# Batalin-Vilkovisky Gauge-Fixing via Homological Perturbation Theory

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# Lagrangian Data

- Space of fields: $\mathcal{A}$

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- Action: $S_0 \in C^\infty_{loc}(\mathcal{A})$

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- Formal measure: $\Omega_0$

$$\Omega_0 = \prod_{a,\mu,x} dA^a_\mu(x) \,.$$

# Symmetry

$$P \subset \Gamma T\mathcal{A} \quad \text{linear subspace s.t., for all } \mathbf{X},\ \mathbf{Y} \in P,$$

$$\mathbf{X}(S_0) = 0\,, \quad \mathrm{div}_{\Omega_0}(\mathbf{X}) = 0\,,$$

$$[\mathbf{X}, \mathbf{Y}] = T(\mathbf{X}, \mathbf{Y}) + dS_0 \neg E(\mathbf{X}, \mathbf{Y})\,, T(\mathbf{X}, \mathbf{Y}) \in P\,, E(\mathbf{X}, \mathbf{Y}) \in \Gamma\Lambda^2 T\mathcal{A}\,.$$

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$$\Omega^0(\mathbb{M}^4, \mathbf{g}) \to P\,,$$
$$\epsilon \mapsto \mathbf{X}_\epsilon\,,$$
$$\mathbf{X}_\epsilon(A) = d\epsilon + [A, \epsilon]\,,$$
$$[\mathbf{X}_\epsilon, \mathbf{X}_{\epsilon'}] = \mathbf{X}_{[\epsilon, \epsilon']} \quad \text{"closed symmetry".}$$

# Quantization and Gauge Fixing

For any observable, $\mathcal{O} \in C^\infty_{loc}(\mathcal{A})^P$ ($\mathbf{X}(\mathcal{O}) = 0$, for $\mathbf{X} \in P$), define the expectation value

$$\langle \mathcal{O} \rangle = \frac{\int\limits_{\mathcal{A}} \mathcal{O}[A] e^{iS_0[A]/\hbar} \Omega_0}{\int\limits_{\mathcal{A}} e^{iS_0[A]/\hbar} \Omega_0} .$$

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Due to non-compact gauge orbits, numerator and denominator diverge.
Idea: "Divide numerator and denominator by the volume of a typical gauge fibre."

# Example

$$\langle\mathcal{O}\rangle = \frac{\int_{\mathcal{A}} \mathcal{O}[A] e^{iS_0[A]/\hbar}\Omega_0}{\int_{\mathcal{A}} e^{iS_0[A]/\hbar}\Omega_0}$$

$$= \frac{\int_{\mathcal{A}} \mathcal{O}[A] e^{iS_0[A]/\hbar}\delta(G(x))\det(\delta G(A))\Omega_0}{\int_{\mathcal{A}} e^{iS_0[A]/\hbar}\delta(G(x))\det(\delta G(A))\Omega_0} \quad e.g. G(A) = \partial^\mu A_\mu(x)$$

$$= \frac{\int_{\hat{\mathcal{A}}} \mathcal{O}[A] e^{i(S_0[A]+\int_{\mathbb{M}^4}\lambda(x)G(x)+\int_{\mathbb{M}^4}\bar{\beta}(x)(\delta G(x))\beta(x))/\hbar}\Omega_{\hat{\mathcal{A}}}}{\int_{\hat{\mathcal{A}}} e^{i(S_0[A]+\int_{\mathbb{M}^4}\lambda(x)G(x)+\int_{\mathbb{M}^4}\bar{\beta}(x)(\delta G(x))\beta(x))/\hbar}\Omega_{\hat{\mathcal{A}}}},$$

$$\hat{\mathcal{A}} = \Omega^1(\mathbb{M}^4, \mathbf{g}) \oplus \Omega^4(\mathbb{M}^4, \mathbf{g}^*) \oplus \Omega^0(\mathbb{M}^4, \mathbf{g}[1]) \oplus \Omega^4(\mathbb{M}^4, \mathbf{g}^*[-1]).$$

Number in brackets: *ghost degree*.

# Example

Gauge-fixed action can be written as

$$S_{gf}(A) = S_0(A) + \mathcal{X}(\Psi)(A)\,,$$

with

$$\Psi(A) = \int_{\mathbb{M}^4} \bar{\beta}(x) G(A)(x) \quad \text{(gauge-fixing fermion)}\,,$$

and $\mathcal{X}$ an extension of the gauge-symmetry to $\hat{\mathcal{A}}$, s.t. $\mathcal{X}^2 = 0$:

$$\mathcal{X}(A) = d\beta + [A, \beta]\,, \qquad \mathcal{X}(\beta) = -\frac{1}{2}[\beta, \beta]\,,$$
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Hence, gauge-fixed action remembers gauge-symmetry:

$$\mathcal{X}(S_{gf}) = \mathcal{X}(S_0) + \mathcal{X}^2(\Psi) = 0\,.$$

# Gauge-fixing for closed symmetries (Becchi, Rouet, Stora, Tyutin (BRST))

Introduce auxiliary fields (Lagrange multipliers and ghosts), $\mathcal{A} \rightsquigarrow \hat{\mathcal{A}}$, such that symmetry extends to a cohomological vector field $\mathcal{X}$ on $\hat{\mathcal{A}}$, i.e., $\mathcal{X}^2 = 0$.

Gauge-fixing is associated with the choice of a suitable representative in the $\mathcal{X}$-cohomology class of $S_0$:

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Expectation values of gauge-invariant functions are invariant under variations of $\Psi$ $\Leftrightarrow$ Expectation value is defined on $\mathcal{X}$-cohomology classes (Observables).

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Mathematically, this means to extend $\hat{\mathcal{A}}$ to $T^*[1]\hat{\mathcal{A}}$, which acquires an odd symplectic structure

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The BRST-operator is encoded in extension of the action

$$S = S_0 + S_1 \,, \quad S_1 = z_i^\dagger \mathcal{X}(z^i) \,.$$

Then,

$$\mathcal{X}(f) = \{S_1, f\} = (-1)^{|z^i|} \left( \frac{S_1 \overleftarrow{\partial}}{\partial z^i} \frac{\overrightarrow{\partial} f}{\partial z_i^\dagger} - \frac{S_1 \overleftarrow{\partial}}{\partial z_i^\dagger} \frac{\overrightarrow{\partial} f}{\partial z^i} \right) \,.$$

# Antifield-Formalism (Zinn-Justin)

Eq. $\mathcal{X}^2 = 0$ reads

$$\{S_1, S_1\} = 0\,,$$

and due to $\mathcal{X}(S_0) = 0$ we get, in addition,

$$\{S, S\} = \{S_1, S_1\} + 2\{S_1, S_0\} = 0 \quad \text{(Classical Master Eq. (CME))}\,.$$

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The gauge-fixing, $\Psi$, is encoded in the Lagrangian submanifold

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We have

$$S_{|\mathcal{L}} = S_0 + z_i^\dagger \mathcal{X}(z^i)_{|\mathcal{L}} = S_0 + \mathcal{X}(\Psi) = S_{gf}\,,$$

and, hence,

$$\int_{\hat{\mathcal{A}}} e^{i(S_0 + \mathcal{X}\Psi)/\hbar} \Omega_{\hat{\mathcal{A}}} = \int_{\mathcal{L}} e^{iS/\hbar} \Omega_{\mathcal{L}}\,.$$

# Batalin-Vilkovisky formalism

A new differential appears

$$\delta_0 := \{S_0, .\} \quad \text{Koszul-Tate differential}\,;$$

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Open symmetries cannot be encoded in a ghost-term $S_1$ that satisfies $\{S_1, S_1\} = 0$ (BRST-operator), but often we find a ghost extension of the action, $S = S_0 + S'$, that satisfies $\{S, S\} = 0$ (Classical Master Eq. (CME)).

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With $\delta' := \{S', .\}$, this reads

$$(\delta_0 + \delta')^2 = \delta'^2 + \delta_0\delta' + \delta'\delta_0 = 0.$$

# Batalin-Vilkovisky formalism

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$$[\mathbf{X}, \mathbf{Y}] = T(\mathbf{X}, \mathbf{Y}) + dS_0 \neg E(\mathbf{X}, \mathbf{Y})\,.$$

Hence, in the open case, $\delta'$ is not a differential, but a *perturbation* of $\delta_0$: $(\delta_0 + \delta')^2 = 0$ ($\rightsquigarrow$ HPT).

# Batalin-Vilkovisky formalism

The cohomology of $\{S, .\}$ contains the classical observables (functions on phase space) at degree 0 and anomalies at degree one.

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Need to incorporate quantum effects!

The expectation value

$$\langle \mathcal{O} \rangle = Z^{-1} \int_{\mathcal{L}} \mathcal{O} e^{iS/\hbar} \Omega_{\mathcal{L}}$$

is defined on cohomology classes of the differential

$$\delta_{BV} = \{S, .\} - i\hbar \Delta_\Omega \,,$$

where the differential

$$\Delta_\Omega = \frac{\partial^2}{\partial z^i \partial z_i^\dagger}$$

depends on a volume form $\Omega$ on the body of $T^*[1]\hat{\mathcal{A}}$.

# Batalin-Vilkovisky formalism

The property $\delta_{BV}^2 = 0$ reads

$$\frac{1}{2}\{S,S\} - i\hbar\Delta_\Omega S = 0 \quad \text{(Quantum Master Eq. (QME))}\,.$$

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Hence, $\delta_{BV}$ is a perturbation of $\delta_0$ (shell), by $\delta_1$ (symmetry) and $\hbar\Delta_\Omega$ (quantum effects).

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Even the geometry behind BV-integrals and the QME becomes clear if one considers $\omega\wedge$ as a perturbation of $d$.

# Homological Perturbation Theory (HPT)

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- $(M, d_M)$ and $(N, d_N)$ are differential graded modules
- $\iota$ and $p$ are chain maps, i.e.,

$$d_M \circ \iota = \iota \circ d_N \, , \quad d_N \circ p = p \circ d_M \, ,$$

and

$$p \circ \iota = id_N \, .$$

Hence, $p$ is a surjection and $\iota$ an injection.

# Homological Perturbation Theory (HPT)

$$(N, d_N) \underset{p}{\overset{\iota}{\rightleftarrows}} (M, d_M) \circlearrowleft h \,.$$

- $h$ is a morphism of degree $-1$ such that

$$\iota \circ p - id_M = hd_M + d_M h \,.$$

Hence, $H(d_M) \cong H(d_N)$, since

$$p\alpha = 0 \,, \quad d_M\alpha = 0 \quad \Rightarrow \quad -\alpha = d_M \circ h(\alpha) \,.$$

Therefore, $h$ is called *homotopy operator*.

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  Therefore, $h$ is called *homotopy operator*.
- Side conditions (can always be satisfied)

  $$h^2 = h \circ \iota = p \circ h = 0\,.$$

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## Theorem (Perturbation Lemma)

*Given a contraction*

$$(N, d_N) \underset{p}{\overset{\iota}{\rightleftarrows}} (M, d_M) \circlearrowleft h$$

*and a perturbation, $\delta$, then, formally, there is a new contraction*

$$(N, d_N + \tilde{\delta}) \underset{\tilde{p}}{\overset{\tilde{\iota}}{\rightleftarrows}} (M, d_M + \delta) \circlearrowleft \tilde{h}\,,$$

$$\tilde{\delta} = \sum_{n \geq 0} p\delta(h\delta)^n \iota\,, \quad \tilde{\iota} = \sum_{n \geq 0} (h\delta)^n \iota\,, \quad \tilde{p} = \sum_{n \geq 0} p(\delta h)^n\,, \quad \tilde{h} = \sum_{n \geq 0} (h\delta)^n h\,.$$

# Open Symmetries

Open symmetry vector fields $\mathbf{X}, \mathbf{Y} \in P$ satisfy

$$\mathbf{X}(S_0) = 0\,, \quad [\mathbf{X}, \mathbf{Y}] = T(\mathbf{X}, \mathbf{Y}) + dS_0 \neg E(\mathbf{X}, \mathbf{Y})\,.$$

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$$\delta_0 S_1 = 0\,, \quad \{S_1, S_1\} = \delta_0\text{-exact}\,.$$

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Choose a contraction onto the cohomology of $\delta_0 := \{S_0, .\}$ (shell):

$$(C^\infty(T^*[1]\hat{\Sigma}), 0) \underset{p}{\overset{\iota}{\rightleftarrows}} (C^\infty(\mathcal{E}), \delta_0) \circlearrowleft h\,, \qquad \mathcal{E} := T^*[1]\hat{\mathcal{A}}\,.$$

Then, $S_1$ defines a differential on $C^\infty(T^*[1]\hat{\Sigma})$:

$$\delta_{BRST} := \{p(S_1), .\}\,.$$

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Want to use the perturbation lemma in order to find a differential on $C^\infty(\mathcal{E})$, i.e., terms of higher order in the anti-fields $S = S_0 + S_1 + S_2 + \dots$, s.t. the CME $\{S, S\} = 0$ is satisfied.

# Open Symmetries

## Theorem

*If* $H^1(\delta_{BRST}) = 0$*, there is a formal solution*

$$S = \sum_{m \geq 0} S_m$$

*of the CME*

$$\{S, S\} = 0\,.$$

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*of the CME*

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*Idea of proof:* CME is a MCE:

$$\delta_0 \left( \sum_{m \geq 1} S_m \right) + \frac{1}{2} \left\{ \sum_{m \geq 1} S_m, \sum_{m \geq 1} S_m \right\} = 0\,.$$

# Open Symmetries

Consider $\{.,.\}$ as a perturbation of $\delta_0$ and use the perturbation lemma in order to get an $L_\infty$-structure on $C^\infty(T^*[1]\hat{\Sigma})$:

$$(\bigwedge_{\mathbb{R}}(C^\infty(T^*[1]\hat{\Sigma}))[1], \{.,.\} + \dots) \underset{\tilde{p}}{\overset{\tilde{\imath}}{\rightleftarrows}} (\bigwedge_{\mathbb{R}}(C^\infty(\mathcal{E}))[1], \delta_0 + \{.,.\}) \circlearrowleft \tilde{h}\,.$$

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We know already that $\{pS_1, pS_1\} = 0$. Some effort is needed to show that contraction can be chosen in such a way that higher order brackets vanish when filled with $pS_1$'s (and that's where obstruction shows up).

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  is defined on $C^{\infty}(T^*[1]\mathcal{A})$.
- Its cohomology is given by

$$H(\delta_0) = C^{\infty}(T^*[1]\mathcal{A}_0)\,,$$

  where $\mathcal{A}_0$ is the locus of stationary points of $S_0$ (zero modes).

# Effective Actions

Choose a contraction

$$(Poly(T^*[1]\mathcal{A}_0), 0) \underset{p}{\overset{\iota}{\rightleftarrows}} (Poly(T^*[1]\mathcal{A}), \delta_0) \circlearrowleft \quad h\,.$$

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- $h$, acting on $T^*[1]\mathcal{A}''$, reads

$$h(f) = \begin{cases} \frac{1}{k(f)} \left\{ \int \frac{1}{2} \phi^\dagger_\mu G^{\mu\nu} \phi^\dagger_\nu, f \right\}, & k \neq 0 \\ 0\,, & k = 0 \end{cases},$$

with $G$ denoting the Green's function of $D$ and

$$k\left( \int f^{\mu_1 \dots \mu_s}_{\nu_1 \dots \nu_t} \phi^\dagger_{\mu_1} \dots \phi^\dagger_{\mu_s} \phi^{\nu_1} \dots \phi^{\nu_t} \right) = s + t\,.$$

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- $h$ is called the propagator.

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- $\hbar\Delta_\Omega$ , where $\Omega$ is a formal measure on $\mathcal{A}$.

# Effective Actions

## Lemma

*The perturbed contraction reads*

$$(Poly(T^*[1]\mathcal{A}_0), \{W, .\} + \hbar\Delta_{\Omega_0}) \underset{\tilde{p}}{\overset{\tilde{\iota}}{\rightleftarrows}} (Poly(T^*[1]\mathcal{A}), \{S, .\} + \hbar\Delta_\Omega) \circlearrowright \tilde{h}\,,$$

*with*

$$e^{iW/\hbar} = Z^{-1} \int\limits_{\mathcal{A}''} e^{i(S_0+S_1)/\hbar}[\Omega''], \quad Z = \int\limits_{\mathcal{A}''} e^{iS_0/\hbar}[\Omega''],$$

$$\Delta_{\Omega_0} := p\Delta_\Omega \iota\,, \quad \Omega = p^*(\Omega_0) \wedge \Omega''\,,$$

$$\tilde{p}(f) = e^{-iW/\hbar} Z^{-1} \int\limits_{\mathcal{A}''} e^{i(S_0+S_1)/\hbar} f[\Omega''].$$

# Effective Actions

*Key step of proof:*

$$\tilde{p}(f) = \sum_{n \geq 0} p\left((\delta_1 + \hbar\Delta_\Omega)h\right)^n f = e^{-\hbar^{-1}W} Z^{-1} \int_{\mathcal{A}} e^{\hbar^{-1}(S_0+S_1)} f[\Omega''] \,.$$

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and we have two differentials on $\Omega^\bullet(\mathcal{E})$ that are perturbations of one another:

$$(\omega \wedge + d)^2 = d^2 + \omega \wedge \omega \wedge + (d\omega)\wedge = 0\,.$$

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$$(\omega \wedge +d)^2 = d^2 + \omega \wedge \omega \wedge +(d\omega)\wedge = 0\,.$$
- Two contractions:
$$(H(\omega\wedge), \Delta) \underset{\twoheadleftarrow}{\overset{\rightarrow}{\rightleftarrows}} (\Omega^\bullet(\mathcal{E}), \omega \wedge +d) \underset{\twoheadleftarrow}{\overset{\rightarrow}{\rightleftarrows}} (\Omega^\bullet(E), d)\,, \qquad (1)$$
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$$(H(\omega\wedge), \Delta) \rightleftarrows (\Omega^\bullet(\mathcal{E}), \omega \wedge + d) \rightleftarrows (\Omega^\bullet(E), d)\,, \qquad (1)$$
where $E$ is the body of $\mathcal{E}$.
- The ordinary integration theory on the r.h.s. defines the *BV integration theory* on the l.h.s.

# The Geometry behind BV

- Via the isomorphism $F^\pi : H(\omega\wedge) \to \Omega^\bullet(E)$, which depends on a projection $\pi : \mathcal{E} \to E$, we define

$$\int_{\mathcal{L}} [s] := \int_{L} F^\pi([s]) \,,$$

where $L$ is the body of the Lagrangian submanifold $\mathcal{L}$ and $\pi(\mathcal{L}) = L$. Elements of $H(\omega\wedge)$ are called *semidensities*.

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- Any semidensity can be written in the form

$$[s] = [f\pi^*(\Omega)] \,, \quad \Omega \in \Omega^{top}(E) \,, \quad f \in C^\infty(\mathcal{E}) \,.$$

Then,

$$F^\pi([f\pi^*(\Omega)]) = \iota_f \Omega \quad \text{"Berezin integration"} \,.$$

# The Geometry behind BV

- In coords, $\Delta$ reads

$$\Delta[f(x,x^\dagger)dx^1\wedge\cdots\wedge dx^m] = \frac{\partial^2 f}{\partial x^i \partial x_i^\dagger}[dx^1\wedge\cdots\wedge dx^m] = (\Delta_\Omega f)[\Omega]\,,$$

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- We have

$$\int_{\mathcal{L}} \Delta[s] = 0\,, \quad \int_{\delta_Q \mathcal{L}} [s] = \int_{\mathcal{L}} \underbrace{[[\Delta, Q]]}_{L_Q}[s] = \int_{\mathcal{L}} Q\Delta[s]\,.$$

Hence, if $\Delta[s] = 0$, $\int_{\mathcal{L}}[s]$ is invariant under Hamiltonian transformations of $\mathcal{L}$.

## The Geometry behind BV

- In physical applications, $\mathcal{L}$ encodes the choice of a gauge-fixing and the semidensity is of the form

$$[s] = \left[e^{iS/\hbar}\Omega\right] , \quad S = S_0 + \text{ghost terms} + \text{higher order } \hbar\text{-terms} .$$

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- Trivial example:

$$\int e^{S_0(x^1)} dx^1 \wedge dx^2 \rightsquigarrow \int_{\mathcal{L}} e^{S_0(x^1) + x_2^\dagger \beta} dx^1 \wedge dx^2 \wedge d\beta^\dagger = \int_{\mathbb{R}} e^{S_0(x^1)} dx^1 .$$

# Thank you!

Part of this material can be found in :
C. A., B. Bleile, J. Fröhlich,
*Batalin-Vilkovisky Integrals in Finite Dimensions*,
arXiv:0812.0464