Physical problem
Wess–Zumino model
Schwinger–Dyson equation
Tools
Hopf algebras
Renormalization group
Mellin transform
Results
Conclusion

High order solution of a non-linear Schwinger-Dyson equation.

Marc Bellon Gustavo Lozano Fidel Schaposnik

Cargèse/Carghjese
31 mars 2009

- Physics Letters B, 650:293–297, 2007, arXiv:hep-th/0703185.
- Nucl. Phys. B 800:517-526, 2008, arXiv:0801.0727v2 [hep-th]

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Tools
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Results
Conclusion

Outline

1. Physical problem
   - Wess–Zumino model
   - Schwinger–Dyson equation
2. Tools
   - Hopf algebras
   - Renormalization group
   - Mellin transform
3. Results
4. Conclusion

The fields

A four dimensional quantum field theory with:

- a complex field A and a chiral fermionic field ψ.
- a Yukawa coupling $gA\psi\psi$
- a quartic self-interaction of $g^2|A|^4$.

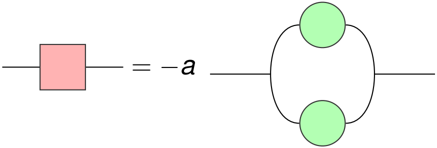
Non-renormalization

The model has exact cancellation of some divergences among diagrams.
This becomes clearer by the introduction of a non-propagating complex field F, which forms a supersymmetry multiplet with A and ψ.

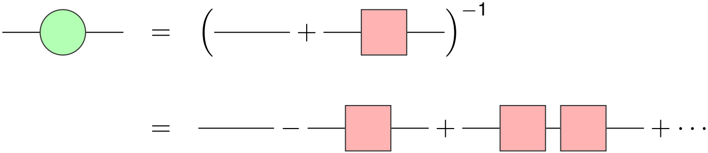
- The quartic interaction is replaced by the coupling gFA^2.
- The three-point functions $A\psi\psi$ and FA^2 are never divergent.
- In the massless case, the three fields get the *same wave function renormalization.*

Schwinger–Dyson equation

We will solve the Schwinger–Dyson equation graphically depicted by:



The square box designs a sum of 1PI diagrams and defines the propagator through:



Why this Schwinger–Dyson equation ?

The iteration of this Schwinger–Dyson equation produces a family of diagrams which:

- includes all diagrams with the unique simple loop primitive divergence.
- dominates in a large N approximation, since they are reinforced by a $n!$ renormalon factor.
- may be subject to compensations between diagrams of different signs after renormalization.

Physical problem
Wess–Zumino model
Schwinger–Dyson equation
Tools
Hopf algebras
Renormalization group
Mellin transform
Results
Conclusion

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Physical problem
Wess–Zumino model
Schwinger–Dyson equation

Tools
Hopf algebras
Renormalization group
Mellin transform

Results

Conclusion

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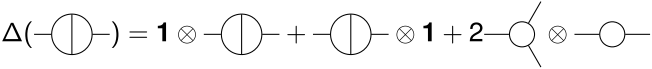
Results
Conclusion

Renormalization Hopf algebra

The combinatorics of renormalization is expressed in terms of a Hopf algebra.
It is a polynomial algebra on one-particle irreducible graphs.

Coproduct:

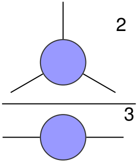


Physical problem
Wess–Zumino model
Schwinger–Dyson equation
Tools
Hopf algebras
Renormalization group
Mellin transform
Results
Conclusion

. . . on Green functions

Introduce the structure constant $a = g^2$ and the “effective structure constant”

$$a_{\text{eff}} = g^2$$



For any Green function $\Gamma = \sum_n \Gamma_n a^n$, we have

$$\Delta\Gamma = \sum_n \Gamma \, a_{\text{eff}}^n \otimes \Gamma_n$$

. . . on Green functions

The preceding form of the coproduct has a number of nice properties:

- It applies to the sums of diagrams generated by a given Schwinger–Dyson equation.
- The same formula applies to products or quotients of Green functions.
- It applies to a_{eff} itself.
- a_{eff} defines a Hopf algebra homomorphism from the dual of formal diffeomorphisms.

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Physical problem
Wess–Zumino model
Schwinger–Dyson equation
Tools
Hopf algebras
Renormalization group
Mellin transform
Results
Conclusion

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- It applies to a_{eff} itself.
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Renormalization

The algebra homomorphisms from the Hopf algebra of diagrams to $\mathbb{C}$ form a group for the convolution:

$$f \star g = (f \otimes g) \circ \Delta$$

An important case: the evaluation maps of the Feynman diagram Φ_{p^2} for the exterior impulsion p.
The renormalization condition is taken at given impulsion p_0^2:

$$\Phi^R_{p_0^2} = \varepsilon$$

The solution is

$$\Phi^R_{p^2} = (\Phi_{p_0^2} \circ S) \star \Phi_{p^2}$$

Φ^R has a well defined limit when the regularizations in Φ are removed and, in the massless case, only depends on the ratio p^2/p_0^2.

Renormalization group

The renormalization group is a simple consequence of the definition of the renormalized evaluation.

$$\begin{aligned}\Phi^R_{q^2/p_0^2} &= (\Phi_{p_0^2} \circ S) \star \Phi_{q^2} \\ &= (\Phi_{p_0^2} \circ S) \star \Phi_{p^2} \star (\Phi_{p^2} \circ S) \star \Phi_{q^2} \\ &= \Phi^R_{p^2/p_0^2} \star \Phi^R_{q^2/p^2}\end{aligned}$$

Changing to the variable $L = \log(q^2/p_0^2)$, we can differentiate to obtain:

$$\frac{\partial}{\partial L}\Phi^R_L = \frac{\partial}{\partial L}\Phi^R_L\Big|_{L=0} \star \Phi^R_L$$

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&= \Phi^R_{p^2/p_0^2} \star \Phi^R_{q^2/p^2}
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Physical problem
Wess–Zumino model
Schwinger–Dyson equation
Tools
Hopf algebras
Renormalization group
Mellin transform
Results
Conclusion

Application to Green functions

We can apply the preceding equation to the Green function or its inverse, since we know the action of the coproduct.
We introduce $\gamma = \frac{\partial}{\partial L}\Phi_L^R(\Gamma)\big|_{L=0}$.

$$\begin{aligned}\frac{\partial}{\partial L}\Phi_L^R(\Gamma) &= \sum_n \frac{\partial}{\partial L}\Phi_L^R(\Gamma^{1-3n})\Big|_{L=0} \Phi_L^R(\Gamma_n) \\ &= \sum_n \gamma(1-3n)\Phi_L^R(\Gamma_n) = \gamma\left(1 - 3a\frac{\partial}{\partial a}\right)\Phi_L^R(\Gamma).\end{aligned}$$

We have a similar result for Γ^{-1}

$$\frac{\partial}{\partial L}\Phi_L^R(\Gamma^{-1}) = \gamma\left(-1 - 3a\frac{\partial}{\partial a}\right)\Phi_L^R(\Gamma^{-1}).$$

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Return to Schwinger–Dyson equation

Evaluate the diagram with two renormalized propagators

- At order n in a, the propagator is a polynomial of order n in $L = \log(p^2/p_0^2)$.
- $\Phi_L^R(\Gamma^{-1}) = \sum_n \gamma_n L^n/n!$
- Generating function for all powers of L by considering propagator $(p^2)^{x-1}$

Generating functions

The evaluation of the diagram

$$\begin{aligned}\Gamma(q^2,x,y) &= +\frac{g^2}{8\pi^4}\int d^4p(p^2)^{x-1}[(q-p)^2]^{y-1} \\ &= +\frac{g^2}{8\pi^2}(q^2)^{x+y}\frac{\Gamma(-x-y)\Gamma(1+x)\Gamma(1+y)}{\Gamma(2+x+y)\Gamma(1-x)\Gamma(1-y)}\end{aligned}$$

The derivative with respect to $\log q^2$

$$\begin{aligned}H(x,y) &= -a\frac{\Gamma(1-x-y)\Gamma(1+x)\Gamma(1+y)}{\Gamma(2+x+y)\Gamma(1-x)\Gamma(1-y)} \\ &= \sum_{p,q} h_{p,q}x^p y^q\end{aligned}$$

Final expression:

$$\gamma = \sum_{p,q} h_{p,q}\gamma_p\gamma_q$$

“Exact” solution

$$
\begin{aligned}
\gamma(a) &= a - 2a^2 + 14a^3 + \Big(-160 + 16\,\zeta(3)\Big)a^4 \\
&+ \Big(2444 - 328\,\zeta(3)\Big)a^5 + \Big(-45792 + 7056\,\zeta(3) + 2016\,\zeta(5)\Big)a^6 \\
&+ \Big(1\,005\,480 - 169\,152\,\zeta(3) - 70\,896\,\zeta(5) + 8960\,\zeta^2(3)\Big)a^7 \\
&+ \Big(-25\,169\,760 + 4\,509\,408\,\zeta(3) + 2\,199\,840\,\zeta(5) + 564\,480\,\zeta(7) - 390\,400\,\zeta^2(3)\Big)a^8 \\
&+ \Big(705\,321\,200 - 132\,548\,640\,\zeta(3) - 69\,922\,848\,\zeta(5) - 29\,005\,632\,\zeta(7) \\
&\quad + 14\,193\,504\,\zeta^2(3) + 6\,397\,056\,\zeta(3)\,\zeta(5)\Big)a^9 \\
&+ \Big(-21\,841\,420\,384 + 4\,261\,047\,424\,\zeta(3) + 2\,354\,993\,856\,\zeta(5) + 1\,194\,909\,696\,\zeta(7) \\
&\quad + \frac{858\,457\,600}{3}\,\zeta(9) - 512\,441\,536\,\zeta^2(3) - 383\,788\,416\,\zeta(3)\,\zeta(5) + \frac{49\,556\,480}{3}\,\zeta^3(3)\Big)a^{10} \\
&+ \Big(740\,194\,188\,032 - 148\,784\,410\,432\,\zeta(3) - 84\,779\,661\,888\,\zeta(5) - 47\,818\,582\,272\,\zeta(7) \\
&\quad - \frac{58\,999\,853\,440}{3}\,\zeta(9) + 19\,225\,297\,088\,\zeta^2(3) + 17\,828\,697\,216\,\zeta(3)\,\zeta(5) + 1\,829\,076\,480\,\zeta^2(5) \\
&\quad + 3\,838\,602\,240\,\zeta(3)\,\zeta(7) - \frac{3\,432\,237\,056}{3}\,\zeta^3(3)\Big)a^{11} \\
&+ \Big(-27\,243\,674\,154\,368 + 5\,610\,375\,120\,768\,\zeta(3) + 3\,266\,192\,145\,024\,\zeta(5) \\
&\quad + 1\,961\,976\,190\,464\,\zeta(7) + 1\,019\,076\,124\,160\,\zeta(9) + 230\,546\,534\,400\,\zeta(11) \\
&\quad - 760\,702\,109\,184\,\zeta^2(3) - 788\,057\,929\,728\,\zeta(3)\,\zeta(5) - 141\,297\,435\,648\,\zeta^2(5) \\
&\quad - 297\,887\,016\,960\,\zeta(3)\,\zeta(7) + 59\,550\,068\,736\,\zeta^3(3) + 34\,512\,334\,848\,\zeta^2(3)\,\zeta(5)\Big)a^{12} + \cdots
\end{aligned}
$$

Numeric solution

$$
\begin{gathered}
\gamma(a) = a - 2\,a^2 + 14\,a^3 - 140.76708954944649 1434\,a^4 \\
+2049.72533576365307439\,a^5 - 35219.8401369368689507\,a^6 \\
+741582.310142069315875\,a^7 - 1.74630317191742523615\times 10^7\,a^8 \\
+4.72719801334671229530\times 10^8\,a^9 - 1.39759545666280992694\times 10^{10}\,a^{10} \\
+4.60146704077682933925\times 10^{11}\,a^{11} - 1.63220296094286720854\times 10^{13}\,a^{12} \\
+6.32651854893093835423\times 10^{14}\,a^{13} - 2.61715263667021333524\times 10^{16}\,a^{14} \\
+1.16791189443603376676\times 10^{18}\,a^{15} - 5.52247245848724267096\times 10^{19}\,a^{16} \\
+\cdots \\
+8.40531761856825275 26\times 10^{454}\,a^{195} - 4.93391103305143676 78\times 10^{457}\,a^{196} \\
+2.91123623467471064 44\times 10^{460}\,a^{197} - 1.72635927382174959 52\times 10^{463}\,a^{198} \\
+1.02898947740083005 71\times 10^{466}\,a^{199} - 6.16363277680185350 21\times 10^{468}\,a^{200} \\
+3.71078785441092892 92\times 10^{471}\,a^{201} + \cdots
\end{gathered}
$$

Properties of the numerical γ

With $\gamma = \sum \gamma_n a^n$, we have

- $\gamma_n \simeq -(3n + 2)\gamma_{n-1}$.
- The singularity in $+\frac{1}{3}$ of the Borel transform is not a pole.

Crossroads

- "Physical" renormalization condition versus Minimal Subtraction.
- Calan–Symanzik versus Wilson renormalization group.

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Physical problem
Wess–Zumino model
Schwinger–Dyson equation
Tools
Hopf algebras
Renormalization group
Mellin transform
Results
Conclusion

Further extensions

- Obtain proofs, maybe in the form of differential equations.
- Obtain some more propagator–coupling duality.
- Effect of additional terms in Schwinger–Dyson equation.
- Case with renormalized vertex function.

Physical problem
Wess–Zumino model
Schwinger–Dyson equation
Tools
Hopf algebras
Renormalization group
Mellin transform
Results
Conclusion

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Physical problem
Wess–Zumino model
Schwinger–Dyson equation
Tools
Hopf algebras
Renormalization group
Mellin transform
Results
Conclusion

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Wess–Zumino model
Schwinger–Dyson equation
Tools
Hopf algebras
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Mellin transform
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Wess–Zumino model
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Tools
Hopf algebras
Renormalization group
Mellin transform
Results
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Schwinger–Dyson equation
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Hopf algebras
Renormalization group
Mellin transform
Results
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Calculation of Broadhurst and Kreimer

In dimension four, the angular average of $1/(p-k)^2$ is the minimum of $1/p^2$ and $1/k^2$.

$$
\begin{aligned}
G(s)^{-1} &= \int_0^s \frac{G(t)}{s} dt + \int_s^\infty \frac{G(t)}{t} dt \\
\frac{\partial}{\partial s} G(s)^{-1} &= \frac{1}{s^2} \int_0^s G(t)\, dt
\end{aligned}
$$

Development in poles

$$H(x,y) = -a(1+xy)\left(\frac{1}{1+x}+\frac{1}{1+y}-1\right)-a\frac{xy}{1-x-y}+\text{poles farther}$$