_X$, and the action of F on $\mathcal{O}_X$ factors as

$$\mathcal{O}_X \xrightarrow{F} \mathcal{O}_{F(X)} \xrightarrow{\pi} \mathcal{O}_X \tag{9.1}$$

where π is the canonical surjection.

Since $\mathbb{P}^5$ is a regular scheme, $\mathcal{O}$ is a flat $F(\mathcal{O})$-algebra (Kunz [24]). So if one applies F to a locally free resolution of $\mathcal{O}_X$ one obtains a locally free resolution of $\mathcal{O}_{F(X)}$. Now the nonclassical (and sometimes degenerate) Enriques surfaces $X \subset \mathbb{P}^5$ of §§7,8 have minimal free resolutions similar to that of (5.4). The factorization (9.1) lifts to the locally free resolutions and gives a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-6) \arrow[r, "d"] \arrow[d, "F"] & \mathcal{O}(-5)^6 \oplus \mathcal{O}(-6) \arrow[r] \arrow[d, "F"] & \mathcal{O}(-4)^{15} \arrow[r] \arrow[d, "F"] & \mathcal{O}(-3)^{10} \arrow[r, "d"] \arrow[d, "F"] & \mathcal{O} \arrow[r, two heads] \arrow[d, "F"] & \mathcal{O}_X \arrow[d, "F"] \\
0 \arrow[r] & \mathcal{O}(-12) \arrow[r, "F(d)"] \arrow[d, "g"] & \mathcal{O}(-10)^6 \oplus \mathcal{O}(-12) \arrow[r] \arrow[d] & \mathcal{O}(-8)^{15} \arrow[r] \arrow[d] & \mathcal{O}(-6)^{10} \arrow[r, "F(d)"] \arrow[d] & \mathcal{O} \arrow[r, two heads] \arrow[d, equal] & \mathcal{O}_{F(X)} \arrow[d, "\pi"] \\
0 \arrow[r] & \mathcal{O}(-6) \arrow[r, "d"] & \mathcal{O}(-5)^6 \oplus \mathcal{O}(-6) \arrow[r] & \mathcal{O}(-4)^{15} \arrow[r] & \mathcal{O}(-3)^{10} \arrow[r, "d"] & \mathcal{O} \arrow[r, two heads] & \mathcal{O}_X
\end{tikzcd}$$

where the upper set of vertical maps are given by the action of Frobenius, and the lower set of vertical maps are $\mathcal{O}$-linear maps lifting π. The action of Frobenius on the cohomology of $\mathcal{O}_X$ can be computed using the hypercohomology of the resolutions in the above diagram. Only the terms farthest to the left contribute to $H^1(\mathcal{O}_X)$. So the action of Frobenius on $H^1(\mathcal{O}_X)$ is the composite map

$$H^5(\mathbb{P}^5, \mathcal{O}(-6)) \xrightarrow{F} H^5(\mathbb{P}^5, \mathcal{O}(-12)) \xrightarrow{g} H^5(\mathbb{P}^5, \mathcal{O}(-6)).$$

The action of Frobenius on the generator of the Čech $\check{H}^5(\mathbb{P}^5, \mathcal{O}(-6))$, computed with respect to the standard cover, is given by

$$\frac{1}{\prod x_i} \longmapsto \frac{1}{(\prod x_i)^2} \longmapsto \operatorname{coeff}\left(\frac{g}{(\prod x_i)^2}, \frac{1}{\prod x_i}\right) = \operatorname{coeff}(g, \prod x_i) \cdot \frac{1}{\prod x_i}$$

We conclude that the Hasse invariant of X is the coefficient of $\prod x_i$ in g. If this coefficient is nonzero, then X is a $\boldsymbol{\mu}_2$-surface. Otherwise it is an $\boldsymbol{\alpha}_2$-surface.

In the case of Reisner's monomial example $X(\Delta)$ in §7 the minimal free resolution is $\mathbb{Z}^6$-multigraded, and the minimal generators of each syzygy module have different squarefree weights. In such a situation, one can lift the projection $\pi : R/F(I)R \to R/I$ to a chain map between minimal free resolutions in which all the matrices are diagonal with nonzero squarefree monomial entries. The map $g : R(-12) \to R(-6)$ must therefore be $\prod x_i$. In particular the Hasse invariant is one. This can also be verified with direct computations using *Macaulay/Macaulay2* [3] [17].

Macaulay/Macaulay2 can also be used in connection with the methods of this chapter to produce examples of smooth Fano models of unnodal nonclassical Enriques surface of both types $\boldsymbol{\mu}_2$ and $\boldsymbol{\alpha}_2$ defined over the field with two elements.

The previous diagram may be globalized by taking the global resolution (8.6), replacing $p^*\Omega^3_{\mathbb{P}^5}(0,-1)$ by its Koszul free resolution, and applying Frobenius. One obtains a diagram of sheaves on $\mathbb{P}^5 \times \mathbb{O}\mathbb{G}_{20}$.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-6,-1) \arrow[r] \arrow[d, "F"] & \mathcal{O}(-5,-1)^6 \oplus \mathcal{O}(-6,-2) \arrow[r] \arrow[d, "F"] & \cdots \arrow[r] & \mathcal{O} \arrow[r, two heads] \arrow[d, "F"] & \mathcal{O}_Z \arrow[d, "F"] \\
0 \arrow[r] & \mathcal{O}(-12,-2) \arrow[r] \arrow[d, "g"] & \mathcal{O}(-10,-2)^6 \oplus \mathcal{O}(-12,-4) \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathcal{O} \arrow[r, two heads] \arrow[d, equal] & \mathcal{O}_{F(Z)} \arrow[d, "\pi"] \\
0 \arrow[r] & \mathcal{O}(-6,-1) \arrow[r] & \mathcal{O}(-5,-1)^6 \oplus \mathcal{O}(-6,-2) \arrow[r] & \cdots \arrow[r] & \mathcal{O} \arrow[r, two heads] & \mathcal{O}_Z
\end{tikzcd}$$

The coefficient of $\prod x_i$ in g is now a section of $\mathbb{O}\mathbb{G}_{20}(1)$. We conclude:

Proposition 9.1. *The closure in $\mathbb{O}\mathbb{G}_{20}$ of the locus parametrizing Fano models of unnodal $\boldsymbol{\alpha}_2$-surfaces is an SL_6-invariant hyperplane section of $\mathbb{O}\mathbb{G}_{20}$.*

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