Anomalies in Quantum Field Theory and Cohomologies of Configuration Spaces

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Plan

Outline

Introduction: why renormalization in configuration spaces?

- The renormalization on configuration space has a direct geometric interpretation allowing generalization on manifolds.
- The Epstein-Glaser renormalization is done for the products of fields, which also facilitates the generalization of the perturbation theory on manifolds.
    - Still this has the disadvantage to be rather complicated technically, especially for concrete calculations.
    - One of the results in this talk is an analog of the Epstein–Glaser approach, which is entirely stated in terms of renormalization of integrals of functions. This approach then has the additional advantage to be independent of concrete models of quantum fields like the φ^4–theory or quantum electrodynamics etc.

Outline

Preliminary definition of renormalization maps

$$\left\{ \begin{array}{c} \text{algebra } \mathscr{O}_n \text{ of} \\ \text{non globally defined} \\ \text{smooth functions} \end{array} \right\} \xrightarrow{R_n} \left\{ \begin{array}{c} \text{space of} \\ \text{globally defined} \\ \text{distributions} \end{array} \right\}$$

$\mathscr{O}_n$ is an algebra of translation invariant functions of n vector arguments

and these functions one can think of as coming from Feynman diagrams.

Preliminary definition of renormalization maps

- We define a sequence of algebras $\mathscr{O}_2, \mathscr{O}_3, \ldots, \mathscr{O}_n, \ldots$

$$\mathscr{O}_n = \text{Span}\left\{ \prod_{1 \leqslant j < k \leqslant n} G_{jk}(\mathbf{x}_j - \mathbf{x}_k) \ : \ G_{jk}(\mathbf{x}) \in \mathscr{O}_2 \right\}.$$

$$\mathbf{x} = (x^1, \ldots, x^D) \in \mathbb{E} \equiv \mathbb{R}^D$$

- $\mathscr{O}_2 \subseteq \mathcal{C}^\infty(\mathbb{E} \setminus \{\mathbf{0}\})$ – "the algebra of propagators".
- Assumption: Polynomials $\cdot\ \mathscr{O}_n \subseteq \mathscr{O}_n\,,\quad \partial_{x_k^\mu}\, \mathscr{O}_n \subseteq \mathscr{O}_n \quad (\partial_{x_k^\mu} = \frac{\partial}{\partial x_k^\mu})$.
- A main example: $\mathscr{O}_2 = \left\{ \frac{p(\mathbf{x})}{(\mathbf{x}^2)^N} : p(\mathbf{x}) - \text{polynomial},\ N \in \mathbb{N} \right\}$,

$$\mathscr{O}_n = \left\{ \frac{p(\mathbf{x}_1 - \mathbf{x}_n, \ldots, \mathbf{x}_{n-1} - \mathbf{x}_n)}{\left(\prod_{j<k} (\mathbf{x}_j - \mathbf{x}_k)^2 \right)^N} : p - \text{polynomial},\ N \in \mathbb{N} \right\}.$$

Preliminary definition of renormalization maps

Another remark: the algebra $\mathscr{O}_n$ consists of translation invariant functions that are regular on the so called configuration space

$$F_n = \big\{(\mathbf{x}_1,\dots,\mathbf{x}_n) \in \mathbb{E}^n \,:\, \mathbf{x}_j \neq \mathbf{x}_k \quad (\forall j \neq k)\big\}\,.$$

From the point of view of the algebraic geometry: $\mathscr{O}_n$ is the ring of regular functions on the quasi-affine manifold that is the complement of union of quadrics

$$F_{n;\mathbb{C}} = \big\{(\mathbf{x}_1,\dots,\mathbf{x}_n) \in \mathbb{C}^{Dn} \,:\, (\mathbf{x}_j - \mathbf{x}_k)^2 \neq 0 \quad (\forall j \neq k)\big\}\,.$$

Preliminary definition of renormalization maps

Renormalization maps are linear maps:

$$R_n : \mathscr{O}_n \to \mathscr{D}'\left(\mathbb{E}^{\times n}/\mathbb{E}\right), \qquad n = 2, 3, \dots .$$

They are supposed to satisfy the following axiomatic conditions $(r1)$–$(r4)$.

$(r1)$ Permutation symmetry:

$$R_n\left(\sigma^* G\right) = \sigma^* R_n(G) \qquad (\forall \sigma \in \mathscr{S}_n),$$

where $\sigma^* F(\mathbf{x}_1, \dots, \mathbf{x}_n) := F(\mathbf{x}_{\sigma_1}, \dots, \mathbf{x}_{\sigma_n})$.

Convention: for every $S \underset{\text{finite}}{\subset} \mathbb{N}$ we shall define an algebra $\mathscr{O}_S \cong \mathscr{O}_n$ $(n = |S|)$:

$$\mathscr{O}_S = \mathrm{Span}\left\{ \prod_{j,k \in S \,;\, j<k} G_{jk}\left(\mathbf{x}_j - \mathbf{x}_k\right) \;:\; G_{jk}\left(\mathbf{x}\right) \in \mathscr{O}_2 \right\}.$$

Then $R_S : \mathscr{O}_S \to \mathscr{D}'\left(\mathbb{E}^S/\mathbb{E}\right)$, $R_S(G) := (\sigma^*)^{-1} R_n\left(\sigma^* G\right)$, $\sigma : \{1, \dots, n\} \cong S$.

Preliminary definition of renormalization maps

($r2$) Preservation of filtrations:

$$\text{Scaling degree of } R_n G \leqslant \text{Scaling degree of } G\,.$$

The scaling degree gives the rate of the singularity for coinciding arguments.

($r3$) Commutativity between the renormalization maps and the multiplication by polynomials:

$$R_n\big(pG\big) = p\,R_n G, \qquad p = p(\mathbf{x}_1 - \mathbf{x}_n, \dots, \mathbf{x}_{n-1} - \mathbf{x}_n) \quad \text{is a polynomial.}$$

Preliminary definition of renormalization maps

Conventions:

Let $\mathfrak{P}$ be a partition of the set $S = \{j_1, \dots, j_n\}$:

$$\Big(\underset{j_1\,j_2\,j_3}{\bullet\bullet\bullet}\Big)\cdots\Big(\cdots\underset{j_k}{\bullet}\cdots\Big)\cdots\Big(\cdots\underset{j_n}{\bullet}\Big) \quad\Leftrightarrow\quad \left\{\begin{array}{l}\text{equivalence relation}\\ \sim_{\mathfrak{P}} \text{ on } S.\end{array}\right.$$

We set: $F_{\mathfrak{P}} = \big\{(\mathbf{x}_{j_1}, \dots, \mathbf{x}_{j_n}) \in \mathbb{E}^S : \mathbf{x}_j \neq \mathbf{x}_k \ \ (\forall j \nsim_{\mathfrak{P}} k)\big\}$.

For $G_S = \prod_{\substack{j,k\in S\\ j<k}} G_{jk}\big(\mathbf{x}_j - \mathbf{x}_k\big)$ we set: $G_S = G_{\mathfrak{P}} \cdot \prod_{S'\in\mathfrak{P}} G_{S'}$,

where

$$G_{\mathfrak{P}} = \prod_{\substack{j\nsim_{\mathfrak{P}} k\\ j<k}} G_{jk}\big(\mathbf{x}_j - \mathbf{x}_k\big), \quad G_{S'} = \prod_{\substack{j,k\in S'\\ j<k}} G_{jk}\big(\mathbf{x}_j - \mathbf{x}_k\big).$$

$(r4)$ For every proper S–partition $\mathfrak{P}$: $R_S G_S\Big|_{F_{\mathfrak{P}}} = G_{\mathfrak{P}} \cdot \prod_{S'\in\mathfrak{P}} R_{S'} G_{S'}$.

In particular: $R_n G\Big|_{F_n} = G$.

Preliminary definition of renormalization maps

Summary:

($r1$) Permutation symmetry.

($r2$) Preservation of the filtrations.

($r3$) Commutativity with the multiplication by polynomials.

($r4$) Recursive relation: $R_S G_S \Big|_{F_{\mathfrak{P}}} = G_{\mathfrak{P}} \cdot \prod_{S' \in \mathfrak{P}} R_{S'} G_{S'}$.

Application to Euclidean perturbative QFT

We need to define products of interactions $I_1(\mathbf{x}_1)\cdots I_n(\mathbf{x}_n)$ as quadratic forms, where $I_k(\mathbf{x}) =$ Wick polynomial of $\varphi(\mathbf{x})$ and its derivatives $\partial^{\mathbf{r}}\varphi(\mathbf{x})$.

Decomposing by the Wick theorem

$$I_1(\mathbf{x}_1)\cdots I_n(\mathbf{x}_n) = \sum_{A_1,\dots,A_n} G_{A_1,\dots,A_n}(\mathbf{x}_1,\dots,\mathbf{x}_n) :\Phi_{A_1}(\mathbf{x}_1)\cdots\Phi_{A_n}(\mathbf{x}_n):, \quad \text{we set:}$$

$$\big(I_1(\mathbf{x}_1)\cdots I_n(\mathbf{x}_n)\big)^{\text{ren}} = \sum_{A_1,\dots,A_n} R_n\big(G_{A_1,\dots,A_n}\big)(\mathbf{x}_1,\dots,\mathbf{x}_n) :\Phi_{A_1}(\mathbf{x}_1)\cdots\Phi_{A_n}(\mathbf{x}_n): .$$

A convenient formula:

$$I_1(\mathbf{x}_1)\cdots I_n(\mathbf{x}_n) = \prod_{j<k} \exp\left(\sum_{\mathbf{r},\mathbf{s}} C_{\mathbf{r},\mathbf{s}}(\mathbf{x}_j-\mathbf{x}_k)\,\frac{\partial}{\partial\varphi_{\mathbf{r}}(\mathbf{x}_j)}\,\frac{\partial}{\partial\varphi_{\mathbf{r}}(\mathbf{x}_k)}\right) :I_1(\mathbf{x}_1)\cdots I_n(\mathbf{x}_n): .$$

$$\varphi_{\mathbf{r}}(\mathbf{x}) := \partial^{\mathbf{r}}_{\mathbf{x}}\varphi(\mathbf{x}), \quad C_{\mathbf{r},\mathbf{s}}(\mathbf{x}_1-\mathbf{x}_2) := \partial^{\mathbf{r}}_{\mathbf{x}_1}\partial^{\mathbf{s}}_{\mathbf{x}_2}\langle\varphi(\mathbf{x}_1)\varphi(\mathbf{x}_2)\rangle .$$

Preliminary definition of renormalization maps

Again:

(*r*1) Permutation symmetry.

(*r*2) Preservation of the filtrations.

(*r*3) Commutativity with the multiplication by polynomials.

(*r*4) Recursive relation: $R_S G_S \Big|_{F_{\mathfrak{P}}} = G_{\mathfrak{P}} \cdot \prod_{S' \in \mathfrak{P}} R_{S'} G_{S'}$.

Since $\mathbb{E}^S \setminus \{\mathbf{0}\} = \bigcup_{\substack{\mathfrak{P} \text{ is a proper} \\ S\text{–partition}}} F_{\mathfrak{P}}$ then by $(r4)$ we have a linear map:

$$\dot{R}_S : \mathscr{O}_S \to \mathscr{D}'_{temp}\big(\mathbb{E}^S \setminus \{\mathbf{0}\}\big),$$

depending on the renormalization maps of lower degree.

Primary renormalization maps

Then to construct R_n we have to compose $\dot{R}_n$ with a linear map

$$P_n : \mathscr{D}'_{temp}\left(\mathbb{E}^{\times n} \setminus \{\mathbf{0}\}\right) \to \mathscr{D}'\left(\mathbb{E}^{\times n}\right),$$

$$R_n = P_n \circ \dot{R}_n ,$$

$$\mathscr{O}_n \xrightarrow{\dot{R}_n} \mathscr{D}'_{temp}\left(\mathbb{E}^{\times n} \setminus \{\mathbf{0}\}\right) \xrightarrow{P_n} \mathscr{D}'\left(\mathbb{E}^{\times n}\right).$$

Axiomatic conditions for P_n:

($p1$) $P_n u \big|_{\mathbb{E}^{\times n} \setminus \{\mathbf{0}\}} = u$, i.e. P_n makes extension of distributions.

($p2$) Preservation of the filtrations.

($p3$) Orthogonal invariance.

($p4$) Commutativity with the multiplication by polynomials.

($p5$) If $u(\mathbf{x}) \in \mathscr{D}'\left(\mathbb{E}^{\times(n-m)}\right)$, $supp\, u = \{\mathbf{0}\}$ and $v(\mathbf{y}) \in \mathscr{D}'_{temp}\left(\mathbb{E}^{\times m} \setminus \{\mathbf{0}\}\right)$:

$$P_n(u \otimes v) = u \otimes P_m v .$$

Change of renormalization

If $\{P_n\}_{n=2}^{\infty}$ ($\Rightarrow \{R_n\}_{n=2}^{\infty}$) and $\{P'_n\}_{n=2}^{\infty}$ ($\Rightarrow \{R'_n\}_{n=2}^{\infty}$) – two renormalizations:

$$R'_S\, G_S \,=\, \sum_{\substack{\mathfrak{P} \text{ is a} \\ S\text{–partition}}} \Big(R_{S/\mathfrak{P}} \otimes \mathrm{id}_{\mathscr{D}'_{\mathfrak{P},0}}\Big) \circ n.f._{\mathfrak{P}} \Big(G_{\mathfrak{P}} \prod_{S' \in \mathfrak{P}} Q_{S'}\, G_{S'}\Big)\,.$$

Here: $Q_S \,=\, \big(P'_S - P_S\big) \circ \dot{R}'_S : \mathscr{O}_S \to \mathscr{D}'_{S,0} := \mathscr{D}'\big[\mathbf{0} \in \mathbb{E}^S\big]$,
$S/\mathfrak{P} := \{\max S' : S' \in \mathfrak{P}\}$.

The change of the renormalization is characterized by a sequence $\{Q_n : \mathscr{O}_n \to \mathscr{D}'_{n,0}\}$, $Q_1 = 1$, satisfying the properties:

- $(c1)$ Permutation symmetry.
- $(c2)$ Preservation of the filtrations.
- $(c3)$ Commutativity with multiplications by polynomials.

The set of all such systems of linear maps form a group with a multiplication

$$Q''_S\, G_S \,=\, \sum_{\substack{\mathfrak{P} \text{ is a} \\ S\text{–partition}}} \Big(Q'_{S/\mathfrak{P}} \otimes \mathrm{id}_{\mathscr{D}'_{\mathfrak{P},0}}\Big) \circ n.f._{\mathfrak{P}} \Big(G_{\mathfrak{P}} \prod_{S' \in \mathfrak{P}} Q_{S'}\, G_{S'}\Big)\,.$$

Change of renormalization

A key role in the derivation of the representation of the universal renormalization group by formal diffeomorphisms on the couplings play the formulas

$$R'_S G_S = \sum_{\substack{\mathfrak{P} \text{ is a} \\ S\text{–partition}}} \left(R_{S/\mathfrak{P}} \otimes \mathrm{id}_{\mathscr{D}'_{\mathfrak{P},0}} \right) \circ n.f._{\mathfrak{P}} \left(G_{\mathfrak{P}} \prod_{S' \in \mathfrak{P}} Q_{S'}\, G_{S'} \right) ,$$

and

$$l_1(\mathbf{x}_1) \cdots l_n(\mathbf{x}_n) = \prod_{1 \leqslant j < k \leqslant n} \exp\left(\sum_{\mathbf{r},\mathbf{s}} C_{\mathbf{r},\mathbf{s}}(\mathbf{x}_j - \mathbf{x}_k) \frac{\partial}{\partial \varphi_{\mathbf{r}}(\mathbf{x}_j)} \frac{\partial}{\partial \varphi_{\mathbf{r}}(\mathbf{x}_k)} \right) \times \, : l_1(\mathbf{x}_1) \cdots l_n(\mathbf{x}_n) : \, .$$

Outline

Cohomological equations

- The source of the anomalies in QFT is the noncommutativity between the renormalization maps and the action of the linear partial differential operators:

$$\big[A, R_n\big]\, G \equiv A R_n(G) - R_n(AG) \neq 0$$

 (A – linear partial differential operator).
- By construction: $\big[p, R_n\big] = 0$.
 $\Rightarrow$ what remains as a source for the anomalies are the commutators

$$\omega_{n;\,\xi} := \big[\partial_{x^{\xi}}, R_n\big] \; \big(= \big[\partial_{x_k^{\mu}}, R_n\big]\,\big), \quad \xi = (k, \mu)\,.$$

Cohomological equations

Applying the main formula

$$R_n = P_n \circ \dot{R}_n .$$

to $\omega_{n;\xi} = \left[\partial_{x^\xi}, R_n\right]$ we obtain a decomposition,

$$\omega_{n;\xi} = \gamma_{n;\xi} + \dot{\omega}_{n;\xi} \qquad (n > 2), \qquad \omega_{2;\xi} \equiv \gamma_{2;\xi} ,$$

$$\gamma_{n;\xi} := \left[\partial_{x^\xi}, P_n\right] \circ \dot{R}_n \qquad (n > 2) ,$$

$$\dot{\omega}_{n;\xi} := P_n \circ \left[\partial_{x^\xi}, \dot{R}_n\right] \qquad (n > 2) ,$$

where $\dot{\omega}_{n;\xi}$ are determined by the recursion and $\gamma_{n;\xi} : \mathscr{O}_n \to \mathscr{D}'_{n,0} = \mathscr{D}'\left[\mathbf{0} \in \mathbb{E}^n / \mathbb{E}\right]$ are simpler linear maps.

Then:

$$\left[\partial_{x^\xi}, \gamma_{2;\eta}\right] - \left[\partial_{x^\eta}, \gamma_{2;\xi}\right] = 0 ,$$

$$\left[\partial_{x^\xi}, \gamma_{n;\eta}\right] - \left[\partial_{x^\eta}, \gamma_{n;\xi}\right] = -\left[\partial_{x^\xi}, P_n\right] \circ \left[\partial_{x^\eta}, \dot{R}_n\right] + \left[\partial_{x^\eta}, P_n\right] \circ \left[\partial_{x^\xi}, \dot{R}_n\right] \quad (n > 2) .$$

Theorem.

Let $n > 2$ and we have a system of primary renormalization maps

$$P_2,\, P_3,\, \dots,\, P_n$$

(which therefore determine renormalization maps $R_2,\, R_3,\, \dots,\, R_n$).

Let $\{\gamma_{n;\,\xi}\}_{\xi}$ be defined by $P_2,\, P_3,\, \dots,\, P_n$ and $\{\gamma'_{n;\,\xi}\}_{\xi}$ be a solution of

$$\big[\partial_{x^{\xi}}, \gamma_{n;\,\eta}\big] - \big[\partial_{x^{\eta}}, \gamma_{n;\,\xi}\big] = -\big[\partial_{x^{\xi}}, P_n\big] \circ \big[\partial_{x^{\eta}}, \dot{R}_n\big] + \big[\partial_{x^{\eta}}, P_n\big] \circ \big[\partial_{x^{\xi}}, \dot{R}_n\big]\,,$$

which differs from $\{\gamma_{n;\,\xi}\}_{\xi}$ by an exact solution:

$$\gamma'_{n;\,\xi} - \gamma_{n;\,\xi} = \big[\partial_{x^{\xi}}, Q_n\big] \qquad (\xi = 1, \dots, D(n-1)),$$

for some linear map $Q_n : \mathscr{O}_n \to \mathscr{D}'_{n,0}$.

Then there exists a primary renormalization map P'_n, which together with $P_2,\, \dots,\, P_{n-1}$ determines a system of renormalization maps $R_2,\, \dots,\, R_{n-1}$ and R'_n and a primary renormalization cocycle coinciding with $\{\gamma'_{n;\,\xi}\}_{\xi}$.

Reduction of the cohomological equations

- An important property of $\gamma_{n;\xi}$: $\left[x^{\eta}, \gamma_{n;\xi}\right] = 0$.
- It follows then that $\gamma_{n;\xi}$ has the following form

$$\gamma_{n;\xi}(G) = \sum_{\mathbf{r}} \frac{(-1)^{|\mathbf{r}|}}{\mathbf{r}!} \, \Gamma_{n;\xi}\big(\mathrm{x}^{\mathbf{r}} G\big) \, \delta^{(\mathbf{r})}(\mathrm{x}),$$

where $\Gamma_{n;\xi} \in \mathscr{O}_n^{\bullet} := \left\{\Gamma \in \mathscr{O}_n' : \exists M \text{ s.t. } \Gamma\big(G\big) = 0 \text{ if } \mathrm{Sc.d.}\, G \leqslant M\right\}$.

- The correspondence $\gamma_{n;\xi} \leftrightarrow \Gamma_{n;\xi}$ is one–to–one.
- $\left[\partial_{x^{\eta}}, \gamma_{n;\xi}\right] \leftrightarrow -\Gamma_{n;\xi} \circ \partial_{x^{\eta}} =: \partial_{x^{\eta}} \Gamma_{n;\xi}$.
- Organizing $\Gamma_{n;\xi}$ into: $\underline{\Gamma}_n = \sum_{\xi} \Gamma_{n;\xi}\, dx^{\xi} \in \Omega^1\big(\mathscr{O}_n^{\bullet}\big) := \mathscr{O}_n^{\bullet} \otimes \Lambda^1 \mathbb{R}$:

$$d\underline{\Gamma}_2 = 0\,, \qquad d\underline{\Gamma}_n = \sum_{m=2}^{n-1} \underline{\Gamma}_{n-m+1} \mathbin{\mathring{\wedge}} \underline{\Gamma}_m \qquad (n > 2)\,,$$

$$\big(\underline{\Gamma}_{n-m+1} \mathbin{\mathring{\wedge}} \underline{\Gamma}_m\big)\big(G_S\big) = \sum_{\substack{S' \subsetneq S \\ |S'| = m}} \sum_{\mathbf{r}'} \frac{1}{\mathbf{r}'!} \underline{\Gamma}_{S/S'}\Big(\partial_{\mathrm{x}'}^{\mathbf{r}'} G_{\mathfrak{P}(S')}\Big|_{\mathrm{x}' = 0}\Big) \wedge \underline{\Gamma}_{S'}\big(\mathrm{x}'^{\mathbf{r}'} G_{S'}\big).$$

Cohomological analysis of renormalization

- We have a natural duality:

$$H^1\big(\mathscr{O}_n^{\bullet}\big) \cong \Big(H^{D(n-1)-1}\big(\mathscr{O}_n\big)\Big)',$$

- Recall that $\mathscr{O}_n =$ the algebra of regular functions on:

$$F_{n;\mathbb{C}} = \big\{(\mathbf{x}_1,\dots,\mathbf{x}_n) \in \mathbb{C}^{Dn} : (\mathbf{x}_j - \mathbf{x}_k)^2 \neq 0 \ \ (\forall j \neq k)\big\}.$$

And so, $H^1\big(\mathscr{O}_n^{\bullet}\big) \cong \Big(H_{DR}^{D(n-1)-1}\big(F_{n;\mathbb{C}}/\mathbb{C}^D\big)\Big)'.$

- If we work instead of with $\mathscr{O}_n$ with $\mathcal{C}^{\infty}\big(F_n/\mathbb{E}\big)$ then:

$$H^1\Big(\mathcal{C}^{\infty}\big(F_n/\mathbb{E}\big)^{\bullet}\Big) \cong \Big(H_{DR}^{D(n-1)-1}\big(F_n/\mathbb{E}\big)\Big)' = 0 \qquad \text{for} \qquad n \geqslant 3.$$

- Idea: look for an intermediate differential–algebraic extension:

$$\mathscr{O}_n \subseteq \widetilde{\mathscr{O}}_n \subseteq \mathcal{C}^{\infty}\big(\mathbf{F}_n/\mathbb{E}\big),$$

which would trivialize the cohomologies: $H^{D(n-1)-1}\big(\widetilde{\mathscr{O}}_n\big) = 0\,.$

The two–dimensional case

- Introducing $(x^1, x^2) \leftrightarrow (z, w)$: $z := x^1 + i\,x^2$ and $w := x^1 - i\,x^2$:

$$\mathscr{O}_n \cong \mathbb{Q}\big[z_1, \dots, z_{n-1}\big]\Big[\Big(\prod_j z_j\Big)^{-1}\Big(\prod_{j<k}(z_j - z_k)\Big)^{-1}\Big]$$
$$\otimes\, \mathbb{Q}\big[w_1, \dots, w_{n-1}\big]\Big[\Big(\prod_j w_j\Big)^{-1}\Big(\prod_{j<k}(w_j - w_k)\Big)^{-1}\Big]$$

- Using $\mathbb{Q}\big[z_1, \dots, z_{n-1}\big] \subset$ Multiple Polylogs$\big(z_1, \dots, z_{n-1}\big)$ we set:

$$\widetilde{\mathscr{O}}_n =$$
$$\Big(\text{Multiple Polylogs}\big(z_1, \dots, z_{n-1}\big) \otimes \text{Multiple Polylogs}\big(w_1, \dots, w_{n-1}\big)\Big)^{\text{Monod–romy}}$$

This requires an extension: $\mathbb{Q} \subset$ Ring of multiple zeta values.

- Problem: $\mathbb{Q}\big[\mathbf{x}_1, \dots, \mathbf{x}_{n-1}\big]\Big[\Big(\prod_j \mathbf{x}_j^2\Big)^{-1}\Big(\prod_{j<k}(\mathbf{x}_j - \mathbf{x}_k)^2\Big)^{-1}\Big] \overset{?}{\subset} \widetilde{\mathscr{O}}_n$.

A strategy for solving the cohomological equations

- Recall: $\underline{\Gamma}_n = \sum_\xi \Gamma_{n;\xi}\, dx^\xi$, where $\Gamma_{n;\xi} : \mathscr{O}_n \to \mathbb{R}$,

$$\gamma_{n;\xi} := \left[\partial_{x^\xi}, P_n\right] \circ \dot{R}_n = \sum_{\mathbf{r}} \frac{(-1)^{|\mathbf{r}|}}{\mathbf{r}!} \left(\Gamma_{n;\xi} \circ \mathrm{x}^{\mathbf{r}}\right) \delta^{(\mathbf{r})}(\mathrm{x}) .$$

  These linear functionals $\Gamma_{n;\xi}$ satisfy the cohomological equations:

$$d\underline{\Gamma}_2 = 0 ,$$

$$d\underline{\Gamma}_n = \mathscr{F}_n\left[\underline{\Gamma}_1, \ldots, \underline{\Gamma}_{n-1}\right] = \sum_{m=2}^{n-1} \underline{\Gamma}_{n-m+1} \overset{\circ}{\wedge} \underline{\Gamma}_m \qquad (n > 2) .$$

- Assume:

$$\exists\ K_n : \Omega^{D(n-1)-1}\left(\widetilde{\mathscr{O}}_n\right) \to \Omega^{D(n-1)-2}\left(\widetilde{\mathscr{O}}_n\right) , \quad K_n \circ d + d \circ K_n = \mathrm{id} .$$

  Then a solution of a cohomological equations is:

$$\underline{\Gamma}_n = \mathscr{F}_n \circ K_n .$$

Outline

Conclusions

- The renormalization in configuration spaces provides a geometric insight to the problem what are the transcendental extensions, which we need for the function spaces that would be appropriate for the description of the correlation functions in perturbative QFT.
- Conjecture: the coefficients of the beta functions in any perturbative QFT on even space–time dimensions are multiple zeta values.