The Arakawa-Kaneko Zeta Function

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Abstract

We present a very natural generalization of the Arakawa-Kaneko zeta function introduced ten years ago by T. Arakawa and M. Kaneko. We give in particular a new expression of the special values of this function at integral points in terms of modified Bell polynomial. By rewriting Ohno’s sum formula, we are in a position to deduce a new class of relations between Euler sums and the values of zeta.

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1 Introduction

The Arakawa-Kaneko zeta function has been introduced ten years ago by T. Arakawa and M. Kaneko in [1]. Let us recall that this is the function $\xi_k$ defined
for any integer $k \geq 1$ by:

$$
\xi_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{t^{s-1}}{e^t - 1} \text{Li}_k(1 - e^{-t}) \, dt
$$

where $\text{Li}_k$ denotes the $k$-th polylogarithm $\text{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$. The integral converges for $\Re(s) > 0$ and the function $\xi_k$ continues analytically to an entire function of the whole $s$-plane. For $k = 1$, $\xi_1(s)$ is nothing else than $s \zeta(s + 1)$ and for $s = 1$, $\xi_1(1) = \zeta(k + 1)$. In [1], Arakawa and Kaneko have expressed the special values of this function at negative integers with the help of generalized Bernoulli numbers $B_{n}^{(k)}$ called “poly-Bernoulli numbers”. Introduced by M. Kaneko in [5], these numbers are defined by the generating function:

$$
\frac{\text{Li}_k(1 - e^{-z})}{1 - e^{-z}} = \sum_{n=0}^{\infty} B_{n}^{(k)} \frac{z^n}{n!}.
$$

In the case where $k = 1$, one finds again - apart from the sign for $B_{1}^{(1)}$ - the classical Bernoulli numbers. Arakawa and Kaneko also provide in [1] (see their corollary 10) a rather complex expression of the special values of the function at positive integers in terms of MZV (multiple zeta value) but, very soon afterwards, a simpler representation of the values of $\xi_k$ at positive integers in terms of MZSV (multiple zeta-star value) has been obtained by Y. Ohno. More precisely, transforming the original expression given in [1] by means of a duality theorem, Ohno establishes in [7] that:

$$
\xi_{k-1}(m) = \sum_{n_1 \geq n_2 \geq \ldots \geq n_m \geq 1} \frac{1}{n_1 n_2 \ldots n_m m^{-1}} = \zeta^*(k, 1, \ldots, 1),
$$

where $\zeta^*(k_1, k_2, \ldots, k_m)$ refers to the sum:

$$
\sum_{n_1 \geq n_2 \geq \ldots \geq n_m \geq 1} \frac{1}{n_1^{k_1} n_2^{k_2} \ldots n_m^{k_m}}.
$$

Subsequently, this expression has found an important continuation in [8] where Ohno states and proves his remarkable sum formula:

$$
\sum_{m=0}^{k-2} \zeta^*(k - m, 1, \ldots, 1) = 2(k - 1)(1 - 2^{1-k})\zeta(k),
$$

and it is also the subject of an interesting commentary in [6].

In this article, we introduce the more general function $\xi_k(s, x)$ defined for $\Re(s) > 0$ and $x > 0$ by:

$$
\xi_k(s, x) = \frac{1}{\Gamma(s)} \int_0^{+\infty} e^{-xt} \frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} t^{s-1} \, dt
$$
which is a very natural extension of the Arakawa-Kaneko zeta function in the same way as the Hurwitz zeta function $\zeta(s, x)$ generalizes the Riemann zeta function: one has $\xi_k(s, 1) = \xi(s)$ and, in the case where $k = 1$, the function $\xi_1(s, x)$ is nothing else than the classical $s \zeta(s + 1, x)$.

Following the same pattern as in [1], we show that this function $\xi_k(s, x)$ continues analytically in the whole complex $s$-plane as an entire function of $s$, and we express its special values at negative integral points by means of generalized Bernoulli polynomials $B_n^{(k)}(x)$ whose values at 0 are precisely the poly-Bernoulli numbers (cf. theorem 2 and remark 3). In this way, we show that:

$$\xi_k(-m, x) = (-1)^m B_m^{(k)}(x) \quad (m = 0, 1, 2, \ldots).$$

Regarding the special values of $\xi_k(s, x)$ at positive integers, we obtain the following representation (cf. theorem 1):

$$\xi_k(m + 1, x) = \sum_{n=0}^{\infty} \frac{n!}{(n+1)^k x(x+1) \ldots (x+n)} P_m(h_n^{(1)}(x), \ldots, h_n^{(m)}(x)), $$

where $P_m(x_1, \ldots, x_m)$ denotes the m-th modified Bell polynomial defined by the generating function:

$$\exp\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right) = \sum_{m=0}^{\infty} P_m(x_1, \ldots, x_m) t^m$$  \hspace{1cm} (*)

and where:

$$h_n^{(m)}(x) = \sum_{j=0}^{n} \frac{1}{(j+x)^m}. \hspace{1cm} (**)$$

This “Hasse formula” extends the representation already given in [3] in the case $k = 1$ (cf. remark 2). Specializing this expression at $x = 1$, we then deduce a new expression for the values of the Arakawa-Kaneko function at positive integers (cf. corollary 1) from which follows the decomposition:

$$\xi_{k-1}(m + 1) = \zeta^{*}(k, 1, 1, \ldots, 1) = \sum_{n=1}^{\infty} \frac{1}{n^k} P_m(H_n, H_n^{(2)}, \ldots, H_n^{(m)})$$

where $H_n, H_n^{(2)}, \ldots, H_n^{(m)}$ denote the harmonic numbers (cf. corollary 2). This leads us to the rewriting of Ohno’s sum formula in the following form:

$$\sum_{m=1}^{k-3} \sum_{n=1}^{\infty} \frac{1}{n^{k-m}} P_m(H_n, H_n^{(2)}, \ldots, H_n^{(m)}) = [(k - 2) - (k - 1)2^{2-k}] \zeta(k),$$

which defines a new class of relations between Euler sums and the values of zeta (cf. corollary 3 and example 3). This class contains in particular (in the simplest case where $k = 4$), the famous relation $\sum_{n=1}^{\infty} \frac{H_n}{n^3} = \frac{5}{4} \zeta(4)$ whose origin goes back to Euler and Goldbach (cf. [4]).
2 A generalized Arakawa-Kaneko function

Proposition 1. Let

\[ F(s, x) = \frac{1}{\Gamma(s)} \int_0^{+\infty} e^{-xt} f(t) t^{s-1} \, dt \]

be a Laplace-Mellin integral with:

\[ f(t) = \sum_{n=0}^{\infty} a_{n+1} (1 - e^{-t})^n \]

where the coefficients \( a_n \) are supposed to satisfy the condition \( |a_n| = O\left(\frac{1}{n}\right) \). The following properties hold:

1) The integral \( F(s, x) \) converges for \( \Re(s) > 0 \) and \( x > 0 \).

2) If \( m \) is a natural number and \( s = m + 1 \) then:

\[ F(m+1, x) = \sum_{n=0}^{\infty} \frac{n! a_{n+1}}{x(x+1) \ldots (x+n)} P_m(h_n^{(1)}(x), \ldots, h_n^{(m)}(x)) \tag{1} \]

where \( P_m \) and \( h_n^{(m)} \) are respectively given by formulas (⋆) and (⋆⋆).

Proof. By our assumption on \( a_n \), there exists a constant \( C > 0 \) and an integer \( N \geq 1 \) such that for all \( t \geq 0 \):

\[ \sum_{n=N}^{\infty} |a_n|(1 - e^{-t})^{n-1} \leq C \sum_{n=N}^{\infty} \frac{(1 - e^{-t})^{n-1}}{n} \leq C \sum_{n=1}^{\infty} \frac{(1 - e^{-t})^{n-1}}{n} = \frac{Ct}{1 - e^{-t}} \]

which ensures the convergence of the integral and authorizes the inversion of \( \int \) and \( \sum \):

\[ F(s, x) = \sum_{n=0}^{\infty} a_{n+1} \int_0^{+\infty} e^{-xt} (1 - e^{-t})^n t^{s-1} \frac{1}{\Gamma(s)} \, dt. \]

Then, formula (1) results from the following lemma 1. \( \square \)

Lemma 1. For \( x > t \), one has:

\[ \int_0^{+\infty} e^{-xt} (1 - e^{-t})^n e^{t \xi} \, d\xi = \frac{n!}{x(x+1) \ldots (x+n)} \times \exp\left( \sum_{m=1}^{\infty} h_n^{(m)}(x) \frac{t^m}{m!} \right). \]

Proof. For \( a > 0 \) and \( b > 0 \), let us start from the classical Euler’s relation:

\[ B(a, b) = \int_0^1 u^{a-1} (1-u)^{b-1} \, du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \]
Putting: \( u = e^{-\xi}, a = x - t \) and \( b = n + 1 \), one deduces:

\[
\int_{0}^{+\infty} e^{-xt}(1 - e^{-\xi})^n e^{\xi t} \, d\xi = \frac{n!}{(x-t)(1+x-t) \ldots (n+x-t)}.
\]

Moreover, one has:

\[
\frac{n!}{(x-t)(1+x-t) \ldots (n+x-t)} = \frac{n!}{x(x+1) \ldots (x+n)} \times \prod_{k=0}^{n} (1 - \frac{t}{k+x})^{-1}
\]

\[
= \frac{n!}{x(x+1) \ldots (x+n)} \times \exp\left(-\sum_{k=0}^{n} \ln(1 - \frac{t}{k+x})\right)
\]

\[
= \frac{n!}{x(x+1) \ldots (x+n)} \times \exp\left(\sum_{k=0}^{n} \sum_{m=1}^{\infty} \frac{t^m}{m(x+k)^m}\right)
\]

\[
= \frac{n!}{x(x+1) \ldots (x+n)} \times \exp\left(\sum_{m=1}^{\infty} h_n^{(m)}(x) \frac{t^m}{m}\right).
\]

Applying now the previous proposition 1 with \( a_{n+1} = \frac{1}{(n+1)^k} \), one has:

\[
f(t) = \sum_{n=0}^{\infty} \frac{(1 - e^{-t})^n}{(n+1)^k} = \frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}}
\]

and we immediately obtain the following theorem:

**Theorem 1.** Let \( k \) an integer \( \geq 1 \). The Laplace-Mellin integral:

\[
\xi_k(s, x) = \frac{1}{\Gamma(s)} \int_{0}^{+\infty} e^{-xt} \text{Li}_k(1 - e^{-t}) \frac{t^{s-1}}{1 - e^{-t}} \, dt
\]

converges for \( \Re(s) > 0 \) and \( x > 0 \). Moreover one has:

\[
\xi_k(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^k} \int_{0}^{+\infty} e^{-xt}(1 - e^{-t})^n \frac{t^{s-1}}{\Gamma(s)} \, dt.
\]

In particular, if \( m \) is a natural number and \( s = m+1 \), then:

\[
\xi_k(m+1, x) = \sum_{n=0}^{\infty} \frac{n!}{(n+1)^k x(x+1) \ldots (x+n)} P_m(h_n^{(1)}(x), \ldots, h_n^{(m)}(x)).
\]

**Remark 1.** In the case \( x = 1 \), one has:

\[
\xi_k(s, 1) = \frac{1}{\Gamma(s)} \int_{0}^{+\infty} e^{-t} \frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} t^{s-1} \, dt = \frac{1}{\Gamma(s)} \int_{0}^{+\infty} \frac{t^{s-1}}{e^t - 1} \text{Li}_k(1 - e^{-t}) \, dt = \xi_k(s).
\]

Thus, in this case, one finds again the original Arakawa-Kaneko zeta function introduced in [1].

5
Remark 2. In the case $k = 1$, one has: 
$$\frac{\text{Li}_1(1-e^{-t})}{1-e^{-t}} = \frac{t}{1-e^{-t}}$$ 
from which follows that:

$$\xi_1(s-1,x) = \int_0^{+\infty} e^{-xt}(\frac{t}{1-e^{-t}}) \frac{t^{s-2}}{\Gamma(s-1)} dt = (s-1)\zeta(s,x).$$

Thus, in this case, identity (2) is nothing else than the representation for the values of the Hurwitz zeta function (called Hasse formula) given in [3]:

$$(m+1)\zeta(m+2,x) = \sum_{n=0}^{\infty} \frac{n!}{(n+1)x(x+1)\ldots(x+n)} P_n(h_n^{(1)}(x),\ldots,h_n^{(m)}(x)).$$

Example 1.

$$\xi_k(1,x) = \sum_{n=0}^{\infty} \frac{n!}{(n+1)^kx(x+1)\ldots(x+n)};$$
$$\xi_k(2,x) = \sum_{n=0}^{\infty} \frac{n!}{(n+1)^kx(x+1)\ldots(x+n)} \sum_{i=0}^{n} \frac{1}{i+x};$$
$$2\xi_k(3,x) = \sum_{n=0}^{\infty} \frac{n!}{(n+1)^kx(x+1)\ldots(x+n)} \left[ (\sum_{i=0}^{n} \frac{1}{i+x})^2 + \sum_{i=0}^{n} \frac{1}{(i+x)^2} \right];$$
$$6\xi_k(4,x) = \sum_{n=0}^{\infty} \frac{n!}{(n+1)^kx(x+1)\ldots(x+n)} \times \left[ (\sum_{i=0}^{n} \frac{1}{i+x})^3 + 3 \sum_{i=0}^{n} \frac{1}{(i+x)^2} \sum_{i=0}^{n} \frac{1}{(i+x)^2} + 2 \sum_{i=0}^{n} \frac{1}{(i+x)^3} \right].$$

Theorem 2. For all integer $k \geq 1$ et real $x > 0$, the function $s \mapsto \xi_k(s,x)$ analytically continues to the whole complex $s$-plane and its values at negative integers are given by:

$$\xi_k(-m,x) = (-1)^m B_m^{(k)}(x) \quad (m = 0, 1, 2, \ldots)$$

(3)

where $B_m^{(k)}(x)$ is defined by the generating function:

$$e^{-xz} \frac{\text{Li}_k(1-e^{-z})}{1-e^{-z}} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{z^n}{n!}.$$ 

Proof. We apply the classical method to analytically continue a function defined as a Mellin transform (cf. [2]). One splits up $\xi_k(s,x)$ as the sum of two integrals:

$$\xi_k(s,x) = \frac{1}{\Gamma(s)} \int_0^1 e^{-xt} \frac{\text{Li}_k(1-e^{-t})}{1-e^{-t}} t^{s-1} dt + \frac{1}{\Gamma(s)} \int_1^{+\infty} e^{-xt} \frac{\text{Li}_k(1-e^{-t})}{1-e^{-t}} t^{s-1} dt.$$ 

6
The second integral converges absolutely for all \( s \in \mathbb{C} \) and \( x > 0 \) and cancels at negative integers (because \( \frac{1}{\Gamma(-m)} = 0 \) for \( m = 0, 1, 2, \ldots \)). For \( \Re(s) > 0 \), the first integral may be written:

\[
\frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{B_n^{(k)}(x)}{n!} \times \frac{1}{n + s},
\]

from which follows that:

\[
\lim_{s \to -m} \xi_k(s, x) = \left( \lim_{s \to -m} \frac{1}{\Gamma(s)(m + s)} \right) \frac{B_m^{(k)}(x)}{m!} = (-1)^m B_m^{(k)}(x).
\]

\[\blacksquare\]

**Remark 3.** From the generating function which define them, the polynomials \( B_n^{(k)}(x) \) are given by:

\[
B_n^{(k)}(x) = \sum_{q=0}^{n} (-1)^{n-q} \binom{n}{q} B_q^{(k)} x^{n-q}
\]

where \( B_n^{(k)} = B_n^{(k)}(0) \) are the poly-Bernoulli number introduced by Kaneko in [5]. In particular, specializing identity (3) at \( x = 1 \), one finds again:

\[
\xi_k(-m, 1) = (-1)^m B_m^{(k)}(1) = \sum_{q=0}^{m} (-1)^q \binom{m}{q} B_q^{(k)}
\]

which is nothing else than the expression given by Arakawa and Kaneko (cf. [1], theorem 6).

3 **New expression of \( \xi_k(m + 1) \)**

Specializing identity (2) at \( x = 1 \), one obtains the

**Corollary 1.** For all natural numbers \( m \geq 0 \) and integers \( k \geq 1 \), one has:

\[
\xi_k(m + 1) = \sum_{n=1}^{\infty} \frac{1}{n^{k+1}} P_m(H_n, H_n^{(2)}, \ldots, H_n^{(m)}) \quad \text{with} \quad H_n^{(m)} = \sum_{j=1}^{n} \frac{1}{j^m}.
\]  

(4)

**Example 2.**

\[
\xi_k(1) = \zeta(k + 1);
\]

\[
\xi_k(2) = \sum_{n=1}^{\infty} \frac{H_n}{n^{k+1}};
\]
\[ \xi_k(3) = \frac{1}{2} \left( \sum_{n=1}^{\infty} \frac{(H_n)^2}{n^{k+1}} + \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^{k+1}} \right) ; \]

\[ \xi_k(4) = \frac{1}{6} \left( \sum_{n=1}^{\infty} \frac{(H_n)^3}{n^{k+1}} + 3 \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^{k+1}} + 2 \sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^{k+1}} \right) ; \]

\[ \xi_k(5) = \frac{1}{24} \left[ \sum_{n=1}^{\infty} \frac{(H_n)^4}{n^{k+1}} + 6 \sum_{n=1}^{\infty} \frac{(H_n)^2 H_n^{(2)}}{n^{k+1}} + 3 \sum_{n=1}^{\infty} \frac{(H_n^{(2)})^2}{n^{k+1}} + 8 \sum_{n=1}^{\infty} \frac{H_n H_n^{(3)}}{n^{k+1}} + 6 \sum_{n=1}^{\infty} \frac{H_n^{(4)}}{n^{k+1}} \right] . \]

4 Rewriting of Ohno’s sum formula and application to Euler sums

From the comparison of (4) with the expression given by Ohno (cf. [7] theorem 2 and [6] paragraph 2):

\[ \xi_{k-1}(m) = \zeta^*(k; 1, \ldots, 1) := \sum_{m-1}^{\infty} \frac{1}{n_1 n_2 \cdots n_m} , \]

one can immediately deduce the following decomposition:

**Corollary 2.** For all natural number \( m \geq 0 \) and integer \( k \geq 2 \),

\[ \zeta^*(k; 1, \ldots, 1) = \sum_{n=1}^{\infty} \frac{1}{n^k} P_m(H_n, H_n^{(2)}, \ldots, H_n^{(m)}) . \]  

(5)

Rewriting now Ohno’s sum formula (cf. [8], theorem 8):

\[ \sum_{m=0}^{k-2} \zeta^*(k-m; 1, \ldots, 1) = 2(k-1)(1-2^{1-k})\zeta(k) , \]

thanks to the preceding decomposition (5), and taking into account that \( \xi_1(k-1) + \xi_{k-1}(1) = k\zeta(k) \), one obtains the following formula which defines a new class of relations between Euler Sums and the zeta values:

**Corollary 3.** For all integers \( k \geq 4 \),

\[ \sum_{m=1}^{k-3} \sum_{n=1}^{\infty} \frac{1}{n^{k-m}} P_m(H_n, H_n^{(2)}, \ldots, H_n^{(m)}) = [(k-2) - (k-1)2^{2-k}]\zeta(k) \]  

(6)
Example 3.
\[ \sum_{n=1}^{\infty} \frac{H_n}{n^3} = \frac{5}{4} \zeta(4) \quad \text{(Euler and Goldbach)}; \]
\[ \sum_{n=1}^{\infty} \frac{H_n}{n^4} + \frac{1}{2} \left[ \sum_{n=1}^{\infty} \frac{(H_n)^2}{n^3} + \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^3} \right] = \frac{5}{2} \zeta(5); \]

\[ \sum_{n=1}^{\infty} \frac{H_n}{n^5} + \frac{1}{2} \left[ \sum_{n=1}^{\infty} \frac{(H_n)^2}{n^4} + \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^4} \right] + \frac{1}{6} \left[ \sum_{n=1}^{\infty} \frac{(H_n)^3}{n^3} + 3 \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^3} + 2 \sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^3} \right] \]
\[ = \frac{59}{16} \zeta(6); \]

\[ \sum_{n=1}^{\infty} \frac{H_n}{n^6} + \frac{1}{2} \left[ \sum_{n=1}^{\infty} \frac{(H_n)^2}{n^5} + \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^5} \right] + \frac{1}{6} \left[ \sum_{n=1}^{\infty} \frac{(H_n)^3}{n^4} + 3 \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4} + 2 \sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^4} \right] \]
\[ + \frac{1}{24} \left[ \sum_{n=1}^{\infty} \frac{(H_n)^4}{n^3} + 6 \sum_{n=1}^{\infty} \frac{(H_n)^2 H_n^{(2)}}{n^3} + 3 \sum_{n=1}^{\infty} \frac{(H_n^{(2)})^2}{n^3} + 8 \sum_{n=1}^{\infty} \frac{H_n H_n^{(3)}}{n^3} + 6 \sum_{n=1}^{\infty} \frac{H_n^{(4)}}{n^3} \right] \]
\[ = \frac{77}{16} \zeta(7). \]

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