Convection, optimal transport, coupled Monge-Ampère systems and magnetic relaxation

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Outline

1. Navier-Sokes Boussinesq equations, their Darcy/Stokes/Hydrostatic limits and the Angenent-Haker-Tannenbaum model in optimal transport theory

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4. A stringy generalization of the DB model leading to a magnetic relaxation model à la Arnold-Moffatt

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3. Coupled Monge-Ampère systems including Hoskins' semi-geostrophic equations and a fully nonlinear chemotaxis model
4. A stringy generalization of the DB model leading to a magnetic relaxation model à la Arnold-Moffatt
5. The cross-Burgers equation: a model of magnetic reversal

The NS-Boussinesq model

Let D be a smooth bounded domain $\mathrm{D} \subset \mathrm{R}^{\mathbf{3}}$ in which moves an incompressible fluid of velocity $\mathrm{v}(\mathrm{t}, \mathrm{x})$ at $\mathrm{x} \in \mathrm{D}, \quad \mathrm{t} \geq 0$, subject to the Navier-Stokes equations

$$\mathbf{NSB} \quad \epsilon(\partial_{\mathbf{t}}\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}) + \mathbf{K}\mathbf{v} + \nabla\mathbf{p} = \mathbf{y} \qquad \nabla \cdot \mathbf{v} = \mathbf{0}$$

where $\boxed{\mathbf{K}\mathbf{v} = \alpha\mathbf{v} - \nu\boldsymbol{\Delta}\mathbf{v}}$ **with $\alpha, \epsilon, \nu > 0$ and $\mathbf{v} = 0$ along $\partial\mathrm{D}$.**

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where $\mathbf{K}\mathbf{v} = \alpha\mathbf{v} - \nu\boldsymbol{\Delta}\mathbf{v}$ **with $\alpha, \epsilon, \nu > 0$ and $\mathrm{v} = 0$ along $\partial\mathrm{D}$.**

The force field $\mathbf{y}$ is subject to the advection equation

$$\partial_{\mathbf{t}}\mathbf{y} + (\mathbf{v} \cdot \nabla)\mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{y})$$

where G is a given smooth function with bounded derivatives.

Classical Convection Theory

Classical Convection Theory corresponds to the special case:

$$\boxed{\mathbf{K} = -\boldsymbol{\Delta}, \quad \mathbf{G} = \mathbf{0}, \quad \mathbf{y}//\mathbf{e_3}, \quad \mathbf{y} = \eta\mathbf{e_3}, \quad \eta = \eta(\mathbf{t}, \mathbf{x}) \in \mathbf{R}}$$

namely:

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$$\boxed{\partial_{\mathbf{t}}\eta + (\mathbf{v} \cdot \nabla)\eta = \mu\boldsymbol{\Delta}\eta}$$

with $\mu \geq \mathbf{0}$**.**

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with $\mu \geq \mathbf{0}$.

For $\mu = \mathbf{0}$, global existence of weak solutions in 3D follows from Leray/Diperna-Lions theory, while global existence of smooth solutions in 2D follows from Hou-Li 2005 and Chae 2006. (See also recent work by Danchin-Paicu.)

Three limits of the NSB model

While keeping unchanged

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and dropping inertia terms, we consider three possible limits:

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Each of these limits, as $\epsilon \to 0$, can be rigorously justified as long as the limit equation admits smooth solutions, YB 2007.
The last case is more subtle and requires an additional CONVEXITY condition that will be discussed below, YB 2008.

The AHT model

In the case $G = 0$, the Darcy-Boussinesq model

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coincides with the Angenent/Haker/Tannenbaum model which was introduced for the numerical resolution of the polar factorization problem arising in optimal transport theory (cf. Y.B. CPAM 1991).

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$$\frac{\mathbf{d}}{\mathbf{dt}}\int_{\mathbf{D}} |\mathbf{y}(\mathbf{t},\mathbf{x}) - \mathbf{x}|^2 \mathbf{dx} = -2\int_{\mathbf{D}} |\mathbf{v}(\mathbf{t},\mathbf{x})|^2\, \mathbf{dx}$$

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$$\frac{d}{dt} \int_D |\mathbf{y}(t,x) - x|^2 dx = -2 \int_D |\mathbf{v}(t,x)|^2 \, dx$$

So, we expect, as $t \to +\infty$, $\mathbf{v} \to 0$, $(\mathbf{y}, \mathbf{p}) \to (\mathbf{y}^\infty, \mathbf{p}^\infty)$, so that

$$\mathbf{y}^\infty = \nabla \mathbf{p}^\infty$$

is a curl-free 'rearrangement' of the given initial vector field $\mathbf{y}_0$.

A convexity condition for the HB model

The Hydrostatic Boussinesq HB system

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Let us consider, for simplicity, the case of 2 space variables $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2})$ and write $\mathbf{v} = (-\partial_1\theta, \partial_2\theta)$, where $\theta(\mathbf{t}, \mathbf{x_1}, \mathbf{x_2}) \in \mathbf{R}$.

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Take the 2D curl of the evolution equation in $\mathbf{y} = (\partial_1\mathbf{p}, \partial_2\mathbf{p})$:

$$\partial_{11}\mathbf{p}\,\partial_{22}\theta + \partial_{22}\mathbf{p}\,\partial_{11}\theta - 2\partial_{12}\mathbf{p}\,\partial_{12}\theta = \partial_1(\mathbf{G_2}) - \partial_2(\mathbf{G_1})$$

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a well posed linear elliptic equation in θ whenever $\mathbf{D^2_x p(t,x) > 0}$

Observables in Boussinesq systems

For each suitable test function f, consider the 'observable'

$$\mathbf{t} \rightarrow \rho_{\mathbf{f}}(\mathbf{t}) = \int_{\mathbf{D}} \mathbf{f}(\mathbf{y}(\mathbf{t},\mathbf{x}))\mathbf{dx}$$

where y is solution to one of the Boussinesq systems
NSB,SB,DB,HB.

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Then, we get

$$\frac{\mathbf{d}}{\mathbf{dt}} \int_{\mathbf{D}} \mathbf{f}(\mathbf{y}(\mathbf{t}, \mathbf{x}))\mathbf{dx} = \int_{\mathbf{D}} (\nabla \mathbf{f})(\mathbf{y}(\mathbf{t}, \mathbf{x})) \cdot \mathbf{G}(\mathbf{x}, \mathbf{y}(\mathbf{t}, \mathbf{x}))\mathbf{dx}$$

since $\partial_{\mathbf{t}} \mathbf{y} + (\mathbf{v} \cdot \nabla)\mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{y})$ where $\nabla \cdot \mathbf{v} = \mathbf{0}, \quad \mathbf{v} // \partial \mathbf{D}.$

Recovery from Observables

In the HB model, the field $\mathbf{y}$ is required to be a gradient $\mathbf{y} = \nabla \mathbf{p}$

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If we a priori assume

$$\boxed{\mathbf{p}(\mathbf{t}, \mathbf{x}) \text{ is a CONVEX function of } \mathbf{x} \in \mathbf{D},}$$

then the field $\mathbf{y}$ is completely determined by the knowledge of all observables

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NB: This is a typical result of OPTIMAL TRANSPORT THEORY

YB, CRAS Paris 1987 and CPAM 1991, Smith and Knott, JOTA 1987, cf. Villani, Topics in optimal transportation, AMS, 2003, see also papers, lecture notes and books by Rachev-Rüschendorf, Evans, Caffarelli, Urbas, Gangbo-McCann, Otto, Ambrosio-Savaré, Villani, Trudinger-Wang and many others contributions...

Global solutions to the HB system

THEOREM (YB 2007, following YB 2002 (unpublished))

Assume $\mathbf{G}$ to be a smooth function with bounded first derivatives. Let $\mathbf{C}$ be the convex cone of all maps $\mathbf{y} \in \mathbf{L^2(D, R^3)}$ such that $\mathbf{y(x) = \nabla p(x)}$ a.e. in $\mathbf{D}$ for some CONVEX function $\mathbf{p}$. We say that $\mathbf{(t \to y(t, \cdot)) \in C^0(R_+, L^2(D, R^3))}$ valued in the cone $\mathbf{C}$ is a solution to the HB system if

$$\frac{\mathbf{d}}{\mathbf{dt}} \int_{\mathbf{D}} \mathbf{f(y(t,x))dx} = \int_{\mathbf{D}} \mathbf{(\nabla f)(y(t,x)) \cdot G(x, y(t,x))dx}, \quad \forall \mathbf{f}$$

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Then, for each $\mathbf{y_0 \in C}$, there is always a GLOBAL solution such that $\mathbf{y(t = 0, \cdot) = y_0}$

Monge-Ampère coupled systems

Under the CONVEXITY assumption, the HB system

$$\mathrm{HB}: \quad \partial_t \mathbf{y} + (\mathbf{v} \cdot \nabla)\mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{y}), \qquad \mathbf{y} = \nabla \mathbf{p}, \qquad \nabla \cdot \mathbf{v} = \mathbf{0}$$

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The coupled MONGE-AMPERE system

$$\partial_{\mathbf{t}}\rho + \nabla\cdot(\rho\mathbf{w}) = \mathbf{0}, \quad \mathbf{w} = \mathbf{G}(\nabla\mathbf{p}^*(\mathbf{t},\mathbf{x}),\mathbf{x}), \quad \rho = \det(\mathbf{D}^2\mathbf{p}^*(\mathbf{t},\mathbf{x}))$$

where p^* is the LEGENDRE transform

$$\mathbf{p}^*(\mathbf{t},\mathbf{x}) = \sup_{\tilde{\mathbf{x}}\in\mathbf{D}} \ \mathbf{x}\cdot\tilde{\mathbf{x}} - \mathbf{p}(\mathbf{t},\tilde{\mathbf{x}})$$

Justification

Use the change of variable

$$\tilde{\mathbf{x}} = \nabla \mathbf{p}^*(\mathbf{t}, \mathbf{x}), \quad \mathbf{d}\tilde{\mathbf{x}} = \mathbf{det}(\mathbf{D^2 p}^*(\mathbf{t}, \mathbf{x}))\mathbf{dx}$$

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Get

$$\int_{\mathbf{D}} \mathbf{f}(\mathbf{y}(\mathbf{t}, \tilde{\mathbf{x}}))\mathbf{d}\tilde{\mathbf{x}} = \int \mathbf{f}(\mathbf{x})\mathbf{det}(\mathbf{D^2 p}^*(\mathbf{t}, \mathbf{x}))\mathbf{dx} = \int \mathbf{f}(\mathbf{x})\rho(\mathbf{t}, \mathbf{x})\mathbf{dx}$$

$$\int_{\mathbf{D}} (\nabla \mathbf{f})(\mathbf{y}(\mathbf{t}, \tilde{\mathbf{x}})) \cdot \mathbf{G}(\tilde{\mathbf{x}}, \mathbf{y}(\mathbf{t}, \tilde{\mathbf{x}}))\mathbf{d}\tilde{\mathbf{x}} = \int (\rho \mathbf{v})(\mathbf{t}, \mathbf{x}) \cdot \nabla \mathbf{f}(\mathbf{x})\mathbf{dx}$$

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Thus

$$\frac{\mathbf{d}}{\mathbf{dt}} \int \mathbf{f}(\mathbf{x})\rho(\mathbf{t}, \mathbf{x})\mathbf{dx} - \int (\rho \mathbf{v})(\mathbf{t}, \mathbf{x}) \cdot \nabla \mathbf{f}(\mathbf{x})\mathbf{dx} =$$

$$= \frac{\mathbf{d}}{\mathbf{dt}} \int_{\mathbf{D}} \mathbf{f}(\mathbf{y}(\mathbf{t}, \tilde{\mathbf{x}}))\mathbf{d}\tilde{\mathbf{x}} - \int_{\mathbf{D}} (\nabla \mathbf{f})(\mathbf{y}(\mathbf{t}, \tilde{\mathbf{x}})) \cdot \mathbf{G}(\tilde{\mathbf{x}}, \mathbf{y}(\mathbf{t}, \tilde{\mathbf{x}}))\mathbf{d}\tilde{\mathbf{x}} = \mathbf{0}$$

Examples of HB systems

Example 1: Setting $\mathrm{G}(\mathrm{x},\mathrm{y}) = (\mathrm{y}_2 - \mathrm{x}_2, \mathrm{x}_1 - \mathrm{y}_1, 0)$ in the HB system, we recover Hoskins' SEMI-GEOSTROPHIC equations.

Then, the CONVEXITY PRINCIPLE EXACTLY corresponds to the CULLEN-PURSER PRINCIPLE.

cf. Cullen-Norbury-Purser 1991, Benamou-Brenier 1998, Cullen-Gangbo 2001, Loeper 2006.

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Example 2: $G(x, y) = (y - x)/\beta$ where $\beta > 0$ is a constant.

Setting $\boxed{\nabla \mathbf{p}^*(\mathbf{t}, \mathbf{x}) = \mathbf{x} - \beta \nabla \psi(\mathbf{t}, \mathbf{x})}$ we get

$$\boxed{\partial_{\mathbf{t}} \rho + \nabla \cdot (\rho \mathbf{w}) = \mathbf{0}, \quad \mathbf{w} = \nabla \psi(\mathbf{t}, \mathbf{x}), \quad \rho = \mathbf{det}(\mathbf{I} - \beta \mathbf{D}^{\mathbf{2}} \psi(\mathbf{t}, \mathbf{x}))}$$

Fully nonlinear Chemotaxis

The resulting system can be seen as a FULLY NON-LINEAR CHEMOTAXIS model.

Indeed, Assuming $|\beta| << 1$, the MONGE-AMPERE equation becomes

$$\rho = \mathbf{det}(\mathbf{I} - \beta \mathbf{D^2}\psi(\mathbf{t}, \mathbf{x})) = \mathbf{1} - \beta \boldsymbol{\Delta}\psi + \mathbf{O}(\beta^{\mathbf{2}})$$

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which approximates the CHEMOTAXIS model (without viscosity) considered by Jäger and Luckhaus Trans. AMS 1992:

$$\partial_{\mathbf{t}}\rho + \nabla \cdot (\rho \mathbf{w}) = \mathbf{0}, \quad \mathbf{w} = \nabla\psi(\mathbf{t}, \mathbf{x}), \quad \rho = \mathbf{1} - \beta \mathbf{\Delta}\psi(\mathbf{t}, \mathbf{x}))$$

Convexity and Entropy conditions

In one space variable the approximation is exact:

$$\partial_t \rho + \partial_x (\rho \mathbf{w}) = \mathbf{0}, \quad \rho = \mathbf{1} - \beta \partial_x \mathbf{w}$$

and can be reduced to the inviscid BURGERS equation

$$\partial_t \mathbf{w} + \partial_x (\mathbf{w}^2/\mathbf{2}) = \frac{\mathbf{w}}{\beta}$$

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It turns out that the Kruzhkov-Oleinik ENTROPY CONDITION is exactly the CONVEXITY CONDITION we used for the HB system.

A Stringy Boussinesq model

A natural generalization of the DB model amounts to consider the set of curves ('strings') valued in $\mathrm{VPM}(\mathrm{D})$ (the set of all volume preserving maps of D)

$$\mathbf{X}: \ \mathbf{s} \in [\mathbf{0}, \mathbf{1}] \to \mathbf{X}(\mathbf{s}, \cdot) \in \mathbf{VPM}(\mathbf{D})$$

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and the corresponding gradient flow for the functional

$$\mathbf{J}[\mathbf{X}] = \frac{\mathbf{1}}{\mathbf{2}} \int_{\mathbf{0}}^{\mathbf{1}} \int_{\mathbf{D}} |\partial_{\mathbf{s}} \mathbf{X}(\mathbf{s}, \mathbf{a})|^{\mathbf{2}} \mathbf{dads}$$

A Stringy Boussinesq model

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The resulting equation reads

$$\partial_{\mathbf{t}} \mathbf{X}(\mathbf{t}, \mathbf{s}, \mathbf{a}) + (\nabla \mathbf{p})(\mathbf{t}, \mathbf{s}, \mathbf{X}(\mathbf{t}, \mathbf{s}, \mathbf{a})) = \partial_{\mathbf{ss}} \mathbf{X}(\mathbf{t}, \mathbf{s}, \mathbf{a})$$

where p is a Lagrange multiplier for the incompressibility constraint.

The stringy DB model

Let us move to Eulerian coordinates by setting

$$\partial_t \mathbf{X}(\mathbf{t},\mathbf{s},\mathbf{a}) = \mathbf{v}(\mathbf{t},\mathbf{s},\mathbf{X}(\mathbf{t},\mathbf{s},\mathbf{a})), \quad \partial_s \mathbf{X}(\mathbf{t},\mathbf{s},\mathbf{a}) = \mathbf{b}(\mathbf{t},\mathbf{s},\mathbf{X}(\mathbf{t},\mathbf{s},\mathbf{a}))$$

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First, we get two differential constraints (since $\mathrm{X}(\mathrm{t},\mathrm{s},\cdot)$ is volume preserving) and a compatibility condition (using $\partial_t \partial_s \mathbf{X} = \partial_s \partial_t \mathbf{X}$):

$$\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{b} = \mathbf{0}\ , \quad \partial_t \mathbf{b} + (\mathbf{v} \cdot \nabla)\mathbf{b} = \partial_s \mathbf{v} + (\mathbf{b} \cdot \nabla)\mathbf{v}$$

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Then, the equation

$$\partial_{\mathbf{t}}\mathbf{X}(\mathbf{t},\mathbf{s},\mathbf{a}) + (\nabla\mathbf{p})(\mathbf{t},\mathbf{s},\mathbf{X}(\mathbf{t},\mathbf{s},\mathbf{a})) = \partial_{\mathbf{ss}}\mathbf{X}(\mathbf{t},\mathbf{s},\mathbf{a})$$

becomes, in Eulerian coordinates, $\mathbf{v} + \nabla\mathbf{p} = \partial_{\mathbf{s}}\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{b}$

Magnetic relaxation

We may interpret the stringy DB model (in Eulerian coordinates)

$$\mathbf{v} + \nabla \mathbf{p} = \partial_{\mathbf{s}} \mathbf{b} + (\mathbf{b} \cdot \nabla) \mathbf{b} \ , \quad \partial_{\mathbf{t}} \mathbf{b} + (\mathbf{v} \cdot \nabla) \mathbf{b} = \partial_{\mathbf{s}} \mathbf{v} + (\mathbf{b} \cdot \nabla) \mathbf{v} \ ,$$

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As $\mathrm{t} \longrightarrow +\infty$, we expect ’equilibrium states’ $(\mathrm{B}, \mathrm{P})(\mathrm{s}, \mathrm{x}) = (\mathrm{b}, \mathrm{p})(\mathrm{t} = \infty, \mathrm{s}, \mathrm{x})$ to solve the Euler equation

$$\nabla \mathbf{P} = \partial_{\mathbf{s}} \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{B}, \quad \nabla \cdot \mathbf{B} = \mathbf{0}$$

The cross-Burgers equation

In the special case when D is the unit ball of $\mathrm{R}^{\mathbf{3}}$, we look for special solutions of the stringy DB model

$$\boxed{\mathbf{X}(\mathbf{t},\mathbf{s},\mathbf{a}) = \mathbf{U}(\mathbf{t},\mathbf{s})\mathbf{a}, \quad \forall \mathbf{a} \in \mathbf{D}}$$

where $\mathrm{U}(\mathrm{t},\mathrm{s})$ is valued in THE ORTHOGONAL GROUP $\mathbf{O}(\mathbf{3})$.

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The cross-Burgers equation

In the special case when D is the unit ball of R^{3}, we look for special solutions of the stringy DB model

$$\mathbf{X}(\mathbf{t},\mathbf{s},\mathbf{a})=\mathbf{U}(\mathbf{t},\mathbf{s})\mathbf{a},\quad \forall \mathbf{a}\in\mathbf{D}$$

where $\mathrm{U}(\mathrm{t},\mathrm{s})$ is valued in THE ORTHOGONAL GROUP $\mathbf{O}(3)$.

We find $\partial_{\mathbf{t}}\mathbf{U}(\mathbf{t},\mathbf{s})+\mathbf{S}(\mathbf{t},\mathbf{s})\mathbf{U}(\mathbf{t},\mathbf{s})=\partial_{\mathbf{ss}}\mathbf{U}(\mathbf{t},\mathbf{s})$ where each $\mathrm{S}(\mathrm{t},\mathrm{s})$ should be a real symmetric 3×3 matrix.

Introducing for each (t,s) the unique vector $\mathbf{B}(\mathbf{t},\mathbf{s})\in\mathbf{R}^{3}$ such that

$$\partial_{\mathbf{s}}\mathbf{U}(\mathbf{t},\mathbf{s})\mathbf{a}=\mathbf{B}(\mathbf{t},\mathbf{s})\times(\mathbf{U}(\mathbf{t},\mathbf{s})\mathbf{a}),\quad \forall \mathbf{a}\in\mathbf{D}$$

we get for $\mathrm{B}(\mathrm{t},\mathrm{s})$ what we call THE CROSS-BURGERS EQUATION

$$\partial_{\mathbf{t}}\mathbf{B}(\mathbf{t},\mathbf{s})+\mathbf{B}(\mathbf{t},\mathbf{s})\times\partial_{\mathbf{s}}\mathbf{B}(\mathbf{t},\mathbf{s})=\partial_{\mathbf{ss}}\mathbf{B}(\mathbf{t},\mathbf{s})$$

Magnetic Reversal

An exact solution of the **CROSS-BURGERS EQUATION** is

$$\mathbf{B}(t,s) = (f(t)\cos(2\pi s), f(t)\sin(2\pi s), g(t))$$

where

$$\frac{df}{dt} = 2\pi(g - 2\pi)f, \quad \frac{dg}{dt} = -2\pi f^2$$

For $g(t=0) > 4\pi, \quad f(t=0) \neq 0$, we can check that $g(t=+\infty) = 4\pi - g(0) < 0, \quad f(t=+\infty) = 0$, even when $|f(t=0)| << 1$.

This looks like a magnetic reversal

Magnetic Reversal 2

$$\mathbf{g(0) = 36.2830925, \;\; f(0) = 0.075, \;\; g(+\infty) = -23.7167}$$

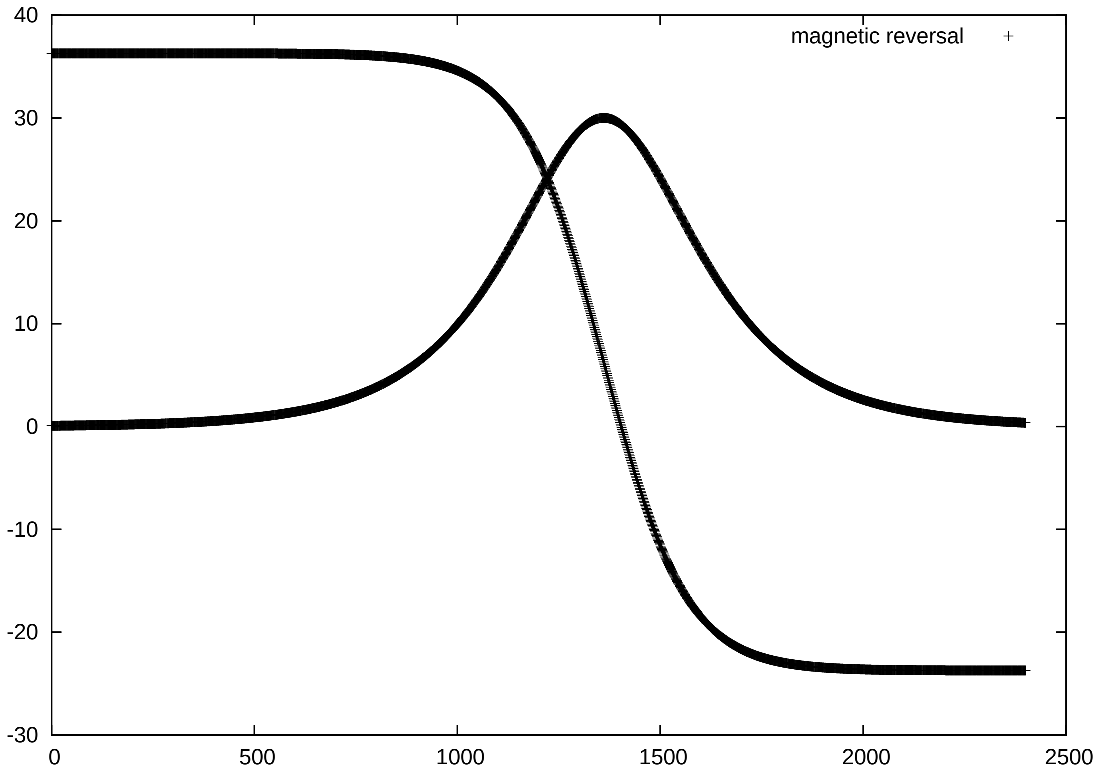

Figure 1: g(t) and f(t) versus t