Density models for credit risk

Monique Jeanblanc, Université d’Évry; Institut Europlace de Finance

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Let us study the case with two random times $\tau_1, \tau_2$.

For $i = 1, 2$, we denote by $(H^i_t, t \geq 0)$ the default process associated with $\tau_i$, i.e., $H^i_t = 1_{\{\tau_i \leq t\}}$.

The filtration generated by the process $H^i$ is denoted $\mathbb{H}^i$ and the filtration generated by the two processes $H^1, H^2$ is $\mathbb{H} = \mathbb{H}^1 \vee \mathbb{H}^2$. 
Any \( \mathbb{H} \)-adapted process \( Z \) admits a representation as

\[
Z_t = h_0(t) \mathbb{1}_{t < \tau_1 \wedge \tau_2} + h_1(t, \tau_1) \mathbb{1}_{\tau_1 \leq t < \tau_2} + h_2(t, \tau_2) \mathbb{1}_{\tau_2 \leq t < \tau_1} + h(\tau_1, \tau_2) \mathbb{1}_{\tau_1 \vee \tau_2 \leq t}
\]

where \( h_0, h_1, h_2, h \) are (deterministic) functions.

We denote by \( G(t, s) = \mathbb{Q}(\tau_1 > t, \tau_2 > s) \) the survival probability of the pair \( (\tau_1, \tau_2) \) and we assume that the joint law of \( (\tau_1, \tau_2) \) admits a density \( f(u, v) \).

We denote by \( \partial_i G \), the partial derivative of \( G \) with respect to the \( i \)-th variable, \( i = 1, 2 \).

Simultaneous defaults are precluded in this framework, i.e,
\( \mathbb{Q}(\tau_1 = \tau_2) = 0 \).
The process $M^1$ defined as

$$M^1_t := H^1_t + \int_0^{t \land \tau_1 \land \tau_2} \frac{\partial_1 G(s, s)}{G(s, s)} \, ds + \int_{t \land \tau_1 \land \tau_2}^{t \land \tau_1} \frac{f(s, \tau_2)}{\partial_2 G(s, \tau_2)} \, ds$$

is a $\mathbb{H}$-martingale.
The processes $H^i_t - \int_0^t \lambda^i_s \, ds$, $i = 1, 2$, are $\mathbb{H}$-martingales, where

\[
\lambda^1_t = \mathbb{P}(\tau_1 \in dt | \mathcal{H}_t, \tau_1 > t) = (1 - H^1_t) \left( (1 - H^2_t) \frac{-\partial_1 G(t, t)}{G(t, t)} - H^2_t \frac{f(t, \tau_2)}{\partial_2 G(t, \tau_2)} \right) =: (1 - H^1_t)(1 - H^2_t)\tilde{\lambda}^1_t + (1 - H^1_t)H^2_t \lambda^1_t(\tau_2)
\]

\[
\lambda^2_t = (1 - H^2_t) \left( (1 - H^1_t) \frac{-\partial_2 G(t, t)}{G(t, t)} - H^1_t \frac{f(\tau_1, t)}{\partial_1 G(\tau_1, t)} \right) = (1 - H^1_t)(1 - H^2_t)\tilde{\lambda}^2_t + H^1_t(1 - H^2_t)\lambda^2_t(\tau_1)
\]

where

\[
\lambda^1_t(s) = -\frac{f(t, s)}{\partial_2 G(t, s)}, \quad \lambda^2_t(s) = -\frac{f(s, t)}{\partial_1 G(s, t)}
\]
The goal is to find the dynamics of $Z_t := \mathbb{E}(h(\tau_1, \tau_2)|\mathcal{H}_t)$ and to give an hedging strategy based on CDSs

The price of the contingent claim $h(\tau_1, \tau_2)$ is

$$Z_t = h(\tau_1, \tau_2)H_t^1 H_t^2 + \psi_{1,0}(\tau_1, t) H_t^1 (1 - H_t^2) + \psi_{0,1}(t, \tau_2) H_t^2 (1 - H_t^1)$$
$$+(1 - H_t^1)(1 - H_t^2)\psi_{0,0}(t)$$

with

$$\psi_{1,0}(u, t) = \frac{-1}{\partial_1 G(u, t)} \int_t^\infty h(u, v) f(u, v) dv$$

$$\psi_{0,1}(t, v) = \frac{-1}{\partial_2 G(t, v)} \int_t^\infty h(u, v) f(u, v) du$$

$$\psi_{0,0}(t) = \frac{1}{G(t, t)} \int_t^\infty du \int_t^\infty dv \ h(u, v) f(u, v)$$
It can be proved that

\[
dZ_t = \left( (h(t, \tau_2) - \psi_{0,1}(t, \tau_2)) H_t^2 + (\psi_{1,0}(t, t) - \psi_{0,0}(t)) (1 - H_t^2) \right) dM^1_t \\
+ \left( (h(\tau_1, t) - \psi_{1,0}(\tau_1, t)) H_t^1 + (\psi_{0,1}(t, t) - \psi_{0,0}(t)) (1 - H_t^1) \right) dM^2_t \\
= \pi_t^1 dM^1_t + \pi_t^2 dM^2_t
\]

where

\[
\psi_{1,0}(u, t) = \frac{-1}{\partial_1 G(u, t)} \int_t^\infty h(u, v) f(u, v) dv \\
\psi_{0,1}(t, v) = \frac{-1}{\partial_2 G(t, v)} \int_t^\infty h(u, v) f(u, v) du \\
\psi_{0,0}(t) = \frac{1}{G(t, t)} \int_t^\infty du \int_t^\infty dv h(u, v) f(u, v)
\]
We consider a CDS

- with a constant spread $\kappa$
- which delivers $\delta(\tau_1)$ at time $\tau_1$ if $\tau_1 < T$, where $\delta$ is a deterministic function.

The value of the CDS is, for $t < \tau_1$

$$V_t = \mathbb{1}_{t<\tau_1} \mathbb{E}(\delta(\tau_1) \mathbb{1}_{\tau_1 \leq T} - \kappa((T \wedge \tau_1) - t) | \mathcal{H}_t) = \tilde{V}_t \mathbb{1}_{\{t<\tau_2\}} + V^1_2(\tau_2) \mathbb{1}_{\{\tau_2 \leq t\}}$$

where

$$\tilde{V}_t = \frac{1}{G(t,t)} \left( - \int_t^T \delta(u) \partial_1 G(u,t) \, du - \kappa \int_t^T G(u,t) \, du \right)$$

$$V^1_2(s) = \frac{-1}{\partial_2 G(t,s)} \left( \int_t^T \delta(u) f(u,s) \, du + \kappa \int_t^T \partial_2 G(u,s) \, du \right).$$
The dynamics of the price of the CDS are

\[ dV_t = (1 - H_t^1) \left( \kappa - \delta(t) \left( (1 - H_t^2) \bar{\lambda}_t^1 + H_t^2 \lambda_t^{1|2}(\tau_2) \right) \right) dt - V_t dM_t^1 + (1 - H_t^1)(V_t^{1|2}(t) - V_{t-})dM_t^2 \]

The dynamics of the cumulative price of the CDS are

\[ dV_{t}^{\text{cum}} = (\delta(t) - V_{t-}^{\text{cum}})dM_t^1 + (1 - H_t^1)(V_t^{1|2}(t) - V_{t-})dM_t^2 \]
Assume now that a CDS written on \( \tau_2 \) is also traded in the market, and that the interest rate is null. We denote by \( V^i, i = 1, 2 \) the prices of the two CDSs.

A self financing strategy consisting in \( \vartheta^i \) shares of CDS’s and \( \vartheta^0 \) shares of savings account has value \( X_t = \vartheta^0_t + \vartheta^1_t V^1_t + \vartheta^2_t V^2_t \) and dynamics

\[
dX_t = \left( -\vartheta^1_t V^1_t - \vartheta^2_t (1 - H^2_t) (V^{2|1}_t(t) - \tilde{V}^2_t) \right) dM^1_t
+ \left( \vartheta^1_t (1 - H^1_t) (V^{1|2}_t(t) - \tilde{V}^1_t) - \vartheta^2_t V^2_t \right) dM^2_t
\]

\[
= (X^1_t - X_{t^-}) dM^1_t + (X^2_t - X_{t^-}) dM^2_t
\]

where we have taken into account that CDSs are paying dividends and \( X^1_t = \vartheta^0_t + (1 - H^2_t) \vartheta^2_t V^{2|1}_t(t) \).
In order to hedge $Z = \mathbb{E}(Z) + \int_0^T \pi_1^t dM_1^t + \int_0^T \pi_2^t dM_2^t$, it remains to solve the linear system (with unknown $\vartheta^i$)

$$-\vartheta_1^t V_{t-}^1 + \vartheta_2^t (1 - H_2^t)(V_t^{2|1}(t) - \tilde{V}_t^2) = \pi_1^t$$

$$\vartheta_1^t (1 - H_1^t)(V_t^{1|2}(t) - \tilde{V}_t^1) - \vartheta_2^t V_{t-} = \pi_2^t$$
Ordered Defaults

Let us now assume that $\tau_1 < \tau_2$, a.s. In that case, $G(t, s) = G(t, t)$ for $s \leq t$,

\[ M_t^1 = H_t^1 + \int_0^{t \wedge \tau_1} \frac{\partial_1 G(s, s)}{G(s, s)} ds = H_t^1 - \int_0^{t \wedge \tau_1} \frac{f_1(s)}{G_1(s)} ds \]

where

\[ G_1(s) = \mathbb{Q}(\tau_1 > s) = G(s, s) = \int_s^\infty f_1(u) du. \]

The process $M^1$ is $\mathbb{H}^1$-adapted, hence is an $\mathbb{H}^1$-martingale and it follows that any $\mathbb{H}^1$-martingale is a $\mathbb{H}$ martingale. Furthermore, the intensity of $\tau_2$ vanishes on the set $t < \tau_1$ and

\[ M_t^2 = H_t^2 + \int_{t \vee \tau_1}^{t \wedge \tau_2} \frac{f(\tau_1, s)}{\partial_1 G(\tau_1, s)} ds. \]
Let $V^i$ be the price of a CDS on $\tau_i$, with spread $\kappa_i$ and recovery $\delta_i$.

The $\mathbb{H}$-dynamics of $V^1$ is

$$dV^1_t = -V^1_t dM^1_t + (1 - H^1_t)(\kappa_1 - \delta_1(t)\tilde{\lambda}_1(t))dt$$

with $\tilde{\lambda}_1(t) = \frac{f_1(t)}{G_1(t)}$.

The $\mathbb{H}$-dynamics of $V^2$ is

$$dV^2_t = -V^2_t dM^2_t + (1 - H^2_t)\kappa_2 dt - (1 - H^2_t)H^1_t \delta_2(t)\lambda^2_t(\tau_1)dt + (V^2_t|_t^1(t) - V^2_t) dM^1_t.$$
More than two defaults

In the filtration generated by three default processes,

\[ V_t^1 = \tilde{V}_t^1 \mathbb{1}_{t<\tau_1 \land \tau_2 \land \tau_3} + V_t^{1|2} (\tau_2) \mathbb{1}_{\tau_2 \leq t < \tau_1 \land \tau_3} + V_t^{1|3} (\tau_3) \mathbb{1}_{\tau_3 \leq t < \tau_1 \land \tau_2} \]

\[ + V_t^{1|23} (\tau_2, \tau_3) \mathbb{1}_{\tau_2 \lor \tau_3 \leq t < \tau_1} \]

where

\[ V_t^1 = \frac{1}{G(t, t, t)} \left( - \int_t^T \delta(u) \partial_1 G(u, t, t) dt - \kappa \int_t^T G(u, t, t) du \right) , \]

\[ V_t^{1|2} (x) = \frac{-1}{\partial_2 G(t, x, t)} \left( \int_t^T \delta(u) \partial_1 \partial_2 G(u, x, t) du + \kappa \int_t^T \partial_2 G(u, x, t) du \right) \]

\[ V_t^{1|3} (y) = \frac{-1}{\partial_3 G(t, t, y)} \left( \int_t^T \delta(u) \partial_1 \partial_3 G(u, t, y) du + \kappa \int_t^T \partial_3 G(u, t, y) du \right) \]

\[ V_t^{1|23} (x, y) = \frac{1}{\partial_2 \partial_3 G(t, x, y)} \left( \int_t^T \delta(u) f(u, x, y) du - \kappa \int_t^T \partial_2 \partial_3 G(u, x, y) du \right) \]
and the price of the CDS follows

\[ dV_t \quad = \quad (1 - H_t^1)\kappa dt - (1 - H_t^1)\delta(t)(1 - H_t^2)(1 - H_t^3)\tilde{\lambda}_t^1 dt \]

\[ -(1 - H_t^1)\delta(t) \left[ (1 - H_t^2)H_t^3\lambda_t^{1|3}(\tau_3) + (1 - H_t^3)H_t^2\lambda_t^{1|2}(\tau_2) \right] dt \]

\[ -(1 - H_t^1)H_t^2H_t^3\delta(t)\lambda_t^{1|23}(\tau_2, \tau_3) dt \]

\[ -V_t - dM_t^1 + (1 - H_t^1) \left( (1 - H_t^3)V_t^{1|2}(t) + H_t^3V_t^{1|32}(\tau_3) - V_{t-} \right) dM_t^2 \]

\[ +(1 - H_t^1) \left( (1 - H_t^2)V_t^{1|3}(t) + H_t^2V_t^{1|23}(\tau_2) - V_{t-} \right) dM_t^3 \]

where the intensities are given by

\[ \tilde{\lambda}_t^1 \quad = \quad \frac{-1}{G(t,t,t)}\partial_1 G(t,t,t) \]

\[ \lambda_t^{1|2}(s) \quad = \quad \frac{-1}{\partial_2 G(t,s,t)}\partial_1 \partial_2 G(t,s,t), \quad \lambda_t^{1|3}(s) \quad = \quad \frac{-1}{\partial_3 G(t,t,s)}\partial_1 \partial_3 G(t,t,s) \]

\[ \lambda_t^{1|23}(s_2, s_3) \quad = \quad \frac{f(t,s_2,s_3)}{\partial_2 \partial_3 G(t,s_2,s_3)} \]
More generally, the value of the contingent claim $h(\tau_1, \tau_2, \tau_3)$ is

$$
\psi_{000}(t) \mathbb{1}_{t < \tau_1 \wedge \tau_2 \wedge \tau_3}
+ \psi_{001}(t, \tau_3) \mathbb{1}_{\tau_3 \leq t < \tau_1 \wedge \tau_2}
+ \psi_{010}(t, \tau_2) \mathbb{1}_{\tau_2 \leq t < \tau_1 \wedge \tau_3}
+ \psi_{100}(t, \tau_1) \mathbb{1}_{\tau_1 \leq t < \tau_2 \wedge \tau_2}
+ \psi_{011}(t, \tau_2, \tau_3) \mathbb{1}_{\tau_2 \vee \tau_3 \leq t < \tau_1}
+ \psi_{110}(t, \tau_1, \tau_2) \mathbb{1}_{\tau_1 \vee \tau_2 \leq t < \tau_3}
+ \psi_{101}(t, \tau_1, \tau_3) \mathbb{1}_{\tau_1 \vee \tau_3 \leq t < \tau_2}
+ h(\tau_1, \tau_2, \tau_3) \mathbb{1}_{\tau_1 \vee \tau_2 \vee \tau_3 \leq t}
$$

where

$$
\psi_{000}(t) = \frac{1}{G(t, t, t)} \int_t^T du \int_t^T dv \int_t^T dwh(u, v, w) f(u, v, w)
$$

$$
\psi_{001}(t, s) = \frac{-1}{\partial_3 G(t, t, s)} \int_t^T du \int_t^T dvh(u, v, s) f(u, v, s)
$$

$$
\psi_{011}(t, s_2, s_3) = \frac{1}{\partial_2 \partial_3 G(t, s_2, s_3)} \int_t^T duh(u, s_2, s_3) f(u, s_2, s_3)
$$

and similar expressions for the remaining terms.
Density Hypothesis, Single default

Let \((\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})\) be a filtered probability space.

A strictly positive and finite random variable \(\tau\) (the default time) is given. We assume the following **density hypothesis**:

\[
G_t(\theta) := \mathbb{P}(\tau > \theta | \mathcal{F}_t) = \int_\theta^\infty f_t(u) du
\]

Let

\[
G_t := G_t(t) = \mathbb{P}(\tau > t | \mathcal{F}_t) = \int_t^\infty f_t(u) du
\]

In what follows, we assume \(G_t > 0\).
The family $f_t(.)$ is called the **conditional density** of $\tau$ given $\mathcal{F}_t$.

Note that

- $G_t(\theta) = \mathbb{E}(G_\theta | \mathcal{F}_t)$ for any $\theta \geq t$
- the law of $\tau$ is $\mathbb{P}(\tau > \theta) = \int_\theta^\infty f_0(u)du$
- for any $t$, $\int_0^\infty f_t(u)du = 1$
- For an integrable $\mathcal{F}_T \otimes \sigma(\tau)$ r.v. $Y_T(\tau)$, one has, for $t \leq T$

  $$\mathbb{E}(Y_T(\tau)|\mathcal{F}_t) = \mathbb{E}(\int_0^\infty Y_T(u)f_T(u)du|\mathcal{F}_t)$$
The process

$$\mathbb{1}_{\{\tau \leq t\}} - \int_0^{t \wedge \tau} \lambda_s^F ds$$

is a $\mathbb{G}$-martingale, where

$$\lambda_s^F = \frac{f_s(s)}{G_s}.$$ 

$G$ admits the multiplicative decomposition

$$G_t = L_t^F e^{-\int_0^t \lambda_s^F ds}$$

where $L^F$ is an $\mathbb{F}$-martingale. Conversely, if $G_t = n_t e^{-\Gamma_t}$ where $n$ is an $\mathbb{F}$-martingale and $\Gamma$ a continuous increasing process, then $\mathbb{1}_{\{\tau \leq t\}} - \Gamma_{t \wedge \tau}$ is a $\mathbb{G}$-martingale.
Pricing formulae

Terminal payoff $X \in \mathcal{F}_T$:

$$
\mathbb{E}(X \mathbb{1}_{\{T < \tau \}} | \mathcal{G}_t) = \mathbb{1}_{t < \tau} \frac{1}{G_t} \mathbb{E}(G_T X | \mathcal{F}_t)
$$

Recovery

$$
\mathbb{E}(Z_\tau \mathbb{1}_{\{t < \tau \leq T \}} | \mathcal{G}_t) = \mathbb{1}_{t < \tau} \frac{1}{G_t} \mathbb{E}(- \int_t^T Z_u dG_u | \mathcal{F}_t) = \mathbb{1}_{t < \tau} \frac{1}{G_t} \mathbb{E}( \int_t^T Z_u f(u) du | \mathcal{F}_t)
$$

$G_t(\theta) := \mathbb{P}(\tau > \theta | \mathcal{F}_t) = \int_\theta^\infty f_t(u) du$
Problem: given a martingale $n$ and an increasing process $\Gamma$ (such that $0 < n_t e^{-\Gamma_t} < 1$), construct $\tau$ such that $G_t = n_t e^{-\Gamma_t}$.

If $n = 1$, this is the Cox model.

In a general case, the problem admits various solutions.

\[
G_t(\theta) := \mathbb{P}(\tau > \theta | \mathcal{F}_t) = \int_{\theta}^{\infty} f_t(u) du
\]
**Immersion property**

Immersion property holds if any $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale. It is equivalent to

$$f_t(s) = f_s(s), \forall t > s$$
**Forward intensity**

The forward intensity \( \lambda_t(\theta) \) of \( \tau \) is given by \( \lambda_t(\theta) = -\partial_\theta \ln G_t(\theta) \)

\[
G_t(\theta) = \exp(-\int_{0}^{\theta} \lambda_t(u)du)
\]

We assume that \( \mathbb{F} \) is a Brownian filtration. There exists \( \Psi(t, \theta) \) such that

1. \( G_t(\theta) = G_0(\theta) \exp \left( \int_{0}^{t} \Psi(s, \theta)dW_s - \frac{1}{2} \int_{0}^{t} \Psi^2(s, \theta)ds \right) \);

2. \( \lambda_t(\theta) = \lambda_0(\theta) - \int_{0}^{t} \psi(s, \theta)dW_s + \int_{0}^{t} \psi(s, \theta)\Psi(s, \theta)ds \);

3. \( G_t = \exp \left( -\int_{0}^{t} \lambda^F_s ds + \int_{0}^{t} \Psi(s, s)dW_s - \frac{1}{2} \int_{0}^{t} \Psi^2(s, s)ds \right) \);

where \( \Psi(t, \theta) = \int_{0}^{\theta} \psi(t, u)du \)
Example: “Cox-like” construction. Here

- $\lambda$ is a non-negative $\mathbb{F}$-adapted process, $\Lambda_t = \int_0^t \lambda_s ds$
- $\Theta$ is a given r.v. independent of $\mathcal{F}_\infty$ with unit exponential law
- $V$ is a $\mathcal{F}_\infty$-measurable non-negative random variable
- $\tau = \inf\{t : \Lambda_t \geq \Theta V\}$.

For any $\theta$ and $t$,

$$G_t(\theta) = \mathbb{P}(\tau > \theta|\mathcal{F}_t) = \mathbb{P}(\Lambda_\theta < \Theta V|\mathcal{F}_t) = \mathbb{P}\left(\exp\frac{-\Lambda_\theta}{V} \geq e^{-\Theta} \middle| \mathcal{F}_t\right).$$

Let us denote $\exp(-\Lambda_t/V) = 1 - \int_0^t \psi_s ds$, with

$$\psi_s = \left(\frac{\lambda_s}{V}\right) \exp - \int_0^s \left(\frac{\lambda_u}{V}\right) du,$$

and define $\gamma_t(s) = \mathbb{E}(\psi_s|\mathcal{F}_t)$. Then, $f_t(s) = \gamma_t(s)/\gamma_0(s)$. 
**Backward construction of the density**

Let \( \varphi(\cdot, \alpha) \) be a family of densities on \( \mathbb{R}^+ \), depending of some parameter and \( X \in \mathcal{F}_\infty \) a random variable. Then

\[
\int_0^\infty \varphi(u, X) \, du = 1
\]

and we can choose

\[
f_t(u) = \mathbb{E}(f_\infty(u) | \mathcal{F}_t) = \mathbb{E}(\varphi(u, X) | \mathcal{F}_t)
\]
**G-martingale characterization**

A càdlàg process $Y^G$ is a **G-martingale** if and only if there exist an $\mathbb{F}$-adapted càdlàg process $Y$ and an $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-optional process $Y_t(.)$ such that

$$Y^G_t = Y_t 1_{\{\tau > t\}} + Y_t(\tau) 1_{\{\tau \leq t\}}$$

and that

- $(Y_t G_t + \int_0^t Y_s(s) f_s(s) ds, t \geq 0)$ is an $\mathbb{F}$-local martingale;
- $(Y_t(\theta) f_t(\theta), t \geq \theta)$ is an $\mathbb{F}$-martingale.
Girsanov theorem

Let $Z^G_t = z_t 1_{\{\tau > t\}} + z_t(\tau) 1_{\{\tau \leq t\}}$ be a positive $G$-martingale with $Z^G_0 = 1$ and let $Z^F_t = z_t G_t + \int_0^t z_t(u) f_t(u) du$ be its $F$ projection. Let $Q$ be the probability measure defined on $\mathcal{G}_t$ by $dQ = Z^G_t dP$. Then, $f^Q_t(\theta) = f_t(\theta) \frac{z_t(\theta)}{Z^F_t}$, and:

(i) the $Q$-conditional survival process is defined by $G^Q_t = G_t \frac{z_t}{Z^F_t}$

(ii) the $(F, Q)$-intensity process is $\lambda^F, Q_t = \lambda^F_t \frac{z_t(t)}{z_t-}$, $dt$- a.s.;

(iii) $L^F, Q_t$ is the $(F, Q)$-local martingale

$$L^F, Q_t = L^F_t \frac{z_t}{Z^F_t} \exp \int_0^t (\lambda^F, Q_s - \lambda^F_s) ds$$
The change of probability measure generated by the two processes

\[ z_t = (L_t^F)^{-1}, \quad z_t(\theta) = \frac{f_{\theta}(\theta)}{f_t(\theta)} \]

provides a model where the immersion property holds true, and where the intensity processes does not change.
Several Defaults

We introduce the *conditional joint survival process* $G_t(u, v)$ by setting, for every $u, v, t$,

$$G_t(u, v) = \mathbb{P}(\tau_1 > u, \tau_2 > v \mid \mathcal{F}_t).$$

We assume that the conditional joint density $f_t(u, v) = \partial_{12} G_t(u, v)$ with respect to $u$ and $v$ exists: $G_t(u, v)$ can be represented as follows

$$G_t(u, v) = \int_u^\infty dx \int_v^\infty dy \ f_t(x, y).$$
The process

\[ M_t^1 = H_t^1 - \int_0^{t \wedge \tau_1 \wedge \tau_2} \tilde{\lambda}_u^1 \, du - \int_{t \wedge \tau_1 \wedge \tau_2}^{t \wedge \tau_1} \lambda^{1|2}(u, \tau_2) \, du, \]

is a \( \mathcal{F} \)-martingale, where

\[ \tilde{\lambda}_t^i = -\frac{\partial_i G_t(t, t)}{G_t(t, t)}, \quad \lambda^{1|2}(t, s) = -\frac{f_t(t, s)}{\partial_2 G_t(t, s)} \]

Toy model:

\[ \tilde{\lambda}_t^i = -\frac{\partial_i G(t, t)}{G(t, t)}, \quad \lambda^{1|2}(t, s) = -\frac{f(t, s)}{\partial_2 G(t, s)} \]
CDS price

Let

\[ V_t = \tilde{V}_t \mathbb{1}_{t < \tau_1 \land \tau_2} + \tilde{V}_t(\tau_2) \mathbb{1}_{\tau_2 < t < \tau_1} \]

The dynamics of the price of a CDS are

\[
dV_t = (1 - H^1_t) \left( \kappa - \delta(t) \left((1 - H^2_t) \tilde{\lambda}^1_t + H^2_t \lambda^1_{t^2}(\tau_2) \right) \right) dt \\
- V_{t-} dM^1_t + (1 - H^1_t)(V^1_{t^2}(t) - V_{t-}) dM^2_t \\
+ (1 - H^1_t)((1 - H^2_t) \sigma^1_t + H^2_t \sigma^1_{t^2}(\tau_2)) d\tilde{W}_t
\]

Toy model

\[
dV_t = (1 - H^1_t) \left( \kappa - \delta(t) \left((1 - H^2_t) \tilde{\lambda}^1_t + H^2_t \lambda^1_{t^2}(\tau_2) \right) \right) dt \\
- V_{t-} dM^1_t + (1 - H^1_t)(V^1_{t^2}(t) - V_{t-}) dM^2_t
\]
\[ \tilde{V}_t = \frac{1}{G_t(t, t)} \left( - \int_t^T \delta(u) \partial_1 G_t(u, t) \, du - \kappa \int_t^T G_t(u, t) \, du \right). \]

Toy Model

\[ \tilde{V}_t = \frac{1}{G(t, t)} \left( - \int_t^T \delta(u) \partial_1 G(u, t) \, du - \kappa \int_t^T G(u, t) \, du \right). \]
Several Defaults

\[ V_t^{1|2}(s) = \frac{1}{\partial_2 G_t(t, s)} \left( - \int_t^T \delta(u) f_t(u, s) \, du - \kappa \int_t^T \partial_2 G_t(u, s) \, du \right). \]

Toy model

\[ V_t^{1|2}(s) = \frac{1}{\partial_2 G(t, s)} \left( - \int_t^T \delta(u) f(u, s) \, du - \kappa \int_t^T \partial_2 G(u, s) \, du \right). \]
**Volatility**

From PRT, there exists $g$ such that

$$G_t(u, v) = G_0(u, v) + \int_0^t g_s(u, v) \, dW_s,$$

The volatility is given by

$$\sigma_t^1 = -\frac{1}{G_t(t, t)} \left( \int_t^T (\delta(u) \partial_1 g_t(u, t) + \kappa g_t(u, t)) \, du + g_t(t, t) \hat{V}_t \right)$$

$$\sigma_t^{1|2}(t, s) = -\frac{1}{\partial_2 G_t(t, s)} \left( \int_t^T \delta(u) \partial_{12} g_t(u, s) \, du + \kappa \int_t^T \partial_2 g_t(u, s) \, du + \hat{V}_t \partial_2 g_t(t, s) \right)$$

and the $G$-Brownian motion $\hat{W}$ satisfies

$$\hat{W}_{t \wedge \tau_1} = W_{t \wedge \tau_1} - \int_0^{t \wedge \tau_1 \wedge \tau_2} \frac{g_s(s, s)}{G_s(s, s)} \, ds - \int_{t \wedge \tau_1 \wedge \tau_2}^{t \wedge \tau_1} \frac{\partial_2 g_s(s, \tau_2)}{\partial_2 G_s(s, \tau_2)} \, ds$$
THANK YOU FOR YOUR ATTENTION