Rearrangement, Convection and Competition

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Outline

1. A toy-model for convection based on rearrangement theory and its interpretation as a competion model

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2. Multidimensional rearrangement theory and generalization of the toy model

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2. Multidimensional rearrangement theory and generalization of the toy model
3. Interpretation of the model as a hydrostatic limit of the Navier-Stokes Boussinesq equations

A reminder

Given a scalar function $z(x),\ x \in D = [0,1]$**, there is a unique non decreasing function** $Z(x) = \text{Rearrange}(z)(x)$ **such that,**

$$\boxed{\int_D f(Z(x))dx = \int_D f(z(x))dx}$$

for all test function f.

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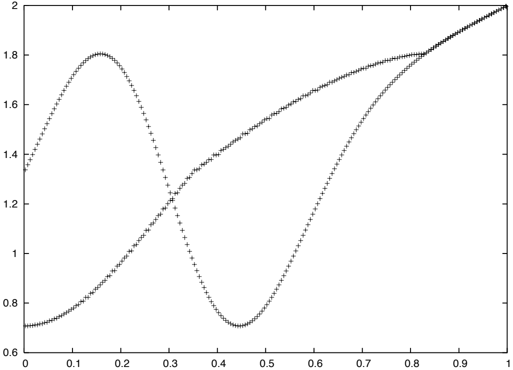
for all test function f.
Notice that in the discrete case when

$$z(x) = z_j, \quad j/N < x < (j+1)/N, \quad j = 0, ..., N-1$$

then $Z(x) = Z_j$ where $(Z_1, ..., Z_N)$ is just $(z_1, ..., z_N)$ sorted in increasing order.

A function and its rearrangement

N $=$ 200 grid points in x



A toy-model for (very fast) convection

Model:
-vertical coordinate only: $\mathbf{x} = \mathbf{x_3} \in \mathbf{D} = [\mathbf{0}, \mathbf{1}]$
-temperature field: $\mathbf{y}(\mathbf{t}, \mathbf{x})$
-heat source term: $\mathbf{G} = \mathbf{G}(\mathbf{t}, \mathbf{x}, \mathbf{y})$

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Time discrete scheme:

-time step $h > 0$, $y(t = nh, x) \sim y_n(x)$, $n = 0, 1, 2, \cdots$

-predictor (heating): $y_{n+1/2}(x) = y_n(x) + h\, G(nh, x, y_n(x))$

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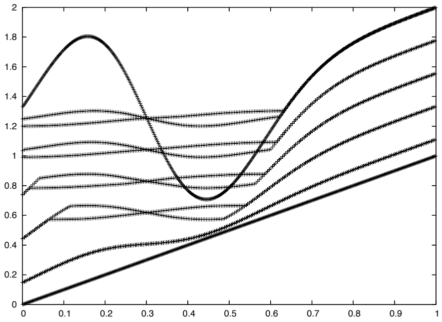
-corrector (fast convection): $y_{n+1} = \mathrm{Rearrange}(y_{n+1/2})$
so that the temperature profile stays monotonically increasing at EACH time step. (This actually corresponds to a succession of stable equilibria modified by the source term.)

Heat profiles with a rough time step

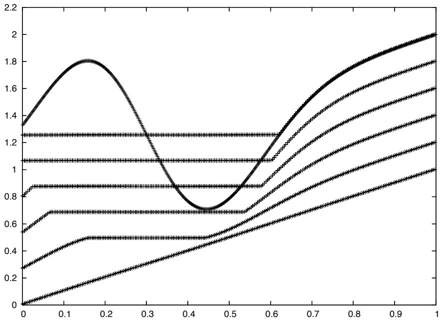
$$\mathbf{G = G(x) = 1 + \exp(-25(x-0.2)^2) - \exp(-20(x-0.4)^2)}$$

$t, x \in [0,1]$ $h = 0.1$ $(= 10$ time steps$)$ 500 grid points in x, heat profile $y(t,x)$ versus x drawn every 2 time steps



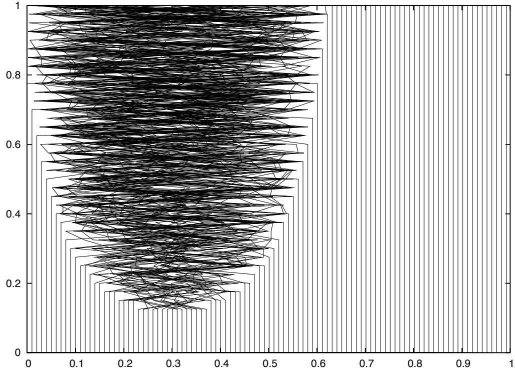
Heat profiles with a fine time step

$$\mathbf{G = G(x) = 1 + \exp(-25(x-0.2)^2) - \exp(-20(x-0.4)^2)}$$
$\mathbf{t, x \in [0,1]}$ $\mathbf{h = 0.005}$ **(= 200 time steps) 500 grid points in x,**
heat profile $\mathbf{y(t,x)}$ versus x drawn every 40 time steps



mixing of the fluid parcels

$\mathbf{t, x \in [0, 1] \quad h = 0.005}$ **500 grid points in x**



Interpretation as a competition model

Model:

N agents (factories, researchers, universities...) in competition,

$\mathbf{x_n(i)} = $ **cumulated production of agent** $\mathbf{i = 1, \cdots, N}$ **at time** $\mathbf{n\underline{h}}$**,**

$\sigma_{\mathbf{n}}(\mathbf{i})$ **rank of agent i at time** $\mathbf{n\underline{h}}$**,**

Model: $\mathbf{x_{n+1}(i) = x_n(i) + h\ G(nh, \sigma_n(i)/N, x_n(i))}$**,**

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Thus the corresponding sorted sequence $y_n = \mathrm{Rearrange}(x_n)$ **satisfies:** $y_{n+1} = \mathrm{Rearrange}(y_n + h\, G)$**, which is just a discrete version of our toy-model.**

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The model means that the production between two different times depends essentially on the ranking. For example $G(x) = 1 - x$, means that the top people slow down their production while the bottom people catch up as fast as possible. It seems that $G(x) = (\sin(1.5\pi x))^2$, for example, is more realistic (bottom people are discouraged while top people get even more competitive!).

Convergence analysis

Theorem

As h $\rightarrow$ 0, the time-discrete scheme has a unique limit y in space $C^0(R_+,\ L^2(D, R^d))$ that satisfies the subdifferential inclusion with convex potential:

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where $C[y] = 0$ if y is non decreasing as a function of x and $C[y] = +\infty$ otherwise.

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where $C[y] = 0$ if y is non decreasing as a function of x and $C[y] = +\infty$ otherwise.
In addition, in the case $G = G(x) = g'(x)$,
the pseudo-inverse $x = u(t, y)$ is an entropy solution to the scalar conservation law

$$\boxed{\partial_t u + \partial_y (g(u)) = 0.}$$

This is an example of the more general L^2 formulation of multidimensional scalar conservation laws, YB ARMA 2009

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$\mathrm{Rearrange}(z)(x) = \nabla p(x)$,
$p(x)$ lsc convex in $x \in R^d$, a.e. differentiable on D, such that

$$\int_D f(\nabla p(x))dx = \int_D f(z(x))dx$$

for all continuous function f such that $|f(x)| \leq 1 + |x|^2$

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for all continuous function f such that $|\mathbf{f(x)}| \leq 1 + |\mathbf{x}|^2$

This is a typical result in optimal transport theory, see YB, CRAS Paris 1987 and CPAM 1991, Smith and Knott, JOTA 1987, cf. Villani, Topics in optimal transportation, AMS, 2003, see also papers, lecture notes and books by Rachev-Rüschendorf, Evans, Caffarelli, Urbas, Gangbo-McCann, Otto, Ambrosio-Savaré, Trudinger-Wang and many others contributions...

Multi-d generalization of the toy-model

Model:

-a smooth bounded domain $\mathbf{x} \in \mathbf{D} \subset \mathbf{R^d}$

-a vector-valued field: $\mathbf{y}(\mathbf{t}, \mathbf{x}) \in \mathbf{R^d}$ (generalized temperature)

-a source term: $\mathbf{G} = \mathbf{G}(\mathbf{t}, \mathbf{x}, \mathbf{y}) \in \mathbf{R^d}$ with bounded derivatives

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as the unique rearrangement with convex potential $y_{n+1} = \nabla p_{n+1}$

Main property of the scheme

Take a smooth function f. Then

$$\int_D f(y_{n+1}(x))dx = \int_D f(y_{n+1/2}(x))dx$$

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$$= \int_D f(y_n(x))dx + h \int_D (\nabla f)(y_n(x)) \cdot G(nh, x, y_n(x))dx \quad + o(h)$$

Convergence of the scheme

Theorem

As $h \to 0$, the time-discrete scheme has converging subsequences.
Each limit y belongs to the space $C^0(R_+,\ L^2(D, R^d))$ and has a convex potential $p(t, \cdot)$ for each $t \geq 0$.

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$$\frac{d}{dt}\int_D f(y(t,x))dx = \int_D (\nabla f)(y(t,x)) \cdot G(t,x,y(t,x))dx$$

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See YB, JNLS 2009. Notice that the system is self-consistent, thanks to the rearrangement theorem. However, our global existence result does not imply stability with respect to initial conditions, except for $d = 1$, where we can use the theory of scalar conservation laws, or $d > 1$ and $G(x) = -x$, where we can use maximal monotone operator theory

Interpretation of the multi-d toy model

The formulation we have obtained for the multidimensional toy model

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for all smooth function f, with $y = \nabla p$,
in some sense means that there exists an underlying divergence-free vector field $v(t,x)$ such that

$$\partial_t y + (v \cdot \nabla)y = G(t,x,y), \qquad \nabla \cdot v = 0, \quad v // \partial D$$

which, continuously in time, rearranges $y(t,x)$ so that y stays a map with a convex potential at any time.

Interpretation of the multi-d toy model

It turns out that the model can be interpreted as a singular limit of the Navier-Stokes Boussinesq equations with vector-valued buoyancy forces. This is what we are now going to explain in the last part of the talk

The NS-Boussinesq model

Let D be a smooth bounded domain $D \subset R^3$ in which moves an incompressible fluid of velocity $v(t,x)$ at $x \in D, \quad t \geq 0$, subject to the Navier-Stokes equations

$$\textbf{NSB} \quad \epsilon^2(\partial_t v + (v \cdot \nabla)v) - \nu \nabla^2 v + \nabla p = y \qquad \nabla \cdot v = 0$$

with $\epsilon, \nu > 0$ and $v = 0$ along ∂D.

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The force field y is subject to the advection equation

$$\partial_t y + (v \cdot \nabla) y = G(t,x,y)$$

where G is a given smooth source term with bounded derivatives.

Remark 1: From the PDE viewpoint, global existence of weak solutions in 3D follows from Leray/Diperna-Lions theory, while global existence of smooth solutions in 2D follows from Hou-Li 2005 and Chae 2006. (See also recent work by Danchin-Paicu.)

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Remark 2: for any suitable test function f, we have INDEPENDTLY of $\epsilon, \mathbf{v}, \nu$

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Remark 3: The presence of $\epsilon << 1$ is equivalent to the action, on a long time interval, of a small source term slowly varying in time and corresponds to the rescaling: $\boxed{G \rightarrow \epsilon G(t\epsilon, x, y), \quad t \rightarrow t/\epsilon}$

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We call Hydrostatic – Boussinesq(HB) the limit equations formally obtained by setting ϵ, ν to zero.

A natural convexity condition for the HB system

The Hydrostatic Boussinesq HB system

$$\mathbf{HB}: \quad \partial_t \mathbf{y} + (\mathbf{v} \cdot \nabla)\mathbf{y} = \mathbf{G}(\mathbf{t}, \mathbf{x}, \mathbf{y}), \quad \nabla \cdot \mathbf{v} = \mathbf{0}, \quad \nabla \mathbf{p} = \mathbf{y}$$

looks strange since there is no direct equation for v.

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Notice that, $(v \cdot \nabla)y = (D_x^2 p \cdot v)$ and $v = \nabla \times A$, for some divergence-free vector potential $A = A(t, x) \in R^3$, when $d = 3$.

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$$\nabla \times (D_x^2 p(t, x) \cdot \nabla \times A) = \nabla \times (G(t, x, \nabla p))$$

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This linear 'magnetostatic' system in A is elliptic whenever p is strongly convex: $0 << D_x^2 p(t, x) << +\infty$

Rigorous derivation of the HB model under strong convexity condiition

Theorem

Assume $D = R^3/Z^3$, (y, p, v) to be a smooth solution of the HB hydrostatic Boussinesq model, with $\boxed{0 << D_x^2 p(t, x) << +\infty}$
Then, as $\nu = \epsilon \to 0$, any Leray solution $(y^\epsilon, p^\epsilon, v^\epsilon)$ to the full NSB Navier-Stokes Boussinesq equations, with same initial condition, converges to (y, p, v).

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Idea of the proof: **Estimate:**

$$\frac{d}{dt}\int_D \{K(t, y^\epsilon(t, x), y(t, x)) + \frac{\epsilon^2}{2}|v^\epsilon - v|^2\}dx$$

$$K(t, y', y) = p^*(t, y') - p^*(t, y) - \nabla p^*(t, y) \cdot (y' - y) \sim |y - y'|^2$$

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where $p^*(t, z) = \sup_{x \in D} x \cdot z - p(t, x)$ is the Legendre-Fenchel transform of p.

Global solutions to the HB system

Under the convexity condition, the HB system just coincides with our multi-d toy model! Thus we conclude:

THEOREM

Assume G to be a smooth function with bounded first derivatives. Let C be the convex cone of all maps $y \in L^2(D, R^3)$ such that $y(x) = \nabla p(x)$ a.e. in D for some CONVEX function p. We say that $(t \to y(t, \cdot)) \in C^0(R_+, L^2(D, R^3))$ valued in the cone C is a solution to the HB system if

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Then, for each $y_0 \in C$, there is always a GLOBAL solution such that $y(t=0,\cdot) = y_0$

Bibliography

1-The 1D toy model

a) convergence to the subdifferential equation in L^2:
YB Methods Appl. Anal. 2004 see also YB Arma 2009 and Bolley, B, Loeper J. Hyp. DE 2005,
b) convergence to Kruzhkov's solutions in L^1:
YB, CRAS 1981 and JDE 1983

2-The HB equations

a) General discussion: YB, JNLS 2009, b) Global existence theory see YB, JNLS 2009, following unpublished note 2002, in the case $G(x) = -x$ and Loeper SIMA 2008 in the case of semigeostrophic equations, namely $G(x) = Jx$, J symplectic
b) Local smooth solutions: G. Loeper 2008 (for semigeostrophic equations)
d) Rigorous derivation of the HB equations
YB and M. Cullen, CMS in press (motivated by semi-geostrophic equations)