RIGOROUS DERIVATION OF THE X-Z SEMIGEOSTROPHIC EQUATIONS *

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Abstract. We prove that smooth solutions of the semigeostrophic equations in the incompressible $x-z$ setting can be derived from the Navier-Stokes equations with the Boussinesq approximation.

Key words. Atmospheric sciences, fluid mechanics, asymptotic analysis

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1. Introduction

We consider the Navier-Stokes equations with the Boussinesq approximation (NSB):

$$\epsilon(\partial_t v + (v\cdot\nabla)v) + \alpha K v + \nabla p = y,\ \ \nabla\cdot v = 0, \tag{1.1}$$

$$\partial_t y + (v\cdot\nabla)y = G(x,y), \tag{1.2}$$

where $x\in D$, D being a smooth bounded domain in R^d $(d=2,3)$, $v=v(t,x)\in R^d$ is the velocity field, $p=p(t,x)$ is the pressure field, $y=y(t,x)\in R^d$ is a vector-valued forcing term, $G(x,y)$ is a given smooth vector-valued source term $D\times R^d\to R^d$, $\epsilon,\alpha>0$ are scaling factors and K is the linear dissipative operator $Kv=-\Delta v$. We assume that the fluid sticks to the boundary: $v=0$ along ∂D.
We now consider the formal limit of these equations obtained by dropping the inertia term and the dissipative term (i.e. setting $\varepsilon=\alpha=0$) in the NSB equations.

$$\nabla p = y,\ \ \nabla\cdot v = 0,\ \ v//\partial D, \tag{1.3}$$

$$\partial_t y + (v\cdot\nabla)y = G(x,y). \tag{1.4}$$

We are going to show that these equations can be justified under a strong uniform convexity assumption on the pressure field p. The situation of interest in this paper is the case when $d=2$ and the source term

$$G(x,y) = (x_2, y_1 - x_1). \tag{1.5}$$

Then (1.3-1.4) are the semigeostrophic Eady model equations in the special incompressible "$x-z$" situation. By $x-z$, we mean that D is part of a vertical section, the second coordinate x_2 of each point $x=(x_1,x_2)\in D$ being the vertical one. The source term in (1.5) represents the effect of the missing third dimension. In this identification, y represents the effects of rotation and stratification, and the relation $\nabla p = y$ in (1.3) expresses geostrophic and hydrostatic balance.

The semigeostrophic model was considered by Hoskins [Ho] to model front formations in atmospheric sciences. The Eady model is defined in chapter 6 of [Cu], and models a quasi-periodic evolution in which fronts form and decay. There has been

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a lot of interest in these equations (see for instance [CNP, BB, CG, CF, Cu]), due to their beautiful geometric structure and their deep links with the Monge-Ampère equation and optimal transport theory [CM, Br1, Br2, Vi]. The rigorous derivation of the full 3 dimensional SG equations is still a challenging problem. The present short note is just the first step toward this goal.

2. Motivation for a convexity assumption In their study of the SG equations, Cullen and Purser have introduced a convexity assumption on the pressure field p, based on a combination of physical and mathematical arguments. Convexity is also natural in the case of the general equations (1.3-1.4), independently of the choice of the source term G, for the following reasons. At first glance, these equations look strange since there is no evolution equation for v. However, y is constrained to be a gradient. Therefore, v can be seen as a kind of Lagrange multiplier for this constraint. (Vaguely speaking, due to the presence of a source term, in order to stay a gradient, the field y needs to be continuously rearranged in a volume-preserving fashion under the action of a time-dependent divergence-free vector field v.) As a matter of fact, it is (formally) very easy to get an equation for v, once $y=\nabla p$ is known. To do that, let us start with the 2 dimensional case and write

$$y(t,x)=(\partial_1 p,\partial_2 p)(t,x_1,x_2),\ \ v(t,x)=(-\partial_2\psi,\partial_1\psi)(t,x_1,x_2)$$

(at least locally), where ψ is a “stream-function”. Then, let us “curl” equation (1.4) and get:

$$-\partial^2_{11}p\,\partial^2_{22}\psi+2\partial^2_{12}p\,\partial^2_{12}\psi-\partial^2_{22}p\,\partial^2_{11}\psi=\partial_1(G_2(x,\nabla p))-\partial_2(G_1(x,\nabla p)). \tag{2.1}$$

This is a linear second order elliptic equation in ψ, whenever p is a given strictly uniformly convex (or concave) function of x, i.e. when $D^2_x p>0$ -in the sense of symmetric matrices- (or <0). In three space dimensions, we get some "magnetostatic" version of equation (2.1). Indeed, since v is divergence-free, we can (at least locally) write $v=\nabla\times A$ for some "potential vector" $A=A(t,x)\in R^3$, that we may assume to be itself divergence-free. Then, by curling equation (1.4), we get a linear system for A when p is convex, namely:

$$\nabla\times(M(t,x)\nabla\times A)=\nabla\times(G(x,\nabla p)). \tag{2.2}$$

This system is elliptic whenever the symmetric matrix $M=D^2_x p(t,x)$ is uniformly positive and bounded, which means that p is convex in a strong sense. In higher dimension, v should be viewed as a $d-1$ form and p as a zero form. The divergence free condition (locally) means that $v=dA$, where A is a $d-2$ form. Then, again taking the curl of equation (1.4), we get the multidimensional generalization of system (2.1): $d(M(t,x)*dA)=d(G(x,dp)$ (where $*$ denotes Hodge duality and $M=D^2p$) which, again, is an elliptic system in A when $D^2_x p$ is uniformly bounded and positive. Thus we see that requiring p to be convex is a natural solvability condition for equations (1.3-1.4).

3. Rigorous derivation from the Navier-Stokes equations

The generalized Cullen-Purser convexity condition plays a crucial role in the rigorous derivation of equations (1.3-1.4) from the NSB equations.

Theorem 3.1. *Let D be a smooth bounded convex domain. Assume G to be smooth with bounded derivatives up to second order. Let $(y^\varepsilon,v^\varepsilon,p^\varepsilon)$ be a Leray-type solution to the NSB equations (1.1,1.2), where $K=-\Delta$, with $\alpha=O(\varepsilon)$. Let $(y=\nabla p,v)$ be* *a smooth solution to equations (1.3,1.4) on a given finite time interval* $[0,T]$. *We assume* $p(t,x)$ *to have a smooth convex extension for all* $x\in R^d$ *so that its Legendre transform*

$$p^*(t,y)=\sup_{x\in R^d} x\cdot y-p(t,x) \tag{3.1}$$

is also smooth for $y\in R^d$ *with Hessian* $D_y^2 p^*(t,y)$ *bounded away from zero and* $+\infty$. *Then, the* L^2 *distance between* y^ε *and* y *stays uniformly of order* $\sqrt{\varepsilon}$ *as* ε *goes to zero, uniformly in* $t\in[0,T]$, *provided it does at* $t=0$ *and the initial velocity* $v^\varepsilon(t=0,x)$ *stays uniformly bounded in* L^2.

Notice that the theorem is meaningful, since the local existence of smooth solutions has been proven by Loeper [Lo] (in the SG case) at least for periodic boundary conditions, provided that that $y(0,x)-x$ is not too large in some appropriate sense.

Proof. For the convergence, we use a relative entropy trick quite similar to the one used by the author for the hydrostatic limit of the 2D Euler equations in a thin domain [Br3]. We introduce the so-called Bregman function (or relative entropy) attached to p^*:

$$\eta_{p^*}(t,z,z')=p^*(t,z')-p^*(t,z)-(\nabla p^*)(t,z)\cdot(z'-z)\sim|z'-z|^2 \tag{3.2}$$

and the related functional:

$$e(t)=\int_D(\epsilon\frac{|v^\varepsilon(t,x)-v(t,x)|^2}{2}+\eta_{p^*}(t,y(t,x),y^\varepsilon(t,x)))\,dx. \tag{3.3}$$

Given a weak solution $(y^\varepsilon,v^\varepsilon)$ to the NSB equations (1.1,1.2) and a solution y of (1.3-1.4), we want to get an estimate of form:

$$\frac{d}{dt}(e(t)+O(\epsilon))\le(e(t)+O(\epsilon))c, \tag{3.4}$$

where c depends only on the limit solution (y,v) on a fixed finite time interval $[0,T]$ on which (y,v) is smooth. From this estimate (3.4), we immediately get that $y-y^\varepsilon$ is of order $O(\sqrt{\epsilon})$ in $L^\infty([0,T],L^2(D))$. So, we are left with proving (3.4). To save time, we do calculations just as if the Leray solutions were smooth solutions. Let us compute

$$I=I_1+I_2+I_3+I_4,$$

$$I_1=\frac{d}{dt}\int_D p^*(t,y^\varepsilon(t,x))\,dx,$$

$$I_2=-\frac{d}{dt}\int_D p^*(t,y(t,x))\,dx,$$

$$I_3=-\frac{d}{dt}\int_D(\nabla p^*(t,y(t,x)))\cdot y^\varepsilon(t,x)\,dx,$$

$$I_4=\frac{d}{dt}\int_D(\nabla p^*(t,y(t,x)))\cdot y(t,x)\,dx.$$

We first get

$$I_1 = \int_D [(\partial_t p^*)(t, y^\varepsilon(t,x)) + (\nabla p^*)(t, y^\varepsilon(t,x)) \cdot G(x, y^\varepsilon(t,x))]\, dx,$$

(using that v^ε is divergence free and parallel to ∂D). Similarly,

$$I_2 = -\int_D [(\partial_t p^*)(t, y(t,x)) + (\nabla p^*)(t, y(t,x)) \cdot G(x, y(t,x))]\, dx.$$

Next,

$$I_3 = -\int_D [(\partial_t \nabla p^*)(t, y(t,x)) \cdot y^\varepsilon(t,x) + (D_y^2 p^*)(t, y(t,x))(\partial_t y(t,x), y^\varepsilon(t,x))]\, dx$$

$$-\int_D (\nabla p^*)(t, y(t,x)) \cdot G(x, y^\varepsilon(t,x))\, dx + I_5,$$

where

$$I_5 = \int_D x \cdot (v^\varepsilon(t,x) \cdot \nabla) y^\varepsilon(t,x)\, dx$$

$$= -\int_D v^\varepsilon(t,x) \cdot y^\varepsilon(t,x)\, dx,$$

(where we have used, for the two last lines, that $(\nabla p^*)(t, y(t,x)) = x$, which follows from Legendre duality). Since v^ε solves the NSB equations, we find

$$I_5 = -\int_D [\varepsilon(\partial_t + v^\varepsilon \cdot \nabla) v^\varepsilon + \nabla p^\varepsilon + \alpha K v^\varepsilon] \cdot v^\varepsilon\, dx$$

$$= -\frac{\varepsilon d}{2dt} \int_D |v^\varepsilon|^2\, dx - \int v^\varepsilon \cdot \alpha K v^\varepsilon\, dx.$$

Similarly

$$I_4 = \int_D [(\partial_t \nabla p^*)(t, y(t,x)) \cdot y(t,x) + (D_y^2 p^*)(t, y(t,x))(\partial_t y(t,x), y(t,x))]\, dx +$$

$$+\int_D \nabla p^*(t, y(t,x)) \cdot G(x, y(t,x))\, dx.$$

Collecting all terms, we get

$$I = I_5 + I_6 + I_7 + I_8 + I_9,$$

where

$$I_6 = \int_D \eta_{\partial_t p^*}(t, y(t,x), y^\varepsilon(t,x))\, dx,$$

(which involves the Bregman functional associated with $\partial_t p^*$ and therefore is bounded by $e(t)c$ where c is a constant depending only on the limit solutions $y=\nabla p$),

$$I_7 = -\int_D [(\nabla p^*)(t,y) - (\nabla p^*)(t,y^\varepsilon)]\cdot G(x,y^\varepsilon)\,dx,$$

$$I_8 = \int (D_y^2 p^*)(t,y)(G(x,y), y-y^\varepsilon)\,dx,$$

$$I_9 = \int (D_y^2 p^*)(t,y)(\partial_t y - G(x,y), y - y^\varepsilon)\,dx,$$

$$= \int (D_y^2 p^*)(t,y)((v\cdot\nabla)y, y^\varepsilon - y)\,dx.$$

We easily see that

$$I_7 + I_8 = \int_D \eta_{\nabla p^*}(t, y(t,x), y^\varepsilon(t,x))\cdot G(x,y)\,dx$$

$$+\int_D [(\nabla p^*)(t,y) - (\nabla p^*)(t,y^\varepsilon)]\cdot(G(x,y) - G(x,y^\varepsilon))\,dx,$$

(which is again bounded by $e(t)c$ where c is a constant depending only on the limit solutions $y=\nabla p$). Let us finally consider the most delicate term I_9. We can write I_9 in index notations as:

$$I_9 = \int \sum_{ijk} \partial^2_{ij} p^*(t,y) v_k \partial_k y_i (y^\varepsilon - y)_j,$$

$$= \int \sum_{ijk} \delta_{jk} v_k (y^\varepsilon - y)_j = \int v\cdot(y^\varepsilon - y),$$

(indeed, p^* is the Legendre transform of p and $y=\nabla p$, thus $D^2 p^*(y) Dy = D^2 p^*(\nabla p) D^2 p = Id$)

$$= \int v\cdot y^\varepsilon,$$

(since y is a gradient and v is divergence free and parallel to ∂D)

$$= \int v\cdot(\varepsilon(\partial_t + v^\varepsilon\cdot\nabla)v^\varepsilon + \alpha K v^\varepsilon),$$

(using the NSB equations)

$$= J_1 + J_2,$$

where

$$J_1 = \frac{d}{dt}\varepsilon\int v^\varepsilon\cdot v,$$

and

$$|J_2| \le \varepsilon(\int |v^\varepsilon|^2 + 1)c \le \varepsilon(\int |v^\varepsilon - v|^2 + 1)c \le (e(t) + \varepsilon)c,$$

where c are constants only depending on the limit solution v. Thus, again collecting all terms, and using that

$$I_5 = -\frac{\varepsilon d}{2dt}\int_D |v^\varepsilon|^2\,dx - \alpha\int v^\varepsilon \cdot K v^\varepsilon\,dx,$$

we have obtained

$$I + \frac{d}{dt}(\frac{\varepsilon}{2}\int |v^\varepsilon - v|^2 + O(\varepsilon)) + \alpha\int v^\varepsilon \cdot K v^\varepsilon\,dx \le (e(t) + O(\varepsilon))c,$$

which leads to the desired inequality (3.4) and completes the proof.

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