Individual displacement in biology: a new model for fish behavior

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Math. Models of Traffic Flow
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Motivation

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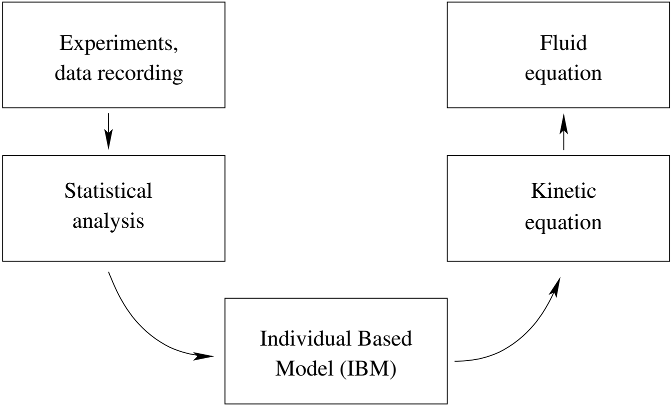
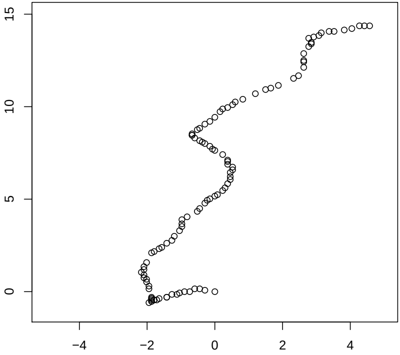
How this structures could emerge from local interactions ?

Method


Experiments,
data recording
Statistical
analysis
Individual Based
Model (IBM)
Kinetic
equation
Fluid
equation

Example : a trajectory of an ant



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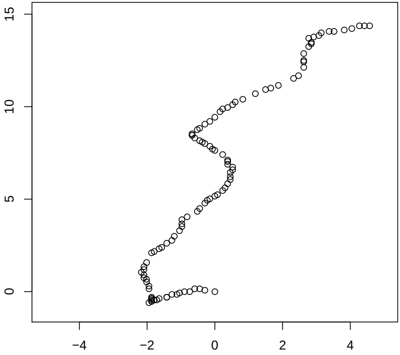

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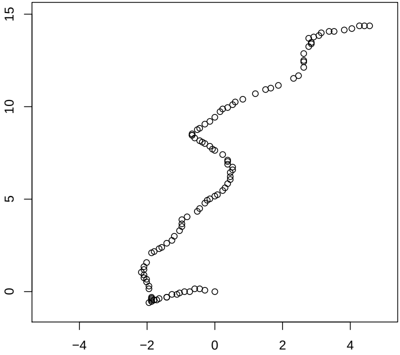

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- The direction of the ant changes abruptly

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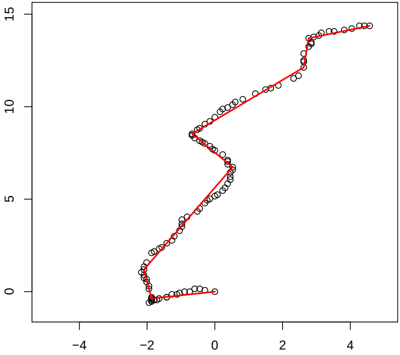

- The norm of the velocity is constant
- The direction of the ant changes abruptly

Mathematically

- Let $\vec{x}$ be the vector position. Since velocity is constant, we have :

$$\frac{d\vec{x}}{dt} = c\vec{\tau}(\theta),$$

with $\vec{\tau}(\theta) = (\cos\theta, \sin\theta)$.

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with $\vec{\tau}(\theta) = (\cos\theta, \sin\theta)$.

- The angle θ of the vector speed is modelled as a jump process :

$$\theta(t) = \sum_{i=1}^{N(t)} S_i,$$

with $N(t)$ a Poisson process of intensity λ and S_i independent jumps.

Model for fish behavior

The jump process is replaced by a diffusive process :

$$\begin{aligned} \frac{d\vec{x}}{dt} &= c\vec{\tau}(\theta) \\ d\theta &= b\, dW_t, \end{aligned}$$

where dW_t is the white noise.

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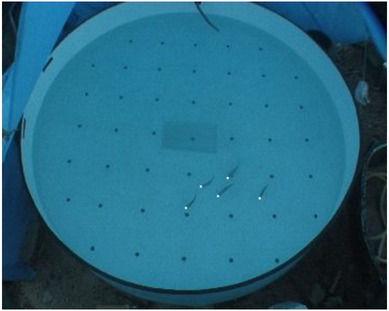
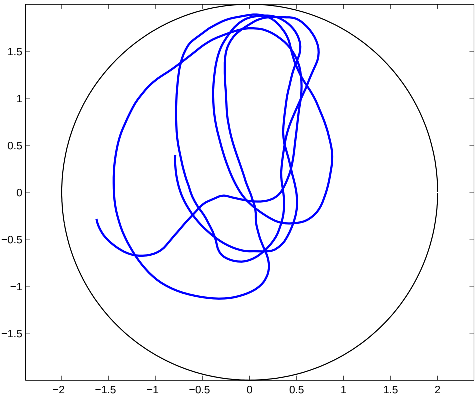
$\Rightarrow$ *Experiments*

Experiments



- The diameter of the basin is 4 meters
- Species studied : Kuhlia mugil (20-25 cm)

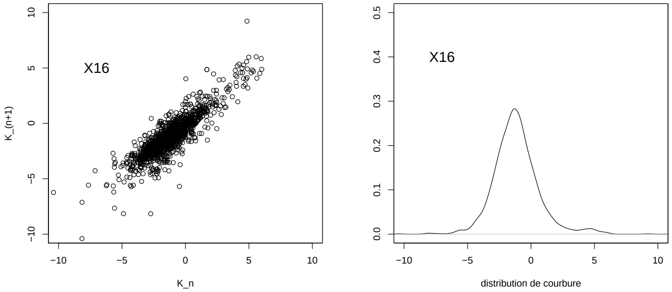
An example of trajectory :



- The norm of the velocity is constant
- The trajectory is smooth, the fish seems to turn constantly

Two key elements in the statistical analysis of the curvature :

- Strong correlation between two time steps
- Gaussian form for the stationnary state



Model

We note κ the curvature. The model proposed is :

$$\kappa_{n+1} = \kappa_n - \alpha\kappa_n + \varepsilon_n,$$

where ε_n is a random noise independent of κ_n.

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Continuous version of the model :

$$d\kappa = -a\kappa dt + bdW_t.$$

In the field of probability, this equation is known as the Ornstein-Uhlenbeck process.

Model

Finally, the model can be written as :

$$\left\{ \begin{array}{lcl} \frac{d\vec{x}}{dt} & = & c\vec{\tau}(\theta) \\ \frac{d\theta}{dt} & = & c\kappa \\ d\kappa & = & -a\kappa\, dt + b\, dW_t \end{array} \right.$$

where c is the velocity.

We call this model "Persistent Turning Walker" (PTW).

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$\Rightarrow$ *Numerical simulation*

Macroscopic model

We start from the stochastic differential equation

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\end{aligned}
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Macroscopic model

We start from the stochastic differential equation in scaled variables

$$\begin{aligned} \frac{d\vec{x}_*}{dt_*} &= \vec{\tau}(\theta) \\ \frac{d\theta}{dt_*} &= \kappa \\ d\kappa &= -\kappa\, dt_* + \sqrt{2}\alpha\, dW_{t_*}, \end{aligned}$$

with : $t_* = \frac{1}{a} t$, $x_* = \frac{c}{a} x$ and $\alpha^2 = \frac{b^2 c^2}{2\alpha^3}$.

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What is the dynamics on a larger scale ?

Probabilistic method

- $d\kappa = -\kappa\, dt + \sqrt{2}\alpha\, dW_t$
  Explicit solution in term of stochastic integral :

$$\Rightarrow \kappa(t) = \mathrm{e}^{-t}\kappa_0 + \sqrt{2}\alpha \mathrm{e}^{-t} \int_0^t \mathrm{e}^s\, dW_s.$$

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- $d\theta = \kappa\, dt$
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  $$\Rightarrow \theta(t) = \theta_0 + \kappa_0 - \kappa(t) + \sqrt{2}\alpha W_t.$$

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  $$\Rightarrow \theta(t) = \theta_0 + \kappa_0 - \kappa(t) + \sqrt{2}\alpha W_t.$$
- $d\vec{x} = \vec{\tau}(\theta) dt$
  Explicit solution : ?

Hyp. : $\vec{x}_0 = (0,0)$, $\theta_0 \sim \mathcal{U}]-\pi,\pi]$, $\kappa_0 \sim \mathcal{N}(0,\alpha^2)$,
θ_0, κ_0 and W_t independent.

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Thm. Under above hypothesis, we have :

$$\begin{aligned} \mathbb{E}\{\vec{x}(t)\} &= (0,0)\,, \qquad \forall\, t \geq 0, \\ \mathrm{Var}\{\vec{x}(t)\} &= 2\int_{s=0}^{t} (t-s) \exp\left(-\alpha^2\left(-1+s+\mathrm{e}^{-s}\right)\right)\, ds. \end{aligned}$$

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\end{aligned}$$

In particular :

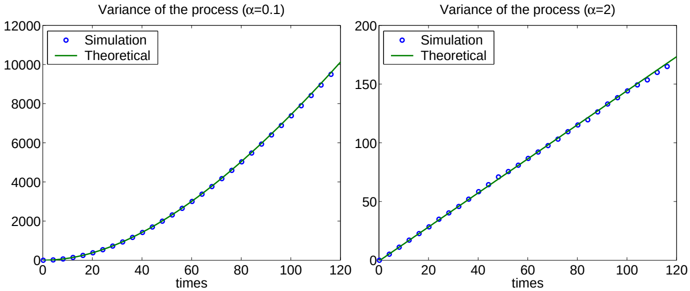
$$\mathrm{Var}\{\vec{x}(t)\} \overset{t\to+\infty}{\sim} 2\mathcal{D}\, t,$$

with :

$$\mathcal{D} = \int_0^{\infty} \exp\left(-\alpha^2(-1+s+\mathrm{e}^{-s})\right)\, ds.$$

Numerical illustration

We use Monte-Carlo method to estimate the mean square displacement :



Summary of the probabilistic method :

- Explicit expression for the mean square displacement.
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However :

- No rigorous proof of diffusive behavior asymptotically.
- If we put interaction terms in the PTW model, the method is not available.

Kinetic method

We start from our PTW model :

$$\begin{aligned} \frac{d\vec{x}}{dt} &= \vec{\tau}(\theta) \\ \frac{d\theta}{dt} &= \kappa \\ d\kappa &= -\kappa\, dt + \sqrt{2}\alpha\, dW_t \end{aligned}$$

We introduce $f(t, x, \theta, \kappa)$ the density distribution of particles.

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Individual dynamics can be translated on f using Kolmogorov (*forward*) equation :

$$\partial_t f + \vec{\tau} \cdot \nabla_{\vec{x}} f + \kappa \partial_\theta f - \partial_\kappa(\kappa f) - \alpha^2 \partial_{\kappa^2} f = 0.$$

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With these new variables, we define $f^\varepsilon(t', x', ...) = f(t, x, ...)$ which satisfies :

$$\varepsilon \partial_t f^\varepsilon + \vec{\tau} \cdot \nabla_{\vec{x}} f^\varepsilon + \frac{1}{\varepsilon} A f^\varepsilon = 0, \tag{1}$$

with $Af^\varepsilon = \kappa \partial_\theta f^\varepsilon - \partial_\kappa(\kappa f^\varepsilon) - \alpha^2 \partial_{\kappa^2} f^\varepsilon$.

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with $Af^\varepsilon = \kappa \partial_\theta f^\varepsilon - \partial_\kappa(\kappa f^\varepsilon) - \alpha^2 \partial_{\kappa^2} f^\varepsilon$.

What is the limit of f^ε as ε tends to 0 ?

We consider an Hilbert expansion of f^ε :

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  where M Gaussian with zero mean and variance α^{2}.

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  where M Gaussian with zero mean and variance α^2.
- ε^0 : $\quad \vec{\tau} \cdot \nabla_x f^0 = Af^1$.
  We introduce an auxiliary function χ satisfying :

$$A\vec{\chi} = \frac{M(\kappa)}{2\pi}\,\vec{\tau}\,. \qquad (2)$$

Then : $f^1 = -\vec{\chi} \cdot \nabla_{\vec{x}} n^0 + cM(\kappa)$.

Integrating eq. (1) in (θ,κ)

$$\int_{\theta,\kappa} \left(\varepsilon\partial_t f^\varepsilon + \vec{\tau}\cdot\nabla_{\vec{x}} f^\varepsilon + \frac{1}{\varepsilon} A f^\varepsilon = 0 \right) d\theta d\kappa$$

we have the equation of mass conservation :

$$\partial_t n^\varepsilon + \nabla_{\vec{x}} \cdot J^\varepsilon = 0,$$

where

$$n^\varepsilon(t,\vec{x}) = \int_{\theta,\kappa} f^\varepsilon \; d\kappa\, d\theta, \quad J^\varepsilon(t,\vec{x}) = \int_{\theta,\kappa} \frac{f^\varepsilon}{\varepsilon}\, \vec{\tau}(\theta)\, d\kappa d\theta.$$

Inserting the Hilbert expansion in the expression of the flux J^ε :

$$\begin{aligned} J^\varepsilon(t,\vec{x}) &= \frac{1}{\varepsilon}\int_{\theta,\kappa} f^\varepsilon \vec{\tau}(\theta)\, d\kappa d\theta \\ &= \int_{\theta,\kappa} f^1\, \vec{\tau}(\theta)\, d\kappa d\theta + O(\varepsilon), \end{aligned}$$

we have at the limit $\varepsilon \longrightarrow 0$:

$$J^0(t,\vec{x}) = -\left(\int_{\theta,\kappa} \vec{\tau} \otimes \vec{\chi}\, d\theta\, d\kappa\right) \nabla_{\vec{x}} n^0.$$

Diffusive equation

Thm. The distribution f^{ε} solution of (1) satisfies :

$$f^{\varepsilon} \overset{\varepsilon \to 0}{\rightharpoonup} n^0 \, \frac{M(\kappa)}{2\pi},$$

with :

$$\begin{aligned} \partial_t n^0 + \nabla_{\vec{x}} \cdot J^0 &= 0, \\ J^0 &= -D \nabla_{\vec{x}} n^0, \end{aligned}$$

where $D = \int_{\theta,\kappa} \vec{\tau} \otimes \vec{\chi} \, d\theta \, d\kappa$ and χ solution of (2).

Value of D

We have to estimate $D = \int_{\theta,\kappa} \vec{\tau} \otimes \vec{\chi} \, d\theta \, d\kappa$ with :

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We introduce a parabolic equation :

$$\partial_t \vec{\chi}_t = -A\vec{\chi}_t + \vec{\tau}(\theta)\,\frac{M(\kappa)}{2\pi}\,, \quad \vec{\chi}(t=0) = 0.$$

We show that : $\vec{\chi}_t \overset{t\to+\infty}{\rightharpoonup} \vec{\chi}$.

Furthermore, we can write :

$$\begin{aligned} D &= \int_{\theta,\kappa} \vec{\tau} \otimes \vec{\chi}\, d\theta\, d\kappa \\ &= \lim_{t\to+\infty} \int_{\theta,\kappa} \vec{\tau} \otimes \vec{\chi}_t\, d\theta\, d\kappa \\ &= \dots \\ &= \frac{\mathcal{D}}{2} \mathrm{Id}, \end{aligned}$$

where Id denotes the 2x2 identity tensor and

$$\mathcal{D} = \int_0^\infty \exp\left(-\alpha^2(-1+s+\mathrm{e}^{-s})\right)\, ds.$$

Fishes in interaction

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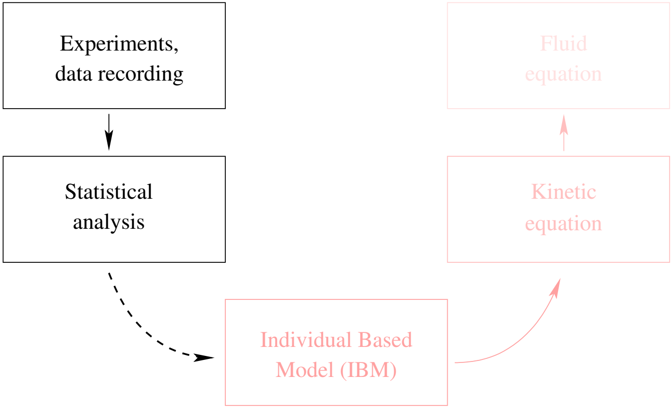
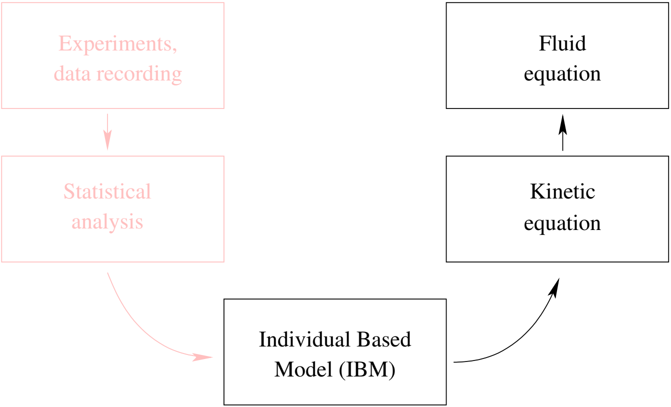
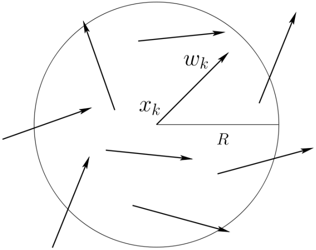


Particles in interaction



Vicsek model



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\begin{aligned}
x_k^{n+1} &= x_k^n + \Delta t\, \omega_k^n \\
\omega_k^{n+1} &= \frac{\sum_{j \in \mathcal{O}_k} \omega_j^n}{\left|\sum_{j \in \mathcal{O}_k} \omega_j^n\right|} \\
&= \bar{\omega}_k^n
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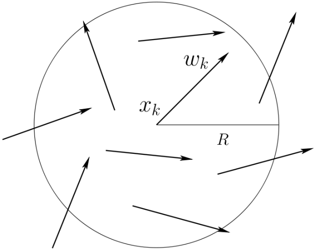

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How can we find a time continuous version of this algorithm ?

Continuous version

Since $|\omega^{n+1}| = |\omega^n|$, we have :

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Furthermore,

$$\frac{1}{\Delta t}(\omega^{n+1} - \omega^n) = \quad \frac{1}{\Delta t}(\mathrm{Id} - \omega^{n+\frac{1}{2}} \otimes \omega^{n+\frac{1}{2}})(\bar{\omega}^n - \omega^n).$$

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Formally, at the limit $\Delta t \to 0$:

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Continuous version

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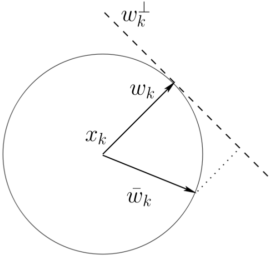
$$\frac{d\omega_k}{dt} = \nu(\mathrm{Id} - \omega_k \otimes \omega_k)\ \bar{\omega}_k$$

where $\bar{\omega}_k = \dfrac{\sum_{j \in \mathcal{O}_k} \omega_j}{\left|\sum_{j \in \mathcal{O}_k} \omega_j\right|}$



Continuous version

$$\frac{dx_k}{dt} = \omega_k$$

$$\frac{d\omega_k}{dt} = \nu(\mathrm{Id} - \omega_k \otimes \omega_k)(\bar{\omega}_k + \sqrt{2D}dW_t)$$

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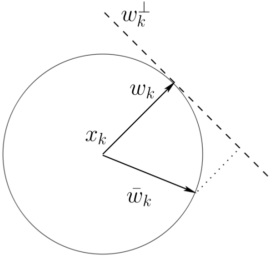

Kinetic equation

The probability density of particles $f(t,x,\omega)$ satisfies :

$$\partial_t f + \omega \cdot \nabla_x f + \nabla_\omega \cdot (Ff) = 0,$$

with :

$$F = \nu\,(\mathrm{Id} - \omega \otimes \omega)\bar{\omega} \quad , \quad \bar{\omega} = \frac{J(x,t)}{|J(x,t)|}$$

$$J(x,t) = \int_{y \in \mathbb{R}^3,\, \upsilon \in \mathbb{S}^2} K(|x-y|)\, \upsilon\, f(y,\upsilon,t)\, dy\, d\upsilon$$

Kinetic equation

The probability density of particles $f(t,x,\omega)$ satisfies :

$$\partial_t f + \omega \cdot \nabla_x f + \nabla_\omega \cdot (Ff) = D\Delta_\omega f,$$

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Large scale

In order to derive a macroscopic equation, we introduce the rescaling :

$$t' = \varepsilon t \quad ; \quad x' = \varepsilon x.$$

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After some computations, we find :

$$f^\varepsilon \overset{\varepsilon \to 0}{\rightharpoonup} f^0 = n(t,x) M_{\Omega(t,x)}$$

with $M_\Omega(\omega) = C \exp(\frac{\nu}{d}\omega \cdot \Omega)$.

Fluid equation

Integrating equation (3), we find formally the hyperbolic system :

$$\partial_t n + \nabla_x \cdot (c_1 n \Omega) = 0,$$
$$n\,(\partial_t \Omega + c_2 (\Omega \cdot \nabla)\Omega) + \lambda\,(\mathrm{Id} - \Omega \otimes \Omega)\nabla_x n = 0,$$

with the coefficients c_1, c_2 and λ depending on d and ν.