Microscopic Models under a Macroscopic Perspective

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Outline:

- General
- Dynamics of the microscopic model (homogeneous case)
- Dynamics of the microscopic model (non-homogeneous case and roadworks)
- Macroscopic view
- Fundamental diagrams
- Future

Basic concept: Take a very simple microscopic model (Bando), study the full dynamics, take a macroscopic view on the results.
**Microscopic Bando model on a circular road (scaled)**

*N* cars on a circular road of length *L*:

Behaviour: $x_j$ position of the *j*-th car

$$\ddot{x}_j(t) = -\left\{ V\left(x_{j+1}(t) - x_j(t)\right) - \dot{x}_j(t) \right\}, \quad j = 1, \ldots, N, \quad x_{N+1} = x_1 + L$$

$V = V(x)$ optimal velocity function:

$V(0) = 0$, $V$ strictly monotonically increasing, $\lim_{x \to \infty} V(x) = V_{\text{max}}$
System for the headways: \( y_j = x_{j+1} - x_j \)

\[
\dot{y}_j = z_j \\
\dot{z}_j = - \{V(y_{j+1}) - V(y_j) - \dot{z}_j\}, \quad j = 1, \ldots, N, \quad y_{N+1} = y_1
\]

Additional condition: \( \sum_{j=1}^{N} y_j = L \)

“quasistationary” solutions: \( y_{s;j} = \frac{L}{N}, \quad z_{s;j} = 0, \quad j = 1, \ldots, N. \)

Linear stability-analysis around this solution gives for the Eigenvalues \( \lambda \):

\[
(\lambda^2 + \lambda + \beta)^N - \beta^N = 0, \quad \beta = V'(\frac{L}{N})
\]

Result (Huijberts (‘02)):

For \( \frac{1}{1 + \cos \frac{2\pi}{N}} > \beta^{max} = \max_x V'(x) \) asymptotic stability

For \( \frac{1}{1 + \cos \frac{2\pi}{N}} = V'(\frac{L}{N}) \) loss of stability
What kind of loss of stability? (I.G., G. Sirito, B. Werner '04):

Eigenvalues as functions of $\beta = V'(\frac{L}{N})$

Bifurcation analysis gives a Hopf bifurcation.
Therefore we have locally periodic solutions.
Are these solutions stable? (i.e. is the bifurcation sub- or super-critical?)
Criterion: Sign of the first Lyapunov-coefficient $l$

Theorem:

$$l = c^2 \left\{ V''' \left( \frac{L}{N} \right) - \frac{(V''(\frac{L}{N}))^2}{V'(\frac{L}{N})} \right\}$$

Conclusion: For the mostly used (Bando et al (95))

$$V(x) = V_{\text{max}} \frac{\tanh(a(x - 1)) + \tanh a}{1 + \tanh a}$$

the bifurcation is supercritical (i.e. stable periodic orbits).

But: “Similar” functions $V$ give also subcritical bifurcations.
**Problem:** It seems to be very sensitive with respect to $V$

**Global bifurcation analysis:** numerical tool (AUTO2000)

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**Conclusion:** Globally “similar” functions $V$ give similar behaviour. The bifurcation is “macroscopically” subcritical

**Conclusion for the application:** the critical parameters from the linear analysis are not relevant
Example 1  

Example 2  

Example 3
More bifurcations:

Eigenvalues as functions of $\beta = V'(\frac{L}{N})$

Conclusion: There are many other (weakly unstable) periodic solutions
(J. Greenberg '04,'07) Solutions with many oscillations finally tend to a solution with one oscillation

(G. Oroz, R.E. Wilson, B. Krauskopf '04, '05) Qualitatively the same global bifurcation diagram for the model with delay
Extension to “standart” microscopic model

Every driver is “aggressive” with weight $\alpha$

$$\ddot{x}_j(t) = -\frac{1 - \alpha}{\tau} \{ V \left( x_{j+1}(t) - x_j(t) \right) - \dot{x}_j(t) \} + \alpha \{ \dot{x}_{j+1}(t) - \dot{x}_j(t) \},$$

$j = 1, \ldots, N, \quad x_{N+1} = x_1 + L$ optimal velocity-function:

loss of stability similar

“Aggressive” drivers stabilize the traffic flow!

unfortunately also the number of accidents increases!

(Olmos & Munos, Condensed matter 2004)
Raiser Lehen Strachen Frei
Symmetry breaking, the above theory is not easily applicable

A solution is called **ponies on a Merry-Go-Round solution** (short **POM**), if there is a $T \in \mathbb{R}$, such that

(i) $x_i(t + T) = x_i(t) + L \quad (i = 1, \ldots, N)$

(ii) $x_i(t) = x_{i-1}(t + \frac{T}{N}) \quad (i = 1, \ldots, N)$

hold (Aronson, Golubitsky, Mallet-Paret ’91). We call $T$ rotation number and $\frac{T}{N}$ the phase (phase shift).
Theorem: The above model has POM solutions for small $\epsilon > 0$.

Velocity of the quasistationary solution (no roadwork) versus roadwork solution (The red line indicates maximum velocity).
**Technique**: Poincare maps

\[ \Pi(\eta) = \Phi_{T(\eta)}(\eta) - \Lambda, \] where \( \Phi \) is the induced flow and \( \Lambda \) reduces the spacial components by \( L \).

Study fixed points of the corresponding Poincare and reduced Poincare maps. Roadworks are (regular) perturbations.
Bifurcation diagram in \((L, \epsilon)\)-plane:

A curve of Neimark-Sacker bifurcations in the \((L, \epsilon)\)-plane for \(N = 5\).
Four different attractors:

\[
\begin{array}{ccc}
\epsilon & I & II \\
\epsilon = 0 & \text{trivial POM } x^0 & \text{Hopf periodic solution} \\
\epsilon > 0 & \text{POM } x^\epsilon & \text{quasi-POM} \\
\end{array}
\]

i.e. here

POM’s are typically perturbed quasistationary solutions
quasi-POM’s are perturbed (Hopf) periodic solutions
Two closed invariant curves ($\epsilon = 0$ and $\epsilon > 0$) of the reduced Poincaré map $\pi$. On the left also the optimal velocity function $V_0$ is given in gray.
The 4 different scenarios:

above: no roadworks, below: with roadworks
Macroscopic view of the 4 different scenarios:

above: no roadworks, below: with roadworks
Macroscopic view of density and velocity:

strong road work influence ($\epsilon = 0.32$)
A real world point of view on the reduced Poincaré map $\pi$ for $N = 10, \epsilon = 0$. 

**Fundamental diagrams:**
Fundamental diagrams I:

Overlapped fundamental diagrams for $N = 10, L = 50, ..., 4$ measuring at a fixed point.
Fundamental diagrams II:

Fundamental diagram of time-averaged flow versus average density for $N = 10, L = 50 \ldots 4$. 
Overlapped fundamental diagrams for $N = 10, L = 50, \ldots, 4, \quad \epsilon = 0.1$ measuring at a fixed point.
Current and future work:

- Is this dynamics contained in macroscopic models?

- Which macroscopic model has the same (rich) dynamics than the basic Bando model

- Micro-macro link (Aw, Klar, Materne, Rascle 2002)