

NON-RATIONALITY OF THE SYMMETRIC SEXTIC FANO THREEFOLD

ARNAUD BEAUVILLE

INTRODUCTION

The symmetric sextic Fano threefold is the subvariety X of \mathbb{P}^6 defined by the equations

$$\sum X_i = \sum X_i^2 = \sum X_i^3 = 0.$$

It is a smooth complete intersection of a quadric and a cubic in \mathbb{P}^5 , with an action of \mathfrak{S}_7 . We will prove that it is not rational.

Any smooth complete intersection of a quadric and a cubic in \mathbb{P}^5 is unirational [E]. It is known that a *general* such intersection is not rational: this is proved in [B] (thm. 5.6) using the intermediate Jacobian, and in [Pu] using the group of birational automorphisms. But neither of these methods allows to prove the non-rationality of any particular such threefold. Our method gives the above explicit (and very simple) counter-example to the Lüroth problem.

Our motivation comes from the recent paper of Prokhorov [P], which classifies the simple finite subgroups of the Cremona group $\text{Cr}_3 = \text{Bir}(\mathbb{P}^3)$. In view of this work our result implies that the alternating group \mathfrak{A}_7 admits only one embedding into Cr_3 up to conjugacy.

Our proof uses the Clemens-Griffiths criterion ([C-G], Cor. 3.26): if X is rational, its intermediate Jacobian JX is the Jacobian of a curve, or a product of Jacobians. The presence of the automorphism group \mathfrak{S}_7 , together with the celebrated bound $\#\text{Aut}(C) \leq 84(g-1)$ for a curve C of genus g , immediately implies that JX is not isomorphic to the Jacobian of a curve. To rule out products of Jacobians we need some more information, which is provided by a simple analysis of the representation of \mathfrak{S}_7 on the tangent space $T_0(JX)$.

PROOF OF THE RESULT

Theorem. *The intermediate Jacobian JX is not isomorphic to a Jacobian or a product of Jacobians. As a consequence, X is not rational.*

The second assertion follows from the first by the Clemens-Griffiths criterion mentioned in the introduction. Since the Jacobians and their products form a closed subvariety of the moduli space of principally polarized abelian varieties, this gives an easy proof of the fact that a general intersection of a quadric and a cubic in \mathbb{P}^5 is not rational.

As mentioned in the introduction, the classification in [P] together with the theorem implies:

Corollary. *Up to conjugacy, there is only one embedding of \mathfrak{A}_7 into the Cremona group Cr_3 , given by an embedding $\mathfrak{A}_7 \subset \text{PGL}_4(\mathbb{C})$.*

(The embedding $\mathfrak{A}_7 \subset \mathrm{PGL}_4(\mathbb{C})$ is the composition of the standard representation $\mathfrak{A}_7 \rightarrow \mathrm{SO}_6(\mathbb{C})$ and the double covering $\mathrm{SO}_6(\mathbb{C}) \rightarrow \mathrm{PGL}_4(\mathbb{C})$.)

The intermediate Jacobian JX has dimension 20. The group \mathfrak{S}_7 acts on JX and therefore on the tangent space $T_0(JX)$; we will first determine this action.

Lemma. *As a \mathfrak{S}_7 -module $T_0(JX)$ is the sum of two irreducible representations, of dimensions 6 and 14.*

Proof : Let V be the standard (6-dimensional) representation of \mathfrak{S}_7 , and $\mathbb{P} := \mathbb{P}(V)$; we will view X as a subvariety of \mathbb{P} , stable under \mathfrak{S}_7 .

By definition $T_0(JX)$ is $H^2(X, \Omega_X^1)$. Every \mathfrak{S}_7 -module is isomorphic to its dual, so we can identify $T_0(JX)$ with $H^1(X, T_X(-1))$ by Serre duality. The exact sequence

$$0 \rightarrow T_X \rightarrow T_{\mathbb{P}|_X} \rightarrow \mathcal{O}_X(2) \oplus \mathcal{O}_X(3) \rightarrow 0$$

twisted by $\mathcal{O}_X(-1)$, gives a cohomology exact sequence

$$0 \rightarrow H^0(X, T_{\mathbb{P}(-1)}|_X) \rightarrow H^0(X, \mathcal{O}_X(1)) \oplus H^0(X, \mathcal{O}_X(2)) \rightarrow H^1(X, T_X(-1)) \rightarrow H^1(X, T_{\mathbb{P}(-1)}|_X).$$

From the Euler exact sequence $0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(1) \otimes_{\mathbb{C}} V \rightarrow T_{\mathbb{P}|_X} \rightarrow 0$ we deduce $H^1(X, T_{\mathbb{P}(-1)}|_X) = 0$ and an isomorphism $V \xrightarrow{\sim} H^0(X, T_{\mathbb{P}(-1)}|_X)$. Thus we find an exact sequence

$$0 \rightarrow V \rightarrow H^0(X, \mathcal{O}_X(1)) \oplus H^0(X, \mathcal{O}_X(2)) \rightarrow T_0(JX) \rightarrow 0,$$

which is equivariant with respect to the action of \mathfrak{S}_7 . As a \mathfrak{S}_7 -module $H^0(X, \mathcal{O}_X(1))$ is isomorphic to V and $H^0(X, \mathcal{O}_X(2))$ to $S^2V/\mathbb{C}.q$, where q corresponds to the quadric containing X . On the other hand we have $S^2V = \mathbb{C} \oplus V \oplus V_{(5,2)}$, where $V_{(5,2)}$ is the irreducible representation of \mathfrak{S}_7 corresponding to the partition $(5, 2)$ of 7 ([F-H], exercise 4.19). Thus we get $T_0(JX) \cong V \oplus V_{(5,2)}$. Since $\dim T_0(JX) = 20$ and $\dim(V) = 6$ we find $\dim V_{(5,2)} = 14$. ■

Proof of the theorem : We first observe that \mathfrak{A}_7 cannot act non-trivially on the Jacobian JC of a curve of genus $g \leq 20$. Indeed by the Torelli theorem we have $\mathrm{Aut}(JC) \cong \mathrm{Aut}(C)$ if C is hyperelliptic and $\mathrm{Aut}(JC) \cong \mathrm{Aut}(C) \times \mathbb{Z}/2$ otherwise. Since \mathfrak{A}_7 is simple we find $\#\mathrm{Aut}(C) \geq \#\mathfrak{A}_7 = 2520$. On the other hand we have $\#\mathrm{Aut}(C) \leq 84(g-1) \leq 1596$, a contradiction.

Now assume that JX is a product $J_1 \times \dots \times J_m$ of Jacobians. Such a decomposition is *unique* up to the order of the factors: it corresponds to the decomposition of the Theta divisor into irreducible components, see [C-G], Cor. 3.23. Thus the group \mathfrak{A}_7 acts on $[1, m]$ by permuting the factors. Let O_1, \dots, O_ℓ be the orbits of this action. For $1 \leq k \leq \ell$ we put $J_{(k)} := J_{m_k}$ with $m_k = \min O_k$; then for each i in O_k J_i is isomorphic to $J_{(k)}$, so our decomposition can be written $JX \cong J_{(1)}^{O_1} \times \dots \times J_{(\ell)}^{O_\ell}$.

Since $\#O_k \leq m \leq 20$, the orbit O_k has 1, 7 or 15 elements ([D-M], thm. 5.2.A). If $\#O_k = 1$, \mathfrak{A}_7 acts on the Jacobian $J_{(k)}$; by the lemma this action is faithful, contradicting the beginning of the proof. Thus $\#O_k = 7$ or 15 for each k , which contradicts the equality $\sum \#O_k \dim(J_{(k)}) = 20$. ■

Remarks. The same kind of argument gives the non-rationality of the threefold $\sum X_i^2 = \sum X_i^3 = 0$ in \mathbb{P}^5 , using the action of \mathfrak{S}_6 . It also gives a simple proof of the non-rationality of the Klein cubic threefold, defined by $\sum_{i \in \mathbb{Z}/5} X_i^2 X_{i+1} = 0$ in \mathbb{P}^4 (and, by the same token, of the general cubic threefold). The automorphism group of the Klein cubic is $\mathrm{PSL}_2(\mathbb{F}_{11})$, of order 660, while its intermediate Jacobian has dimension 5. It is easily seen as above that a 5-dimensional principally polarized abelian variety with an action of $\mathrm{PSL}_2(\mathbb{F}_{11})$ cannot be a Jacobian or a product of Jacobians (see also [Z] for a somewhat analogous, though more sophisticated, proof).

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LABORATOIRE J.-A. DIEUDONNÉ, UMR 6621 DU CNRS, UNIVERSITÉ DE NICE, PARC VALROSE, F-06108 NICE CEDEX 2, FRANCE

E-mail address: arnaud.beauville@unice.fr